



## IDENTITIES ON $b$ -GENERALIZED DERIVATIONS AND CENTRALIZERS OF RINGS

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### Abstract

The current research motivated by the structural study of  $b$ -generalized derivations. The  $b$ -generalized derivation is the generalized version of generalized derivation on an extended ring. By  $b$ -generalized derivations  $\mathfrak{F}$  we mean a map from a ring  $\mathcal{F}$  to Martindale ring of quotient  $Q_m^r(\mathcal{F})$  such that  $\mathfrak{F}(vy) = \mathfrak{F}(v)y + bvd(y)$ , with associated map  $d$  for every  $y, v \in \mathcal{F}$ . We establish commutativity theorems by investigating some differential identities involved with  $b$ -generalized derivations and centralizers. Moreover, we obtain a non-commutative structure under certain specific conditions. Suitable examples are given to justify the concept and in favor of better understanding.

### 1. Introduction

Throughout the manuscript, the notation  $\mathcal{Z}(\mathcal{F})$  stands for the center of an associative ring  $\mathcal{F}$ . The symmetric right ring of quotient represents as  $Q_s^r(\mathcal{F})$  and  $Q_m^r(\mathcal{F})$  stands for Martindale right ring of quotients. The center will be represented by  $Z(Q_m^r(\mathcal{F})) = Z(Q_s^r(\mathcal{F})) = \mathfrak{Z}$  and will be called as *extended centroid* of  $\mathcal{F}$ . The known fact is  $\mathcal{F} \subseteq Q_s^r(\mathcal{F}) \subseteq Q_m^r(\mathcal{F})$  and the over ring  $Q_m^r(\mathcal{F})$  is prime if  $\mathcal{F}$  is prime with given center  $\mathfrak{Z}$ . For thorough study, we refer the reader to [6].

The symbol  $[l, k]$  specifies the commutator of  $l, k \in \mathcal{F}$ , which is represented by the mathematical formula  $lk - kl$ . If  $pr = 0$  yields  $r = 0$  for all  $r \in \mathcal{F}$  and  $p > 1$  is a fixed integer, then  $\mathcal{F}$  is a  $p$ -torsion free ring. A ring  $\mathcal{F}$  is a prime if  $r\mathcal{F}t = \{0\}$  gives that either  $t = 0$  or  $r = 0$ . It is called *semiprime* if it fulfills the requirement that  $c\mathcal{F}c = \{0\}$  implies  $c = 0$ .

A mapping  $d$  from  $\mathcal{F}$  to  $\mathcal{F}$ , that is additive, and fulfills  $d(ce) = d(c)e + cd(e)$ , for each of  $c, e \in \mathcal{F}$ , then it is regarded as a derivation on  $\mathcal{F}$ . A map  $\mathfrak{G} : \mathcal{F} \rightarrow \mathcal{F}$  having additivity and is termed as a generalized

derivation if there exists a derivation  $d$  of  $\mathcal{F}$  such that, for all  $c, w \in \mathcal{F}$ ,  $\mathfrak{G}(cw) = \mathfrak{G}(c)w + cd(w)$  ( $d$  will be called as *derivation associated* with  $\mathfrak{G}$ ). Derivations and generalized inner derivations are fundamental examples. Operator algebras have been the main focus of research on generalized derivations. Consequently, any research from an algebraic perspective might be intriguing, one can see [1-5, 9, 13, 17] for more details.

Commutativity in semiprime and prime rings was studied by a number of authors, who discovered derivations and generalized derivations that meet the necessary algebraic criteria on relevant subsets of the rings. If  $K$  is a nonzero ideal of a semiprime ring  $\mathcal{F}$  and  $d$  is a derivation on  $\mathcal{F}$  such that  $d([\zeta, \tau]) = \mp[\zeta, \tau]$ , then  $K \subseteq Z(\mathcal{F})$  for every  $\zeta, \tau \in K$ , as demonstrated by Daif and Bell [8].

When the generalized derivation  $\mathfrak{G}$  appears on one sided ideal of  $\mathcal{F}$ , the authors in [1] looked into the previously mentioned result and proved that: let  $K$  be a nonzero left ideal of a semiprime ring  $\mathcal{F}$ . If  $\mathcal{F}$  possesses a generalized derivation  $\mathfrak{G}$ , having derivation  $d$  such that  $d(K) \neq (0)$ , then  $\mathcal{F}$  has a nonzero central ideal, if (i)  $\mathfrak{G}(uw) \mp wu \in Z(\mathcal{F})$  and (ii)  $\mathfrak{G}(uw) \mp [u, w] \in Z(\mathcal{F})$ , hold for each  $u, w \in \mathcal{F}$ .

