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SOME APPLICATIONS OF WEYL CALCULUS TO BURCHNALL-CHAUNDY THEORY, IV

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Abstract

Suppose that A, B are differential operators with polynomial coefficients and they commute. Then there is a polynomial $f(\lambda, \mu)$ such that $f(A, B) = 0$. The set of all such polynomials is a principal ideal J. In this paper, we develop an algorithm for finding the generator of J in some particular cases.

1. Introduction

The paper is in continuation of the papers [2-4].

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Burchnall and Chaundy in [1] proved that two commuting ordinary differential operators P, Q with variable coefficients satisfy nontrivial polynomial identity $f(P, Q) = 0$ with constant coefficients.

We consider two commuting operators A, B with polynomial coefficients and write them using classical representation (Weyl representation also can be used), with differentiation ∂ after multiplication by x. Replacing ∂ with ξ, we obtain symbols of these operators, $a(x, \xi)$, $b(x, \xi)$. For each monomial, we use the weighted degree. The weights of x, and ξ are, respectively, nonnegative integers m, and n not equal to 0 simultaneously. The weighted degree of monomial $x^k \xi^l$ is $mk + nl$. From this moment, the word degree applied to symbols will mean weighted degree, and degree of an operator is equal to degree of its symbol.

Combining all monomials of the same degree together, we can represent $a(x, \xi)$, $b(x, \xi)$ in the form

$$a(x, \xi) = \sum_{i=0}^{r} a_i(x, \xi), \quad b(x, \xi) = \sum_{i=0}^{s} b_i(x, \xi), \tag{1}$$

where $a_i(x, \xi)$, $b_i(x, \xi)$ are polynomials of degree i, none of $a_r(x, \xi)$, $b_s(x, \xi)$ is identically zero, and $r > 0$. Notation $\sigma_0(A)$ is used to denote the principal symbol of A - that is the terms of the highest degree; so $\sigma_0(A) = a_r(x, \xi)$, $\sigma_0(B) = b_s(x, \xi)$. Also, $a_i(x, \xi)$, $b_i(x, \xi)$ satisfy identities:

$$a_i(t^m x, t^n \xi) = t^i a_i(x, \xi), \quad b_i(t^m x, t^n \xi) = t^i b_i(x, \xi). \tag{2}$$

We can represent $a_r(x, \xi)$ in the form:

$$a_r(x, \xi) = d(x, \xi)^{k_0}, \quad k_0 \geq 1, \tag{3}$$

where $d(x, \xi)$ is a polynomial that cannot be represented as a power (greater than one) of another polynomial. Each monomial in $d(x, \xi)$ has the same degree that we denote by d_0 and

$$r = d_0 \cdot k_0. \tag{4}$$

We now formulate the main theorem on which many constructions are based. It is proved in [3].

Theorem 1. *Assume* (3) *and let* $r > 0$. *If* $[A, B] = 0$, *then* $s = 0$ *and* $B = const$, *or* $s > 0$ *and for some positive integer* l_0,

$$b_s(x, \xi) = C \cdot d(x, \xi)^{l_0}, \quad s = d_0 \cdot l_0. \tag{5}$$

Calling the number l_0 to be the order of B, $ord(B) = l_0$, $ord(A) = k_0$. In the remaining part of the paper, assume that $s > 0$ and also that $C = 1$. Note that, we can reduce to this case by replacing *B* with B/C.

We present the proof a theorem from [4] because it will be convenient for the reader later.

Theorem 2. *There is a nontrivial polynomial relation* $f(A, B) = 0$.

Proof. For monomial $A^\alpha B^\beta$ with nonnegative integers α, β, the principal symbol is $d(x, \xi)^{k_0\alpha + l_0\beta}$. Therefore, we will try equation of the form

$$f(A, B) = \sum_{k_0\alpha + l_0\beta \le w} c_{\alpha\beta} A^\alpha B^\beta = 0, \tag{6}$$

where the number *w* is selected later. The number *w* is the weighted degree of $f(A, B)$, where the weight of *A* is equal to the $ordA = k_0$, and the weight of *B* is equal to the $ordB = l_0$; also $ord(A^\alpha B^\beta) = k_0\alpha + l_0\beta$.

Hereafter, the word degree will mean the weighted degree.

