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HIGHER-ORDER CAHN-HILLIARD MODELS WITH SINGULAR NONLINEAR TERMS

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Abstract

Our aim in this article is to study the well-posedness for a class of higher-order (in space) anisotropic Cahn-Hilliard models with singular nonlinear terms. More precisely, we prove the existence and uniqueness of variational solutions, based on a variational inequality, as well as the existence of the global attractor.

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1. Introduction

The Cahn-Hilliard equation (after John W. Cahn and John E. Hilliard), see [8],

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = 0, \tag{1.1}$$

is an equation of mathematical physics which describes the process of phase separation, spinodal decomposition: the material quickly becomes inhomogeneous, forming a fine-grained structure in which each of the two components appears more or less alternatively. In a second stage, which is called coarsening, occurs at a slower time scale and is less understood, these microstructures coarsen. Such phenomena play an essential role in the mechanical properties of the material, e.g., strength. We refer the reader to, e.g., [3, 4, 6, 7, 11, 12, 14-17, 21, 22, 24, 25, 27, 31, 36, 37, 38-40] for more details.

Here, u is the order parameter (one usually considers a rescaled density of atoms or concentration of one of the material's components which takes values between -1 and 1, -1 and 1 corresponding to the pure states; the density of the second component is $1 - u,$ meaning that the total density is a conserved quantity). Furthermore, f is the derivative of a double-well potential F whose wells correspond to the phases of the material. A thermodynamically relevant the derivative f is the following logarithmic function which follows from a mean-field model (see, e.g., [18, 30]):

$$f(s) = -2\lambda_1 s + \lambda_2 \ln \frac{1+s}{1-s}, \quad 0 < \lambda_2 < \lambda_1, \quad s \in (-1, 1). \tag{1.2}$$

These logarithmic nonlinear terms are still relevant for (1.1).

The Cahn-Hilliard equation is based on the so-called Ginzburg-Landau free energy,

$$\Psi_{GL} = \int_\Omega \left(\frac{1}{2} | \nabla u |^2 + F(u) \right) dx, \tag{1.3}$$

where Ω is the domain occupied by the system (we assume here that it is a bounded and regular domain of $\mathbb{R}^3$, with boundary Γ). In (1.3), the term $|\nabla u|^2$ models short-ranged interactions. It is however interesting to note that such a term is obtained by truncation of higher-order ones (see [8]); it can also be seen as a first-order approximation of a nonlocal term accounting for long-ranged interactions (see [35]).

In [5] the authors proposed higher-order phase-field models in order to account for anisotropic interfaces (for other approaches which, however, do not provide an explicit way to compute the anisotropy). More precisely, these authors proposed the following modified (total) free energy, in which we omit the temperature:

$$\Psi_{HOGL} = \int_\Omega \left(\frac{1}{2} \sum_{i=1}^{k} \sum_{|\alpha|=i} a_\alpha |\mathcal{D}^\alpha u|^2 + F(u) \right) dx, \tag{1.4}$$

where, for $\alpha = (k_1, k_2, k_3) \in (\mathbb{N} \cup \{0\})^3$,

$$|\alpha| = k_1 + k_2 + k_3$$

and, for $\alpha \neq (0, 0, 0)$,

$$\mathcal{D}^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3^{k_3}}$$

(we agree that $\mathcal{D}^{(0,0,0)} v = v$). The corresponding higher-order equation then reads

$$\frac{\partial u}{\partial t} - \Delta \sum_{i=1}^{k} (-1)^i \sum_{|\alpha|=i} a_\alpha \mathcal{D}^{2\alpha} u - \Delta f(u) = 0. \tag{1.5}$$

In [28] the authors studied the corresponding higher-order Allen-Cahn models, namely,

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{k} (-1)^i \sum_{|\alpha|=i} a_\alpha \mathcal{D}^{2\alpha} u + f(u) = 0,$$

endowed with the Dirichlet boundary conditions

$$\mathcal{D}^{\alpha} u = 0 \text{ on } \Gamma, |\alpha| \leq k - 1. \tag{1.6}$$

Our aim in this article is to study the model consisting of the higher-order anisotropic equation

$$\frac{\partial u}{\partial t} - \Delta \sum_{i=1}^{k} (-1)^i \sum_{|\alpha|=i} a_\alpha \mathcal{D}^{2\alpha} u - \Delta f(u) + \lambda \Delta u = 0, \quad \lambda \in \mathbb{R}. \tag{1.7}$$