A mapping  $\mathfrak{H} : \mathcal{F} \rightarrow \mathcal{F}$  (not necessarily additive) is said to be a *left (right) centralizer* if it satisfies  $\mathfrak{H}(uw) = \mathfrak{H}(u)w$  ( $\mathfrak{H}(uw) = u\mathfrak{H}(w)$ ) for  $w, u \in \mathcal{F}$ . Centralizers play an important role in finding the commutativity of structures like rings, algebra and their subsets. The theory of centralizers and their crucial role in the development of ring structure can be found in [12].

Following [11], by a  $b$ -generalized derivation  $\mathfrak{F}$ , we mean a map from a ring  $\mathcal{F}$  to the maximal right ring of quotient  $Q_m^r(\mathcal{F})$  such that  $\mathfrak{F}(wy) = \mathfrak{F}(w)y + bwd(y)$ , with associated map  $d$  for every  $w, y \in \mathcal{F}$  and for some  $0 \neq b \in Q_s^r(\mathcal{F}) \subseteq Q_m^r(\mathcal{F})$  provided  $\mathfrak{F}$  is additive. It is obvious to see that

for  $b = 1$  the map  $\mathfrak{F}$  will be ordinary generalized derivation in case  $\mathfrak{F}$  is additive. Hence,  $b$ -generalized derivations are extended version of generalized derivations. Few examples are listed below:

**Example 1.1.** Consider  $F$  is a field having char 2. Suppose that  $\mathcal{F} = \left\{ \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} \mid u, v, w \in F \right\}$ . Then construct  $\mathfrak{F} : \mathcal{F} \rightarrow \mathcal{F}$  as  $\mathfrak{F}(v) = e_{11}v + e_{11}ve_{22}$  for all  $v \in \mathcal{F}$ . Take a mapping  $d : \mathcal{F} \rightarrow \mathcal{F}$  as  $d(v) = e_{11}ve_{22}$  for  $v \in \mathcal{F}$ , then it is easy to see that  $\mathfrak{F}$  will be a  $b$ -generalized derivation with derivation  $d$ , for any fixed  $b \in \mathcal{F}$ .

**Example 1.2.** Suppose that  $\mathcal{F} = \left\{ \begin{pmatrix} 0 & u & v \\ 0 & 0 & w \\ 0 & 0 & 0 \end{pmatrix} \mid u, v, w \in \mathbb{Z} \right\}$  and the

mappings  $\mathfrak{F}, d : \mathcal{F} \rightarrow \mathcal{F}$  defined by  $\mathfrak{F} \begin{pmatrix} 0 & u & v \\ 0 & 0 & w \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & v \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ , and

$d \begin{pmatrix} 0 & u & v \\ 0 & 0 & w \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & w \\ 0 & 0 & 0 \end{pmatrix}$ , since  $b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ,  $\mathfrak{F}$  is a  $b$ -generalized

derivation associated with derivation  $d$ .

**Example 1.3.** Let  $\mathcal{F} = \left\{ \begin{pmatrix} u & 0 & v \\ 0 & 0 & w \\ 0 & 0 & 0 \end{pmatrix} \mid u, v, w \in \mathbb{Z} \right\}$ , and the mappings

$\mathfrak{F}, d : \mathcal{F} \rightarrow \mathcal{F}$  defined by  $\mathfrak{F} \left\{ \begin{pmatrix} u & 0 & v \\ 0 & 0 & w \\ 0 & 0 & 0 \end{pmatrix} \right\} = \begin{pmatrix} u & 0 & v \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ,  $d \left\{ \begin{pmatrix} u & 0 & v \\ 0 & 0 & w \\ 0 & 0 & 0 \end{pmatrix} \right\}$

$= \begin{pmatrix} u & 0 & v \\ 0 & 0 & w \\ 0 & 0 & 0 \end{pmatrix}$ , since  $b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ ,  $\mathfrak{F}$  is a  $b$ -generalized derivation

associated with derivation  $d$ .

