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LUCAS PERMUTATIONS AND TOGGLES ON FIBONACCI PERMUTATIONS

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Abstract

In this article, we discuss Fibonacci permutations and give a new definition of Lucas permutations. We also define toggle maps on the set of Fibonacci permutations. By using this definition, we establish a bijection between toggle group generated by these maps and permutation group S_{F_n}, where F_n is the nth Fibonacci number.

1. Introduction

The notion of *toggles* was introduced on *order ideals* of a poset by Cameron and Fon-Der-Flaass in [1] and the introduction of the toggle group on the independent sets is due to Joseph and Roby [7]. The toggle group is

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generated by some involutions called *toggles*. Various studies are known on toggle groups [6, 9, 11, 12].

An independent set of a graph is a subset of vertices that does not contain a pair of adjacent vertices.

The present work is motivated by the relation between permutations and Fibonacci sequence [10] together with the relation between toggle groups and permutation groups [9].

The aim of this paper is to give a new definition of Lucas permutations and establish a bijection between the toggle group generated by toggle maps on the set of Fibonacci permutations and the permutation group S_{F_n}, where F_n is the nth Fibonacci number.

The article is organized as follows: In Section 2, we recall some basic notions and set preliminaries on the permutation group S_n. In Section 3, we discuss Fibonacci permutations and give a new definition of Lucas permutations. The last section is devoted to toggles on the set of Fibonacci permutations where we establish a bijection between toggle group generated by toggle maps on the set of Fibonacci permutations and the permutation group S_{F_n}.

2. Preliminaries

Let us denote by S_n the group of permutations of $\{1, 2, ..., n\}$. That is, a permutation σ in S_n is represented by $\sigma = a_1 a_2 \cdots a_n$, where $a_i = \sigma(i)$ for all $1 \leq i \leq n$.

Example 2.1. Consider

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 6 & 1 & 4 & 5 \end{pmatrix}.$$

Then σ is denoted by $\sigma = 236145$.

Each permutation σ of S_n will be associated with a word. The way to associate a word with a permutation is the following as in [10]. Before this, we need some concepts as a rise and a descent.

Definition 2.2. Let $i \in \{1, 2, ..., n\}$ and $\sigma = a_1 a_2 \cdots a_n$ be a permutation of S_n. Then i is said to be a *rise* (resp. *descent*) if and only if $a_i < a_{i+1}$ (resp. $a_{i+1} < a_i$).

Example 2.3. Let $\sigma = 236145$ be in S_6. Then 1 is a rise while 3 is a descent.

The set of the rises is $\{1, 2, 4, 5\}$ and the set of the descents is $\{3\}$.

Definition 2.4. Let $\sigma \in S_n$. Then the word associated with σ consists of $(n-1)$ characters $\omega = z_1 z_2 \cdots z_{n-1}$; $z_i \in \{+, -\}$ such that

$$z_i = \begin{cases} +, & \text{if } a_i < a_{i+1}, \\ -, & \text{otherwise,} \end{cases} \quad i = 1, 2, ..., (n-1).$$

That is, the word associated with $\sigma = 236145$ is $\omega = + + - + +$.

Each word $\omega = z_1 z_2 \cdots z_{n-1}$ contains a double rise (resp. descent) if it contains two consecutive $+$ (resp. $-$).

Let $\beta = 346125$ be in S_6. Then the word associated with β is $\omega_1 = + + - + +$. It results that σ and β have the same word $w = w_1$. It is clear that for each word, one can associate at least one permutation. The way to compute the cardinality of the subset $P(\omega)$ of S_n, which corresponds to a given word ω, has been done [2, 3, 8, 13]. The determination of the elements of the subset $P(\omega)$ has also been done [10]. We recall here the method of the computation of the cardinality of $P(\omega)$ as in [13] and how to enumerate the elements as in [10].

2.1. Cardinality of the set $P(\omega)$

Let ω be a word of $\{+, -\}^{n-1}$ and $P(\omega)$ the subset of S_n which corresponds to ω. Then the cardinality of $P(\omega)$ is given by Propositions 1 and 2 of [13], and we give here the technical computation.

Let $\omega = + - + + -$ be a word and let us rewrite ω as follows: $\omega = rdrrd$, where r is taken to be a rise and d a descent. For Table 1 below, each entry of r or d corresponds to a row and notice that for the row that the entry is r, we go from the left to the right and for the entry d, we go from the right to the left.