As far as the mathematical analysis of the standard Cahn-Hilliard equation with such nonlinear terms is concerned, a complete picture (well-posedness, regularity of solutions, and existence of finite-dimensional attractors) was given in [29] (see also [20, 23]). In particular, one has the existence and uniqueness of solution satisfying the separation property

$$|u(t, x)| < 1 \text{ a.e. } (t, x). \tag{1.8}$$

Furthermore, in one and two space dimensions, one has the stronger (strict) separation property

$$\|u(t)\|_{L^\infty(\Omega)} \leq \delta = \delta(\tau, T), \, t \in [\tau, T], \, 0 < \tau < T, \, \delta \in (0, 1). \tag{1.9}$$

We can note that the proof of the above strict separation property uses in an essential way the comparison principle for second-order parabolic equations, so that the techniques used in [29] cannot be applied to (1.4). Thus, a priori, for this equation, we should only expect the weak separation (1.7).

In this article, we are not able to prove the existence of a classical solution to (1.4) when f is singular. However, we are able to prove the existence of a (weaker) variational solution. This notion of a variational solution was introduced in [18] for the Cahn-Hilliard equation with singular nonlinear terms and dynamic boundary conditions and is based on a variational inequality (see also [33] for a different, though related, approach based on duality techniques). It was also applied with success in other situations in [26, 38].

This article is organized as follows. We introduce, in the next section, the mathematical setting and proper approximated problems. Then, in Section 3, we derive a priori estimates on the solutions to the approximated problems. Finally, in Section 4, we give the definition of a variational solution and prove the existence and uniqueness of such solutions.

2. Setting of the Problem

We consider, in a bounded and regular domain $\Omega \subset \mathbb{R}^3$, with boundary Γ, the following initial and boundary value problem, for $k \in \mathbb{N}$, $k \geq 2$ (the case $k = 1$ can be treated as in [13]):

$$\frac{\partial u}{\partial t} - \Delta \sum_{i=1}^{k} (-1)^i \sum_{|\alpha|=i} a_\alpha \mathcal{D}^{2\alpha} u - \Delta f(u) + \lambda \Delta u = 0, \quad \lambda \in \mathbb{R}, \tag{2.1}$$

$$\mathcal{D}^\alpha u = 0 \text{ on } \Gamma, |\alpha| \leq k, \tag{2.2}$$

$$u|_{t=0} = u_0. \tag{2.3}$$

We assume that

$$a_\alpha > 0, \quad |\alpha| = k, \tag{2.4}$$

and we define the elliptic operator A_k by

$$\langle A_k v, w \rangle_{H^{-k}(\Omega), H_0^k(\Omega)} = \sum_{|\alpha|=k} a_\alpha ((\mathcal{D}^\alpha v, \mathcal{D}^\alpha w)), \tag{2.5}$$

where $H^{-k}(\Omega)$ is the topological dual of $H_0^k(\Omega)$ and $((\cdot, \cdot))$ denotes the usual L^2-scalar product, with associated norm $\|\cdot\|$. More generally, we denote by $\|\cdot\|_X$ the norm on the Banach space X; we also set $\|\cdot\|_{-1} =$

$\| (-\Delta)^{-\frac{1}{2}} . \|$, where $(-\Delta)^{-1}$ denotes the inverse minus Laplace operator associated with Dirichlet boundary conditions. We can note that

$$(v, w) \in H_0^k(\Omega)^2 \mapsto \sum_{|\alpha|=k} a_\alpha((\mathcal{D}^\alpha v, \mathcal{D}^\alpha w)),$$

is bilinear, symmetric, continuous and coercive, so that

$$A_k : H_0^k(\Omega) \to H^{-k}(\Omega)$$

is indeed well defined. It then follows from elliptic regularity results for linear elliptic operators of order $2k$ (see [1] and [2]) that A_k is a strictly positive, selfadjoint and unbounded linear operator with compact inverse, with domain

$$D(A_k) = H^{2k}(\Omega) \cap H_0^k(\Omega),$$

where, for $v \in D(A_k)$,

$$A_k v = (-1)^k \sum_{|\alpha|=k} a_\alpha \mathcal{D}^{2\alpha} v.$$

Furthermore, we note that $D(A_k^{\frac{1}{2}}) = H_0^k(\Omega)$ and, for $(v, w) \in D(A_k^{\frac{1}{2}})^2$,

$$((A_k^{\frac{1}{2}} v, A_k^{\frac{1}{2}} w)) = \sum_{|\alpha|=k} a_\alpha((\mathcal{D}^\alpha v, \mathcal{D}^\alpha w)).$$

Finally, we note that (see, e.g., [36]) $\| A_k . \|$ (resp., $\| A_k^{\frac{1}{2}} . \|$) is equivalent to the usual H^{2k}-norm (resp., H^k-norm) on $D(A_k)$ (resp., $D(A_k^{\frac{1}{2}})$).