We see that equation (6) is equivalent to $(w + 1)$ linear homogeneous equations for $c_{\alpha\beta}$ whereas the number of unknowns $c_{\alpha\beta}$ is equal to the number of points (α, β) with nonnegative coordinates that are below or on the line $k_0\alpha + l_0\beta = w.$ It is clear that the second number, that we will denote as $N(w)$, increases as w^2 so for some w, the number of unknowns is greater than the number of equations and there is a nontrivial solution of (6) and thus the proof.

The principal symbol of $f(A, B)$ is a scalar multiple of $d(x, \xi)^w$ and the scalar is a linear function $L_w(c_{\alpha\beta})$ of coefficients $c_{\alpha\beta}$, so we get our first equation $L_w(c_{\alpha\beta}) = 0.$ If this equation is satisfied, then (by Theorem 1) the principal symbol of $f(A, B)$ is a scalar multiple of $d(x, \xi)^{w-1}$ and the scalar is a linear function $L_{w-1}(c_{\alpha\beta})$ of coefficients $c_{\alpha\beta}$, so our second equation will be $L_{w-1}(c_{\alpha\beta}) = 0.$ Now, assume that the first two equations are satisfied. Then the principal symbol of $f(A, B)$ is a scalar multiple of $d(x, \xi)^{w-2}$ and the scalar is a linear function $L_{w-2}(c_{\alpha\beta})$ of coefficients $c_{\alpha\beta}$, and this gives the third equation $L_{w-2}(c_{\alpha\beta}) = 0$, and so on. Finally, we get a system of equations:

$$L_w(c_{\alpha\beta}) = 0,\ L_{w-1}(c_{\alpha\beta}) = 0,\ L_{w-2}(c_{\alpha\beta}) = 0,\ \ldots,\ L_0(c_{\alpha\beta}) = 0, \tag{7}$$

having $(w + 1)$ equations. For some w, there is a nontrivial solution and this proves the theorem.

We can now introduce the ideal $J = J(A, B) = \{f(\lambda, \mu) | f(A, B) = 0\}.$ Note that J is the principal ideal. The fact should be known, and the author gave the proof in Theorem 3 in [4]. We denote the generating polynomial by $f_0(\lambda, \mu)$, which is irreducible.

2. Preliminary Lemmas

In the main part of the paper, it is assumed that k_0, l_0 are relatively prime numbers. We represent a positive integer s as a nonnegative integer combination of k_0, l_0 in the form

$$s = \alpha k_0 + \beta l_0, \quad \alpha, \beta \geq 0. \tag{8}$$

Such numbers s are called *representable*. It is known that all sufficiently large numbers s are representable. The maximal nonrepresentable number is $(k_0 \cdot l_0 - k_0 - l_0)$ but there many more. For example, if $k_0 = 5$, $l_0 = 3$, then nonrepresentable numbers are 1, 2, 4, 7. We will need later a special formula for nonrepresentable numbers.

Lemma 1. *Suppose that s is a nonrepresentable positive integer. Then*

$$s = u \cdot k_0 - v \cdot l_0, \quad 0 < u < l_0, \quad 0 < v < k_0 \tag{9}$$

or

$$s = \tilde{v} \cdot l_0 - \tilde{u} \cdot k_0, \quad 0 < \tilde{u} < l_0, \quad 0 < \tilde{v} < k_0 \tag{10}$$

and each representation is unique.

Proof. We prove only the first formula, the proof of the second is similar.

From the assumption $\gcd(k_0, l_0) = 1$, it follows that we can write s in the form

$$s = u \cdot k_0 - v \cdot l_0, \quad u, v \geq 0,$$

and the value 0 for u or v is impossible because s is an unrepresentable positive integer; for the same reason, u cannot be a multiple of l_0. Suppose that $u > l_0$. Then we can rewrite the previous formula as

$$s = (u - l_0) \cdot k_0 - (v - k_0) \cdot l_0,$$

and in this formula, we should have $(v - k_0) \geq 0$ because otherwise s would be representable. Repeating this operation several times, we finally come to the formula (u, v denote now the new coefficients)

$$s = u \cdot k_0 - v \cdot l_0, \quad 0 < u < l_0, \quad v > 0.$$

In this new formula, we should have $v < k_0$, because otherwise the same transformation would give us impossible equality

$$s = (u - l_0) \cdot k_0 - (v - k_0) \cdot l_0.$$

The existence of two representations

$$s = u \cdot k_0 - v \cdot l_0 = u' \cdot k_0 - v' \cdot l_0$$

implies that $(u - u')k_0 = (v - v')l_0$ from which it follows that $(u - u')$ is divisible by l_0, but it is impossible because both $0 < u, u' < l_0$.