The nature of mappings and behaviour of ring may or may not be commutative explore in [14, 15]. Recently, author's in [16] obtained interesting results for  $b$  generalized derivations on a dense left ideal of  $Q_m^r$  and established the commutativity of underlying ring. We establish commutativity theorems by investigating some differential identities involved  $b$ -generalized derivations and centralizers.

## 2. Main Results

Consider  $0 \neq b$  is a fixed element in  $\mathcal{F}$  throughout. We start with the following lemmas:

**Lemma 2.1** [10]. *The center of  $\mathcal{F}$  includes the center of a nonzero ideal (one-sided) for a semiprime ring  $\mathcal{F}$ . Any commutative ideal (one-sided) is immediately enclosed  $Z(\mathcal{F})$ .*

**Lemma 2.2** [7]. *Let  $\mathcal{I}$  be a nonzero left ideal of a prime ring  $\mathcal{F}$ . If  $\mathcal{F}$  admits a nonzero derivation  $d$  such that  $[d(v), v] \in Z(\mathcal{F})$  for every  $v \in \mathcal{F}$ , then  $\mathcal{F}$  is commutative.*

**Lemma 2.3** [6]. *For any  $\kappa \in (Q_s^r(\mathcal{F}))$ , there exists a dense right ideal  $J$  such that  $\kappa J = \{0\}$  (or  $J\kappa = \{0\}$ ). Then  $\kappa = 0$ .*

**Theorem 2.1.** *Let a prime ring be  $\mathcal{F}$ ,  $Q_s^r$  be the symmetric quotient ring and  $Q_m^r$  be the Martindale ring of quotients. If  $\mathfrak{F}$  is a  $b$ -generalized derivation with associated derivation  $d$  and  $\mathfrak{H}$  is a centralizer such that  $\mathfrak{F}(vy) \pm \mathfrak{H}(yv) = 0, \forall v, y \in \mathcal{F}$ , then one of the following holds:*

- (1)  $\mathcal{F}$  is commutative.
- (2)  $\mathfrak{F}$  acting as centralizer provided  $d = 0$ .

**Proof.** First consider the case

$$\mathfrak{F}(vy) + \mathfrak{H}(yv) = 0, \text{ for every } v, y \in \mathcal{F}. \tag{2.1}$$

Replace  $y$  by  $yz$  in (2.1) to have

$$\mathfrak{F}(vy)z + bvyd(z) + \mathfrak{H}(yz)v = 0 \text{ for every } v, y, z \in \mathcal{F}. \quad (2.2)$$

Reword (2.2) by adding and subtracting the term  $\mathfrak{H}(y\mathfrak{v}z)$  to find

$$(\mathfrak{F}(vy) + \mathfrak{H}(y\mathfrak{v}))z + bvyd(z) + \mathfrak{H}(y)[v, z] = 0 \text{ for every } v, y, z \in \mathcal{F}. \quad (2.3)$$

In view of (2.2), (2.3) takes the form

$$bvyd(z) + \mathfrak{H}(y)[v, z] = 0 \text{ for every } v, y, z \in \mathcal{F}. \quad (2.4)$$

Particularly, we can write (2.4) as

$$bzyd(z) = 0 \text{ for every } y, z \in \mathcal{F}. \quad (2.5)$$

Substitute  $ry$  in place of  $y$  in (2.5) and multiply by  $u$  from right to get

$$bzryd(z)u = 0, \text{ where } y, z, u, r \in \mathcal{F}. \quad (2.6)$$

Put  $y = ryu$  in (2.5) to obtain

$$bzryud(z) = 0 \text{ for every } y, z, r, u \in \mathcal{F}. \quad (2.7)$$

Subtract (2.6) and (2.7) to find

$$bzry[d(z), u] = 0 \text{ for every } z, y, r, u \in \mathcal{F}. \quad (2.8)$$

Using property of symmetric ring of quotient, we can substitute  $z = ybz$  in (2.8) and commute with  $w$ , we have

$$\begin{aligned} [bybzry[d(z), u], w] &= 0 \text{ for every } y, z, w, u, r \in \mathcal{F}, \\ by[bzry[d(z), u], w] + [by, w]bzry[d(z), u] &= 0 \\ &\text{for every } y, z, w, u, r \in \mathcal{F}. \end{aligned} \quad (2.9)$$

From (2.8) and (2.9), we can have

$$[by, w]bzry[d(z), u] = 0 \text{ for every } y, z, w, u, r \in \mathcal{F}. \quad (2.10)$$