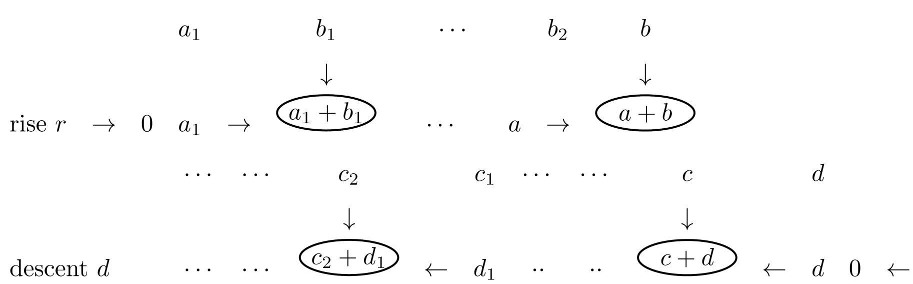
Let $\omega = + - + + -$. Then ω is written as $\omega = rdrrd$.

Table 1

					1			
r			$\rightarrow$	0	1			
d				1	1	0	$\leftarrow$	
r		$\rightarrow$	0	1	2	2		
r	$\rightarrow$	0	0	1	3	5		
d		9	9	9	8	5	0	$\leftarrow$

Hence $P(\omega) = 9 + 9 + 9 + 8 + 5 = 40$. We have 40 permutations in S_6 having $\omega = + - + + -$ as a corresponding word. The arrows indicate the direction of the summation according to whether there is r or d.

These summations can be seen in the following:



Example 2.5. Let $\omega = + - +$ be a word. Then the cardinality of $P(\omega)$ is obtained as follows:

$$\begin{array}{cccccccc} & & & & & 1 & & \\ r & & \rightarrow & & 0 & 1 & & \\ d & & & & 1 & 1 & 0 & \leftarrow \\ r & \rightarrow & & 0 & 1 & 2 & 2 & \end{array}$$

$$P(\omega) = 1 + 2 + 2 = 5.$$

2.2. The set $P(\omega)$

In [10], Panayotopoulos proposed a method to determine the permutations which correspond to a given word. We present here this method which allows us to enumerate all permutations of $P(\omega)$ for a given word ω. This method is based on the factorial tree *T*. In this tree, every path corresponds to a permutation of S_n. The vertices which belong to the $(n - 1)$ levels $N_1, N_2, \ldots, N_{n-1}$ of the tree T correspond to the letters $z_1, z_2, \ldots, z_{n-1}$, of the word $\omega = z_1 z_2 \cdots z_{n-1}$. This method uses branch and bound technique on a sub-tree of *T*. It is based on the following sets:

$$E_0(j) = \{j + 1, j + 2, \ldots, n\},$$

$$\prod\nolimits_0 (j) = \{1, 2, \ldots, j - 1\},$$

$\Gamma(j)$ is the subset of $\{1, 2, \ldots, n\}$, composed with elements which are the terminal vertices of the arcs with initial vertex *j*.

$E(j)$ is the subset of $E_0(j)$, composed with elements which are vertices of the path with first vertex 0 and last vertex *j*.

$\prod(j)$ is the subset of $\prod_0(j)$, composed with elements which are vertices of the path with first vertex 0 and the last vertex *j*.

The following proposition as in [10, Proposition 1] gives the elements of $P(\omega)$.

Proposition 2.6. *The paths of T for which*:

(i) *for the first vertex* 0:

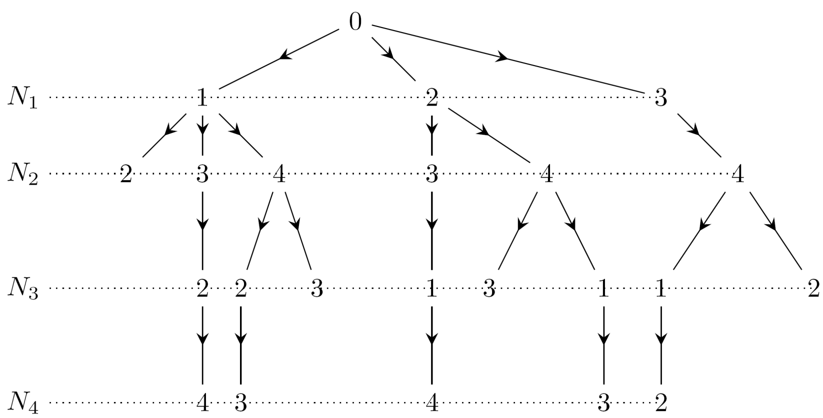
$$\Gamma(0) = \begin{cases} \{1, 2, ..., (n-1)\}, & \text{if } z_1 = +, \\ \{2, 3, ..., n\}, & \text{otherwise}, \end{cases}$$

(ii) *for every level and every vertex j*:

$$\Gamma(j) = \begin{cases} E_0(j) - E(j), & \text{if the level of } j \text{ corresponds to } z = +, \\ \prod_0 (j) - \prod (j), & \text{otherwise}, \end{cases}$$

give the elements of $P(\omega)$.