Another technical lemma, we need, is Lemma 2. Before the proof, one comment is necessary.

In $\alpha\beta$-plane, the line $\alpha k_0 + \beta l_0 = k_0 l_0$ contains exactly two points: $(0, k_0)$, $(l_0, 0)$. Every other line $\alpha k_0 + \beta l_0 = i < k_0 l_0$ contains no more than one point. This is proved in the same way as the proof of uniqueness in Lemma 1.

Lemma 2. *The polynomial* $(\lambda^{l_0} - \mu^{k_0})$ *is irreducible*.

Proof. Suppose that $(\lambda^{l_0} - \mu^{k_0}) = g(\lambda, \mu) \cdot h(\lambda, \mu)$ and assign to λ, μ the weights k_0, l_0, respectively. Then $(\lambda^{l_0} - \mu^{k_0})$ is homogeneous with weight $k_0 \cdot l_0$. Therefore, $g(\lambda, \mu)$ should be homogeneous with weight $i < k_0 \cdot l_0$, and for every monomial $\lambda^\alpha \mu^\beta$ from $g(\lambda, \mu)$, we should have $\alpha k_0 + \beta l_0 = i$. However, the line $\alpha k_0 + \beta l_0 = i$ contains no more than one integer point, so $g(\lambda, \mu)$ itself is a monomial; the same is true for $h(\lambda, \mu)$. Clearly, this is impossible.

3. Construction of a Generator $f_0(\lambda, \mu)$

In this section, we construct a polynomial $f(\lambda, \mu)$ such that

$$f(A, B) = \sum_{k_0\alpha + l_0\beta \le k_0 \cdot l_0} c_{\alpha\beta} A^\alpha B^\beta = 0$$

and find the algorithm for construction of the generator. The order of

$c_{\alpha\beta}A^{\alpha}B^{\beta}$ is $k_0\alpha + l_0\beta$. Therefore, we start with the highest order $k_0 \cdot l_0$, that is the line $\alpha k_0 + \beta l_0 = k_0 l_0$. There are only two points on this line and to eliminate the terms of the highest order, we will select $A^{l_0} - B^{k_0}$ as the beginning of $f(A, B)$. The difference $A^{l_0} - B^{k_0}$ has the order $s < k_0 \cdot l_0$. If s is a representable number, then there is a combination (and only one) $c_{\alpha\beta}A^{\alpha}B^{\beta}$ such that $ord(A^{l_0} - B^{k_0} - c_{\alpha\beta}A^{\alpha}B^{\beta}) < s$. If the new order (smaller than s) is a representable number, then we repeat this operation, and so on. It may happen that we will reach the order zero, that is constant, without any problems. Then we found some polynomial $f(\lambda, \mu)$ such that $f(A, B) = 0$. The leading terms of this polynomial are $(\lambda^{l_0} - \mu^{k_0})$ and they are irreducible, so $f(\lambda, \mu)$ is irreducible and is a scalar multiple of the generator $f_0(\lambda, \mu)$. Our goal is to prove that this is the case. We prove that in the process of reducing the order, we will never encounter unrepresentable order. We begin this with a little bit longer proof. Suppose that we encounter unrepresentable order s_1. Then, we can express what we did in the form of the following equation:

$$A^{l_0} - B^{k_0} - P_1(A, B) = c_1 \cdot R_{s_1}, \quad \sigma_0(R_{s_1}) = d^{s_1}, \quad \deg P_1 < k_0 \cdot l_0. \quad (11)$$

Using Lemma 2, we can find two numbers u_1, v_1 such that $u_1 \cdot k_0 - v_1 \cdot l_0 = s_1$. Create an operator $B^{v_1} \cdot R_{s_1} - A^{u_1}$. From Lemma 2, it follows that $ord(B^{v_1} \cdot R_{s_1} - A^{u_1}) < k_0 \cdot l_0$. We begin now the process of order reduction starting with $B^{v_1} \cdot R_{s_1} - A^{u_1}$. In addition to subtraction of expressions like $c_{\alpha\beta}A^{\alpha}B^{\beta}$, we allow subtraction of expressions like $c \cdot R_{s_1}$ with arbitrary constant c. The process is summarized below:

$$B^{v_1} \cdot R_{s_1} - A^{u_1} - P_2(A, B) - w_{21}R_{s_1} = c_2 \cdot R_{s_2},$$

$$\sigma_0(R_{s_2}) = d^{s_2}, \quad \deg P_2 < k_0 \cdot l_0. \tag{12}$$

Using Lemma 2, we can find two numbers u_2, v_2 such that $u_2 \cdot k_0 - v_2 \cdot l_0 = s_2$. Create an operator $B^{v_2} \cdot R_{s_2} - A^{u_2}$. From Lemma 2, it follows that $ord(B^{v_2} \cdot R_{s_2} - A^{u_2}) < k_0 \cdot l_0$. We begin now the process of order reduction starting with $B^{v_2} \cdot R_{s_2} - A^{u_2}$. In addition to subtraction of expressions like $c_{\alpha\beta}A^{\alpha}B^{\beta}$, we allow subtraction of expressions like $c \cdot R_{s_1}$, $c \cdot R_{s_2}$ with arbitrary constant c. The process is summarized below:

$$B^{v_2} \cdot R_{s_2} - A^{u_2} - P_3(A, B) - w_{31}R_{s_1} - w_{32}R_{s_2} = c_3 \cdot R_{s_3},$$

$$\sigma_0(R_{s_3}) = d^{s_2}, \quad \deg P_3 < k_0 \cdot l_0. \tag{13}$$

Every step in this process creates a new unrepresentable number. So finally, we get an equation without such number; this is the last $(t + 1)$th equation:

$$B^{v_t} \cdot R_{s_t} - A^{u_t} - P_{t+1}(A, B) - \sum_{i=1}^{t} w_{t+1,i}R_{s_i} = 0, \quad \deg P_{t+1} < k_0 \cdot l_0, \tag{14}$$

where the constant in the right hand side was included in $P_{t+1}(A, B)$.

This system of equations can be considered as a homogeneous system with respect to variables 1, $R_{s_1}, R_{s_2}, \dots, R_{s_t}$ in such order. Any expression without $R_{s_1}, R_{s_2}, \dots, R_{s_t}$ will be considered as a coefficient of variable 1. All expressions belong to algebra generated by A, B and this forms a domain, so determinant of the matrix of this system is zero. This is a relation between A and B. Write down the matrix for $t = 4$:

$$\begin{matrix} A^{l_0} - B^{k_0} - P_1(A, B) & -c_1 & 0 & 0 & 0 \\ -A^{u_1} - P_2(A, B) & B^{v_1} - w_{2,1} & -c_2 & 0 & 0 \\ -A^{u_2} - P_3(A, B) & -w_{3,1} & B^{v_2} - w_{3,2} & -c_3 & 0 \\ -A^{u_3} - P_4(A, B) & -w_{4,1} & -w_{4,2} & B^{v_3} - w_{4,3} & -c_4 \\ -A^{u_4} - P_5(A, B) & -w_{5,1} & -w_{5,2} & -w_{5,3} & B^{v_4} - w_{5,4} \end{matrix}$$

In general, the structure is the same. In the first column, the only polynomial of degree $k_0 l_0$ is $A^{l_0} - B^{k_0}$, and all others are of lower degree. The leading term of determinant (we replace A, B with $\lambda, \mu)$ is

$$(\lambda^{l_0} - \mu^{k_0}) \cdot \mu^N, \quad N = v_1 + v_2 + \cdots + v_t. \tag{15}$$

This leading term is divisible by the leading term of unknown generator $f_0(\lambda, \mu)$.

Now we repeat the whole process interchanging the roles of A, B. That is instead of using (9), we use (10), and operators of the type $B^v \cdot R_s - A^u$ are replaced with operators of the type $A^u \cdot R_s - B^v$. As a result, we get another determinant with the leading term

$$(\lambda^{l_0} - \mu^{k_0}) \cdot \lambda^M \tag{16}$$

which is also divisible by the leading term of unknown generator $f_0(\lambda, \mu)$. Thus the leading term of $f_0(\lambda, \mu)$ is a scalar multiple of $(\lambda^{l_0} - \mu^{k_0})$, and hence we can assume that they are the same. This means that in the process of reduction of order, we never encounter unrepresentable order, because all the terms $c_{\alpha\beta} A^\alpha B^\beta$ in the generator and in the process of reduction are the same. It follows from the fact that any line $\alpha k_0 + \beta l_0 = i < k_0 l_0$ contains no more than one point.

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