This implies that

$$[by, w]b\mathcal{F}y[d(z), u] = 0 \text{ for every } y, z, w, u \in \mathcal{F}. \quad (2.11)$$

Primeness of  $\mathcal{F}$  gives that either  $[by, w]b = 0$  or  $y[d(z), u] = 0 \forall y, w, u, z \in \mathcal{F}$ . Using Brauer's concept that if  $\mathcal{F}$  is the union of two additive subgroups say  $\mathcal{L}$  and  $K$  such that  $\mathcal{L} = \{w \in \mathcal{F} \mid [by, w]b = 0\}$  and  $K = \{z \in \mathcal{F} \mid y[d(z), u] = 0\}$ . Then either  $\mathcal{L} = \mathcal{F}$  or  $\mathcal{F} = K$ . Now, we consider the following two cases:

(1) If  $\mathcal{L} = \mathcal{F}$ , then  $[by, w]b = 0 \forall y \in \mathcal{F}$ .

Put  $w = vw$  to get

$$[by, v]wb + v[by, w]b = 0,$$

$$[by, v]wb = 0 \forall v, y, w \in \mathcal{F}.$$

This implies that  $[by, v]\mathcal{F}wb = 0 \forall v, y, w \in \mathcal{F}$ . Primeness yields that either  $wb = 0$  or  $[by, v] = 0$ . If  $wb = 0$ , then by Lemma 2.3, we have  $b = 0$ . This is a contradiction. Next consider  $[by, v] = 0$ . Put  $yz$  in place of  $y$  to find  $by[z, v] + [by, v]z = 0$ , which gives that  $by[z, v] = 0$  for each  $y, v, z \in \mathcal{F}$ . Again using primeness, we get  $[z, v] = 0 \forall v, z \in \mathcal{F}$  because  $by \neq 0$ . Hence,  $\mathcal{F}$  is commutative.

(2) If we take  $\mathcal{F} = K$ , then we have  $y[d(z), u] = 0 \forall u, z, y \in \mathcal{F}$ . This implies that  $[d(z), u] = 0, \forall z, u \in \mathcal{F}$ . Hence,  $\mathcal{F}$  is commutative by application of Lemma 2.2, if  $d \neq 0$  on  $\mathcal{F}$ . Now, conclude the last part of the theorem for the condition  $[d(z), u] = 0$ , for each  $z, u \in \mathcal{F}$ . In this case, if  $d = 0$  on  $\mathcal{F}$ , then by definition  $\mathfrak{F}$  reduces to the expression  $\mathfrak{F}(v\omega) = \mathfrak{F}(v)\omega$ , for every  $v, \omega \in \mathcal{F}$ , a left centralizer. □

**Theorem 2.2.** *Let a prime ring be  $\mathcal{F}$ ,  $Q_s^r$  be the symmetric quotient ring,  $Q_m^r$  be the Martindale ring of quotients, and  $\mathfrak{H}$  be a left centralizer on  $\mathcal{F}$ . If  $\mathfrak{F}$  is a  $b$ -generalized derivation associated with a derivation  $d$  satisfying  $\mathfrak{F}(v\omega) \mp \mathfrak{H}(v\omega) = 0$  for every  $v, \omega \in \mathcal{F}$ , then one of the following holds:*

- (1)  $\mathcal{F}$  is commutative.
- (2)  $\mathfrak{F}$  acts as a centralizer provided  $d = 0$ .

**Proof.** By the hypothesis, first we consider the case

$$\mathfrak{F}(v\omega) - \mathfrak{H}(v\omega) = 0, \text{ for every } v, \omega \in \mathcal{F}. \tag{2.12}$$

Replace  $\omega$  by  $\omega z$  in (2.12) to obtain

$$\mathfrak{F}(v\omega z) - \mathfrak{H}(v\omega z) = 0, \text{ for every } v, \omega, z \in \mathcal{F}. \tag{2.13}$$

On simplification of (2.13), we get

$$\mathfrak{F}(v\omega)z + bv\omega d(z) - \mathfrak{H}(v\omega)z = 0, \text{ for every } v, \omega, z \in \mathcal{F}. \tag{2.14}$$

This implies that

$$(\mathfrak{F}(v\omega) - \mathfrak{H}(v\omega))z + bv\omega d(z) = 0, \text{ for every } v, \omega, z \in \mathcal{F}. \tag{2.15}$$

In view of (2.12), (2.15) yields that

$$bv\omega d(z) = 0, \text{ where } v, \omega, z \in \mathcal{F}. \tag{2.16}$$

After this, following the similar steps as in Theorem 2.1 from equation (2.5), we conclude the desired result.