Example 2.7. For the word $\omega = + - +$, we have the following tree:



For every level N_1, N_2, N_3, N_4 and every vertex *j*, we obtain the following subsets:

$$N_1 : \Gamma(1) = E_0(1) - E(1) = \{2, 3, 4\} - \{\} = \{2, 3, 4\},$$

$$N_2 : \Gamma(2) = \prod_0 (2) - \prod (2) = \{1\} - \{1\} = \varnothing,$$

$$N_3 : \Gamma(3) = E_0(3) - E(3) = \{4\} - \{4\} = \varnothing,$$

$$\Gamma(2) = E_0(2) - E(2) = \{3, 4\} - \{4\} = \{3\}, \ldots.$$

We have $P(\omega) = \{1324, 1423, 2314, 2413, 3412\}$.

3. Fibonacci and Lucas Permutations

In this section, we discuss Fibonacci and Lucas permutations and give some properties of these permutations.

3.1. Fibonacci permutations

We give in the following the definition of Fibonacci permutation as in [4].

Definition 3.1. Let σ be a permutation of S_n. Then the permutation σ is said to be a *Fibonacci permutation* if and only if $|\sigma(i) - i| \leq 1$ for all $i \in \{1, 2, ..., n\}$.

Example 3.2. Let $\sigma_2 = 13254 \in S_5$. Then the permutation σ_2 is a Fibonacci permutation as for $i \in \{1, 2, 3, 4, 5\}$, $|\sigma_2(i) - i| \leq 1$.

The word associated with a Fibonacci permutation will be called a *Fibonacci word*. According to Definition 3.1, it is clear that a Fibonacci word does not contain a double descent. That is, if we take a Fibonacci word $\omega = z_1 z_2 \cdots z_{n-1}$, then there is no two consecutive $(-)$.

Now what will be the word associated with the composition of two Fibonacci permutations? For this answer, we recall Proposition 4 of [4].

Proposition 3.3. *The word of the composition of any two Fibonacci permutations is a Fibonacci word.*

Proof. Let α, β be two Fibonacci permutations of S_n. Then we need to prove that $\forall i \in \{1, ..., n\}$, $\alpha\beta(i+1) < \alpha\beta(i) \Rightarrow \alpha\beta(i+1) < \alpha\beta(i+2)$.

Let us first assume that $\beta(i) < \beta(i+1)$.

By Definition 3.1, $|\alpha\beta(i+1) - \beta(i+1)| \leq 1$ and $|\alpha\beta(i) - \beta(i)| \leq 1$.

Using these two inequalities and the assumption, we obtain

$$\beta(i+1) \leq \beta(i) + 1 \quad \text{and} \quad \beta(i) < \beta(i+1) \leq \beta(i) + 1.$$

Then we get $\beta(i+1) = \beta(i) + 1$.

Combining the last equation and Definition 3.1, we obtain

$$\alpha\beta(i+1) = \beta(i). \tag{1}$$

Since $\beta(i+1) - 1 = \beta(i) < \beta(i+2)$ and notice that α is a Fibonacci permutation, $-1 \le \alpha\beta(i+2) - \beta(i+2) \le 1$ which gives

$$\beta(i+2) - 1 \le \alpha\beta(i+2).$$

We obtain

$$\beta(i) < \beta(i+2) - 1 \le \alpha\beta(i+2). \tag{2}$$

From (1) and (2), we get $\alpha\beta(i+1) < \alpha\beta(i+2)$.

Now we assume that $\beta(i+1) < \beta(i)$.

By Definition 3.1, $-1 \le \beta(i) - i \le 1$ and $-1 \le \beta(i+1) - (i+1) \le 1$, then $i \le \beta(i+1) < \beta(i) \le i+1$. Hence $\beta(i) = i+1$ and $\beta(i+1) = i$. Thus $\beta(i+2) \in \{(i+2), (i+3)\}$.

– If $\beta(i+2) = i+2$, then

$$\alpha\beta(i+2) \ge \beta(i+2) - 1 = i+1 = \beta(i+1) + 1 \ge \alpha\beta(i+1).$$

– If $\beta(i+2) = i+3$, then

$$\alpha\beta(i+2) \ge \beta(i+2) - 1 = i+2 > i+1 = \beta(i+1) + 1 \ge \alpha\beta(i+1).$$

Hence, $\alpha\beta(i+1) < \alpha\beta(i+2)$. □

It was shown [8, Proposition 4] that two Fibonacci permutations which have the same word are equal and it was also proved in Proposition 6 of [4] that for every Fibonacci word ω, there exists a Fibonacci permutation associated with this word ω. Therefore, there exists a bijection between the set $\mathcal{F}$ of Fibonacci permutations and the set $\mathcal{W}$ of Fibonacci words. The following algorithm (see [4]) shows how to associate a Fibonacci word with Fibonacci permutation.

Algorithm:

(1) Start with element 1.

(2) If $z_i = +$ put the element $(i+1)$ in the $(i+1)$th position.

(3) If $z_i = -$ put the element $(i+1)$ in the ith position.

Example 3.4. Let $w = + - + + - +$ be a Fibonacci word. The Fibonacci permutation σ associated with w is constructed as follows:

1

12

132

1324

13245

132465

1324657.

Thus $\sigma = 1324657$.

3.2. Cardinality of the set $\mathcal{F}$

Now, we discuss the cardinality of the set $\mathcal{F}$ of the Fibonacci permutations in S_n. Let the Fibonacci word $\omega = z_1 z_2 \cdots z_{n-1}$ of length $(n-1)$. Let f_{n-1} denote the number of different ways to extend the word ω of length $(n-1)$. If ω begins with $z_1 = -$, then we know immediately that $z_2 = +$ because ω does not have double descent. Thus, there are f_{n-3} different ways to extend ω. Now suppose that ω begins with $z_1 = +$. Then there are f_{n-2} ways to extend ω. Indeed, $f_{n-1} = f_{n-2} + f_{n-3}$ and $f_1 = 2$, $f_1 = 3$. Setting $F_n = f_{n-1}$, this gives $F_n = F_{n-1} + F_{n-2}$ with $F_n = 2$, $F_n = 3$. It is clear that this sequence coincides with the Fibonacci sequence $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$. Hence the cardinality of the set $\mathcal{F}$ is F_n, the nth Fibonacci number.

Theorem 3.5. *Let* $n \geq 1$ *be an integer and* $\mathcal{F}$ *be the set of Fibonacci permutations in permutation group* S_n. *Then the cardinality of the set* $\mathcal{F}$ *is* F_n, *where* F_k *is the kth Fibonacci number of the sequence* $F_k = F_{k-1} + F_{k-2}$ *with* $F_0 = F_1 = 1$.

Let $\omega = z_1 z_2 \cdots z_{n-1}$ be a Fibonacci word. According to Proposition 5 of [4], there is a unique Fibonacci permutation in the set $P(\omega)$. This uniqueness and Proposition 3.3 allow us to say that the set $\mathcal{F}$ is closed under composition.

It is well known that Fibonacci and Lucas sequences are twin sequences. Therefore, we need to talk about Lucas permutations.

3.3. Lucas permutations

In this part, we state what will be the Lucas permutations and give some properties.

Definition 3.6. Let γ be a permutation of S_n. Then the permutation γ is said to be *Lucas permutation* if and only if $|\gamma(i) - i| \leq 1$ for all $i \in \{1, ..., n\}$ and its associated word $\tilde{\omega} = z_1 z_2 \cdots z_{n-1}$ does not begin and end by a descent.

As defined, a Lucas permutation is also a Fibonacci permutation. Indeed, the set $\mathcal{L}$ of Lucas permutations is a subset of the set $\mathcal{F}$ of Fibonacci permutations. For this, all properties obtained for Fibonacci permutations stay truth with the set $\mathcal{L}$.

Example 3.7. Let $\gamma_1 = 21435$ and $\gamma_2 = 21354$ be in S_5. Then γ_1 and γ_2 are both Fibonacci permutations but only γ_1 is Lucas permutation because their associated words are, respectively, $w_1 = - + - +$ and $w_2 = - + + -$.