**Theorem 2.3.** *Let a prime ring be  $\mathcal{F}$ ,  $Q_s^r$  be the symmetric ring of quotients and  $Q_m^r$  be Martindale ring of quotients. If  $\mathfrak{F}$  is a  $b$ -generalized derivation with associated derivation  $d$  and  $\mathfrak{H}$  is a centralizer such that  $\mathfrak{F}(v)\mathfrak{F}(y) \pm \mathfrak{H}(vy) = 0$  for all  $v, y \in \mathcal{F}$ , then either  $\mathfrak{F} = 0$ , or  $\mathcal{F}$  is commutative.*

**Proof.** By the hypotheses, first we consider the case

$$\mathfrak{F}(v)\mathfrak{F}(y) \pm \mathfrak{H}(vy) = 0 \text{ for all } v, y \in \mathcal{F}. \tag{2.17}$$

Replacing  $y$  by  $yz$  in (2.17) to obtain

$$\mathfrak{F}(v)\mathfrak{F}(yz) \pm \mathfrak{H}(vyz) = 0 \text{ for all } v, y, z \in \mathcal{F}. \tag{2.18}$$

On simplification of (2.18), we get

$$\mathfrak{F}(v)\mathfrak{F}(y)z + \mathfrak{F}(v)byd(z) + \mathfrak{H}(vy)z = 0 \text{ for all } v, y, z \in \mathcal{F}. \quad (2.19)$$

In view of (2.17) and (2.19), it reduces to

$$\mathfrak{F}(v)byd(z) = 0 \text{ for all } v, y, z \in \mathcal{F}. \quad (2.20)$$

Replacing  $y$  by  $yr$  in (2.20) for all  $r \in \mathcal{F}$ , we get

$$\mathfrak{F}(v)bryd(z) = 0 \text{ for all } r, v, y, z \in \mathcal{F}. \quad (2.21)$$

Replacing  $y$  by  $yz$  in (2.21) for all  $z \in \mathcal{F}$ , so we obtain

$$\mathfrak{F}(v)bryzd(z) = 0 \text{ for all } r, v, y, z \in \mathcal{F}. \quad (2.22)$$

Multiplying (2.21) from right by  $z$ , then we get

$$\mathfrak{F}(v)bryd(z)z = 0 \text{ for all } r, v, y, z \in \mathcal{F}. \quad (2.23)$$

Subtracting (2.22) from (2.23), we get

$$\mathfrak{F}(v)bry[d(z), z] = 0 \text{ for all } r, v, y, z \in \mathcal{F}. \quad (2.24)$$

This implies that

$$\mathfrak{F}(v)b\mathcal{F}yzd(z) = 0 \text{ for all } v, y, z \in \mathcal{F}. \quad (2.25)$$

Primeness of  $\mathcal{F}$  gives that either  $\mathfrak{F}(v)b = 0$  or  $y[d(z), z] = 0$  for all  $v, y, z \in \mathcal{F}$ . Since  $\mathcal{F}$  is seen as the union of two proper additive subgroups namely  $\mathfrak{K}_1^+ = \{v \in \mathcal{F} \mid \mathfrak{F}(v)b = 0\}$  and  $\mathfrak{K}_2^+ = \{z \in \mathcal{F} \mid y[d(z), z] = 0\}$ . We draw the conclusion that any one of the conditions is true  $\mathfrak{K}_1^+ = \mathcal{F}$  or  $\mathfrak{K}_2^+ = \mathcal{F}$ .

**Case 1.** In case  $\mathfrak{K}_1^+ = \mathcal{F}$ ,  $\mathfrak{F}(v)b = 0$  for each  $v \in \mathcal{F}$ . Applying Lemma 2.3, we find  $b = 0$ , a contradiction. Hence,  $\mathfrak{F}(v) = 0$  for each  $v \in \mathcal{F}$ .

**Case 2.** For the case  $\mathfrak{K}_2^+ = \mathcal{F}$  and  $d \neq 0$ , it follows that  $y[d(z), z] = 0$ , for every  $y, z \in \mathcal{F}$ . Now replace  $y$  by  $yr$  for all  $r \in \mathcal{F}$ , and we get  $yr[d(z), z] = 0$  for all  $r, y, z \in \mathcal{F}$ . In particular it follows that

$y\mathcal{F}[d(z), z] = 0$  for each  $y, z \in \mathcal{F}$ . Because of primeness  $\mathcal{F}$  splits into two cases either  $y = 0$  which is a contradiction, or  $[d(z), z] = 0$  for each  $z \in \mathcal{F}$ . Hence by Lemma 2.2,  $\mathcal{F}$  is commutative.  $\square$

**Theorem 2.4.** *Let a prime ring be  $\mathcal{F}$ ,  $Q_s^r$  be the symmetric ring of quotients and  $Q_m^r$  be the Martindale ring of quotients. If  $\mathfrak{F}$  is a  $b$ -generalized derivation with associated derivation  $d$  and  $\mathfrak{H}$  is a centralizer such that  $\mathfrak{F}(vy) \pm \mathfrak{H}([v, y]) = 0$  for all  $v, y \in \mathcal{F}$ , then  $\mathcal{F}$  is commutative.*

**Proof.** By the hypotheses, first we consider the case

$$\mathfrak{F}(vy) \pm \mathfrak{H}([v, y]) = 0 \text{ for all } v, y \in \mathcal{F}. \quad (2.26)$$

Replacing  $y$  by  $yz$  in (2.26), our hypothesis reduces to

$$(\mathfrak{F}(vy) + \mathfrak{H}([v, y]))z + bvyd(z) + \mathfrak{H}(y[v, z]) = 0 \text{ for all } v, y, z \in \mathcal{F}. \quad (2.27)$$

By using our hypothesis in (2.27), we get

$$bvyd(z) + \mathfrak{H}(y[v, z]) = 0 \text{ for all } v, y, z \in \mathcal{F}. \quad (2.28)$$

Replacing  $z$  by  $v$  in (2.28) for all  $v \in \mathcal{F}$ , we get

$$bvyd(v) = 0 \text{ for all } v, y \in \mathcal{F}. \quad (2.29)$$

Replacing  $y$  by  $yr$  in (2.29) for all  $r \in \mathcal{F}$ , so we obtain

$$bvryd(v) = 0 \text{ for all } r, v, y \in \mathcal{F}. \quad (2.30)$$

Replacing  $y$  by  $yz$  in (2.30) for all  $z \in \mathcal{F}$ , yields that

$$bvryzd(v) = 0 \text{ for all } r, v, y, z \in \mathcal{F}. \quad (2.31)$$

Multiplying (2.30) from right by  $z$ , then we get

$$bvryd(v)z = 0, \text{ where } r, v, y, z \in \mathcal{F}. \quad (2.32)$$

Subtracting (2.31) from (2.32), we get

$$bvry[d(v), z] = 0 \text{ for all } r, v, y, z \in \mathcal{F}. \quad (2.33)$$

This implies that

$$bv\mathcal{F}y[d(v), z] = 0 \text{ for all } v, y, z \in \mathcal{F}. \tag{2.34}$$

Therefore, either  $bv = 0$  or  $y[d(v), z] = 0$  for all  $v, y, z \in \mathcal{F}$ , by primeness of  $\mathcal{F}$ . Since  $\mathcal{F}$  is seen as the union of two proper additive subgroups namely  $\mathfrak{K}_1^+ = \{vb = 0 | v \in \mathcal{F}\}$  and  $\mathfrak{K}_2^+ = \{z \in \mathcal{F} | y[d(v), z] = 0\}$ . We draw the conclusion that any one of the conditions is true  $\mathfrak{K}_1^+ = \mathcal{F}$  or  $\mathfrak{K}_2^+ = \mathcal{F}$ .

**Case 1.** In case  $\mathfrak{K}_1^+ = \mathcal{F}$ ,  $vb = 0$  for each  $v \in \mathcal{F}$ . Applying Lemma 2.3, we find  $b = 0$ , a contradiction for each  $0 \neq v \in \mathcal{F}$ .