3.4. Cardinality of the set $\mathcal{L}$

Let us denote $\tilde{\omega} = z_1 z_2 \cdots z_{n-1}$ the word associated with a Lucas permutation and l_{n-1} the number of different ways to extend the word $\tilde{\omega}$. If

$\widetilde{\omega}$ begins with $z_1 = -$, then knowing that $\widetilde{\omega}$ does not contain a double descent, necessarily $z_2 = +$. By Definition 3.6, necessarily $z_{n-1} = +$. Thus, there are F_{n-3} ways to extend $\widetilde{\omega}$. Now, if $\widetilde{\omega}$ begins with $z_1 = +$, then there are F_{n-1} ways to extend $\widetilde{\omega}$. Therefore, $l_{n-1} = F_{n-1} + F_{n-3}$ with $l_2 = 3$ and $l_3 = 4$. Using the relation between Fibonacci and Lucas sequences, that is, $L_n = F_{n+1} + F_{n-1}$, where L_n is the nth Lucas number with $L_n = L_{n-1} + L_{n-2}$ and $L_0 = 1$, $L_1 = 3$. Then $l_{n-1} = F_{n-1} + F_{n-3} = L_{n-2}$ with $L_1 = l_2 = 3$ and $L_2 = l_3 = 4$. It is clear that this sequence coincides with the Lucas sequence $L_0 = 1$, $L_1 = 3$ and $L_n = L_{n-1} + L_{n-2}$. Hence the cardinality of the set $\mathcal{L}$ is L_{n-2}, the $(n-2)$th Lucas number. Then we have the following:

Theorem 3.8. *Let* $n \geq 2$ *be an integer and* $\mathcal{L}$ *be the set of Lucas permutations in* S_n. *Then the cardinality of* $\mathcal{L}$ *is* L_{n-2}, *where* L_k *is the kth Lucas number with* $L_k = L_{k-1} + L_{k-2}$ *and* $L_0 = 1$, $L_1 = 3$.

Remark 3.9. In [5], the authors discuss Lucas permutations but their definition stays different to Definition 3.6 which we give in this article.

4. Toggle on the Set $\mathcal{F}$

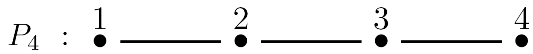
It is known that there is a bijection between the set $\mathcal{F}$ of Fibonacci permutations and its corresponding set $\mathcal{W}$ of Fibonacci words.

For $w = z_1 z_2 \cdots z_{n-1} \in \mathcal{W}$, it is clear that w does not contain a double descent, that is, no two consecutive $(-)$ in w.

Let P_n denote the path graph with vertex set $[n] := \{1, 2, ..., n\}$ and edge set $\{(i, i+1), i \in [n-1]\}$. Then we give the following definition as in [7].

Definition 4.1. An *independent set* of a path graph P_n is a subset of the vertices that does not contain a pair of adjacent vertices.

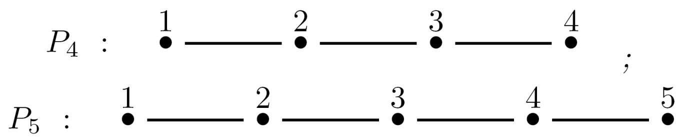
Example 4.2. The path graph with four vertices is



The sets $\{1, 3\}$, $\{2, 4\}$, $\{1, 4\}$ are independent sets but $\{2, 3\}$ is not an independent set because the vertices 2 and 3 are adjacent in P_4.

Let us denote I_n the set of independent sets of P_n. Then it is well known that the cardinality of I_n is F_{n+1}, where F_k is the kth Fibonacci number with $F_0 = F_1 = 1$ and $F_{k+1} = F_k + F_{k-1}$.

Example 4.3. Consider the following two path graphs P_4 and P_5:



The set of independent sets of P_4 is

$$I_4 = \{\varnothing, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 3\}, \{1, 4\}, \{2, 4\}\};$$

$$|I_4| = 8 = F_5.$$

The set of independent sets of P_5 is

$$I_5 = \{\varnothing, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 3\}, \{1, 4\}, \{1, 5\},$$
$$\{2, 4\}, \{2, 5\}, \{3, 5\}, \{1, 3, 5\}\};$$

$$|I_5| = 13 = F_6.$$

Now we can regard a Fibonacci word $w = z_1 z_2 \cdots z_{n-1}$ as a path graph P_{n-1}. An independent set corresponds to the set of indices of those z_i with $z_i = -$. It follows that, according to the cardinality of the set $\mathcal{F}$, there exists a bijection between the set I_{n-1} of independent sets and the set $\mathcal{F}$ of Fibonacci permutations. That is, each Fibonacci permutation can be seen as an independent set. This allows us to talk about toggles on the set $\mathcal{F}$. We recall the notion of toggle maps on independent sets.