**Case 2.** For the case  $\mathfrak{K}_2^+ = \mathcal{F}$  and  $d \neq 0$ , it follows that  $y[d(v), z] = 0$ , for every  $y, v, z \in \mathcal{F}$ . Now replace  $y$  by  $yr$  for all  $r \in \mathcal{F}$ , and we get  $yr[d(z), z] = 0$  for all  $r, y, z \in \mathcal{F}$ . In particular it follows that  $y\mathcal{F}[d(z), z] = 0$  for each  $y, z \in \mathcal{F}$ . Because of primeness  $\mathcal{F}$  splits into two cases either  $y = 0$  which is a contradiction, or  $[d(z), z] = 0$  for each  $z \in \mathcal{F}$ . Hence by Lemma 2.2,  $\mathcal{F}$  is commutative. □

**Theorem 2.5.** *Let a prime ring be  $\mathcal{F}$ ,  $Q_s^r$  be the symmetric ring of quotients and  $Q_m^r$  be the Martindale ring of quotients. If  $\mathfrak{F}$  is a  $b$ -generalized derivation with associated derivation  $d$  and  $\mathfrak{H}$  is a centralizer such that  $\mathfrak{F}([v, y]) \pm \mathfrak{H}([v, y]) = 0$  for all  $v, y \in \mathcal{F}$ , then  $\mathcal{F}$  is commutative.*

**Proof.** We are given that

$$\mathfrak{F}([v, y]) \pm \mathfrak{H}([v, y]) = 0 \text{ for each } v, y \in \mathcal{F}. \tag{2.35}$$

Replace  $y$  by  $yv$  in (2.35) to obtain

$$\mathfrak{F}([v, y]v) \pm \mathfrak{H}([v, y]v) = 0 \text{ for every } v, y \in \mathcal{F}. \tag{2.36}$$

On simplification of (2.36), we find

$$\{\mathfrak{F}([v, y]) \pm \mathfrak{H}([v, y])\}v + b[v, y]d(v) = 0 \text{ for every } v, y \in \mathcal{F}. \quad (2.37)$$

In view of (2.36), (2.37) takes the form

$$b[v, y]d(v) = 0 \text{ for each } v, y \in \mathcal{F}. \quad (2.38)$$

After equation (2.38), an application of Theorem 2.4, completes the proof after following similar footsteps with some necessary manipulations.  $\square$

After visualize all the above differential identities, we deduce some interesting relationship between  $b$ -generalized derivation  $\mathfrak{F}$  and a left centralizer  $\mathfrak{H}$  defined on a given  $\mathcal{F}$ . We prove the following:

**Theorem 2.6.** *Consider a ring  $\mathcal{F}$ , a multiplicative left centralizer  $\mathfrak{H}$ , and a  $b$ -generalized derivation  $\mathfrak{F}$  connected to  $d$ . Then  $G$  is a  $b$ -generalized derivation associated with  $d$ , if the map  $G : \mathcal{F} \rightarrow \mathcal{F}$  is defined as  $G(v) = \mathfrak{F}(v) \mp \mathfrak{H}(v)$  for all  $v \in \mathcal{F}$ .*

**Proof.** Consider, for each  $v \in \mathcal{F}$ ,  $G(v) = \mathfrak{F}(v) \mp \mathfrak{H}(v)$ . Then, for every  $v, y \in \mathcal{F}$ ,  $G(vy) = \mathfrak{F}(vy) \mp \mathfrak{H}(vy) = \mathfrak{F}(v)y + bvd(y) \mp \mathfrak{H}(v)y$ , which implies that for every  $v, y \in \mathcal{F}$ ,

$$G(vy) = (\mathfrak{F}(v) \mp \mathfrak{H}(v))y + bvd(y) = G(v)y + bvd(y).$$

In this case,  $G$  is a multiplicative  $b$ -generalized derivation related to  $d$ .  $\square$

### 3. Conclusion

The current study shows the structural development of  $b$ -generalized derivations on prime ring. We conclude the following facts as an application: (1)  $\mathfrak{F}$  is a zero map under some restricted conditions; (2)  $\mathfrak{F}$  acts as a left centralizer (independent from the condition  $b \neq 0$ ); and (3) set up the relationship between  $b$ -generalized derivation  $\mathfrak{F}$  and a left centralizer  $\mathfrak{H}$  defined on a given ring  $\mathcal{F}$ .

Future studies will find it intriguing to observe how these mappings function in different contexts. We also explain an instance in which  $\mathfrak{H}$  serves as a centralizer and  $\mathcal{F}$  is commutative. Our outcomes are significant in generalized matrix algebra, additive mapping, and related topics in addition to the established results.

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