Definition 4.4. Let $n \geq 1$ be an integer and I_{n-1} be the set of independent sets of P_{n-1}. Then the *toggle* at vertex k, $k \in [n-1]$, is the map τ_k on I_{n-1} defined as follows:

$$\tau_k : I_{n-1} \to I_{n-1}$$

$$I \mapsto \begin{cases} I \backslash \{k\}, & \text{if } k \in I, \\ I \cup \{k\}, & \text{if } k-1, k, k+1 \notin I, \\ I, & \text{otherwise.} \end{cases}$$

Example 4.5. Consider the independent set I_4 of P_4 given by

$I_4 = \{\varnothing, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 3\}, \{1, 4\}, \{2, 4\}\}$, and

$$\begin{array}{cc} \tau_2 : I_4 \to I_4 & \tau_4 : I_4 \to I_4 \\ \varnothing \mapsto \{2\} & \varnothing \mapsto \{4\} \\ \{1\} \mapsto \{1\} & \{1\} \mapsto \{1, 4\} \\ \{2\} \mapsto \varnothing & \{2\} \mapsto \{2, 4\} \\ \{3\} \mapsto \{3\} & \{3\} \mapsto \{3\} \\ \{4\} \mapsto \{2, 4\} & \{4\} \mapsto \varnothing \\ \{1, 3\} \mapsto \{1, 3\} & \{1, 3\} \mapsto \{1, 3\} \\ \{1, 4\} \mapsto \{1, 4\} & \{1, 4\} \mapsto \{1\} \\ \{2, 4\} \mapsto \{4\} & \{2, 4\} \mapsto \{2\}. \end{array}$$

Let $w = z_1 z_2 \cdots z_{n-1}$ be a Fibonacci word. Denote by $w_i^\star$, $\star \in \{+, -\}$, the word obtained from w by changing z_i by $z_i = \star$. For example, if $w = + - + -$, then $w_3^- = + - - -$ and $w_4^+ = + - + +$. Now we define the analogous of toggles on the set $\mathcal{F}$.

Definition 4.6. Let $n \geq 1$ be an integer, $\mathcal{F}$ the set of Fibonacci permutations and $\mathcal{W}$ its set of Fibonacci words. The toggle at k, $k \in [n-1]$, is the map T_k on $\mathcal{F}$ defined as follows:

$$T_k : \mathcal{F} \to \mathcal{F}$$

$$\sigma_w \mapsto \begin{cases} \sigma_{w_k^+}, & \text{if } z_k = -, \\ \sigma_{w_k^-}, & \text{if } z_{k-1} = +,\ z_k = +,\ z_{k+1} = +, \\ \sigma_w, & \text{otherwise,} \end{cases}$$

where σ_w denotes a Fibonacci permutation σ with w its associated Fibonacci word.

Example 4.7. Let $\mathcal{F}$ be the set of Fibonacci permutations of S_5. Then

(1) For the toggle T_2 on $\mathcal{F}$,

$$- + + - \overset{T_2}{\mapsto} - + + -$$

$$+ - + + \overset{T_2}{\mapsto} + + + +$$

$$+ + - + \overset{T_2}{\mapsto} + + - +$$

$$+ + + - \overset{T_2}{\mapsto} + - + -.$$

(2) Next, for the toggle map T_3 on $\mathcal{F}$,

$$- + + - \overset{T_3}{\mapsto} - + + -$$

$$+ - + + \overset{T_3}{\mapsto} + - + +$$

$$+ + - + \overset{T_3}{\mapsto} + + + +$$

$$+ + + - \overset{T_3}{\mapsto} + + + -.$$

As defined, the toggle map T_k is involutive. The toggle maps $T_1, T_2, ..., T_{n-1}$ generate a group G_{n-1} called the *toggle group*. This group acts on the set $\mathcal{F}$. Thus the toggle group is a subgroup of the permutation group on $\mathcal{F}$. Indeed, we give the analogous of Theorem 3.7 of [9].

Theorem 4.8. *Let* $n \geq 1$ *be an integer. Then the toggle group* G_{n-1} *generated by the toggle maps* $T_1, T_2, ..., T_{n-1}$ *is isomorphic to the permutation group* S_{F_n}, *where* F_n *is the nth Fibonacci number with* $F_n = F_{n-1} + F_{n-2}$ *and* $F_0 = F_1 = 1$.

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