Similarly, we can define the linear operator $\overline{A}_k = -\Delta A_k$

$$\overline{A}_k : H_0^{k+1}(\Omega) \to H^{-k-1}(\Omega)$$

which is a strictly positive, selfadjoint and unbounded linear operator with compact inverse, with domain

$$D(\overline{A}_k) = H^{2k+2}(\Omega) \cap H_0^{k+1}(\Omega),$$

where, for $v \in D(\overline{A}_k)$,

$$\overline{A}_k v = (-1)^{k+1} \Delta \sum_{|\alpha|=k} a_\alpha \mathcal{D}^{2\alpha} v.$$

Furthermore, $D(\overline{A}_k^{\frac{1}{2}}) = H_0^{k+1}(\Omega)$ and, for $(v, w) \in D(\overline{A}_k^{\frac{1}{2}})^2$,

$$((\overline{A}_k^{\frac{1}{2}} v, \overline{A}_k^{\frac{1}{2}} w)) = \sum_{|\alpha|=k} a_\alpha ((\nabla \mathcal{D}^\alpha v, \nabla \mathcal{D}^\alpha w)).$$

Besides, $\| \overline{A}_k . \|$ (resp., $\| \overline{A}_k^{\frac{1}{2}} . \|$) is equivalent to the usual H^{2k+2}-norm (resp., H^{k+1}-norm) on $D(\overline{A}_k)$ (resp., $D(\overline{A}_k^{\frac{1}{2}})$).

Finally, we consider the operator $\widetilde{A}_k = (-\Delta)^{-1} A_k$, with

$$\widetilde{A}_k : H_0^{k-1}(\Omega) \to H^{-k+1}(\Omega);$$

note that, as $-\Delta$ and A_k commute, then the same holds for $(-\Delta)^{-1}$ and A_k, so that $\widetilde{A}_k = A_k(-\Delta)^{-1}$.

We have the

Proposition 2.1. *The operator* $\widetilde{A}_k$ *is a strictly positive, selfadjoint and unbounded linear operator with compact inverse, with domain*

$$D(\widetilde{A}_k) = H^{2k-2}(\Omega) \cap H_0^{k-1}(\Omega),$$

where, for $v \in D(\widetilde{A}_k)$

$$\widetilde{A}_k v = (-1)^k \sum_{|\alpha|=k} a_\alpha \mathcal{D}^{2\alpha}(-\Delta)^{-1} v.$$

Moreover, $D(\widetilde{A}_k^{\frac{1}{2}}) = H_0^{k-1}(\Omega)$ *and, for* $(v, w) \in D(\widetilde{A}_k^{\frac{1}{2}})^2$,

$$((\widetilde{A}_k^{\frac{1}{2}} v, \widetilde{A}_k^{\frac{1}{2}} w)) = \sum_{|\alpha|=k} a_\alpha((\mathcal{D}^\alpha(-\Delta)^{-\frac{1}{2}} v, \mathcal{D}^\alpha(-\Delta)^{-\frac{1}{2}} w)).$$

In addition, $\| \widetilde{A}_k . \|$ $($*resp.,* $\| \widetilde{A}_k^{\frac{1}{2}} . \|)$ *is equivalent to the usual* H^{2k-2}*-norm* $($*resp.,* H^{k-1}*-norm*$)$ *on* $D(\widetilde{A}_k)$ $($*resp.,* $D(\widetilde{A}_k^{\frac{1}{2}}))$.

Proof. First of all, we note that $\widetilde{A}_k$ clearly is linear and unbounded. Then, since $(-\Delta)^{-1}$ and A_k commute, it easily follows that $\widetilde{A}_k$ is selfadjoint.

Next, the domain of $\widetilde{A}_k$ is defined by

$$D(\widetilde{A}_k) = \{v \in H_0^{k-1}(\Omega), \widetilde{A}_k v \in L^2(\Omega)\}.$$

Noting that $\widetilde{A}_k v = f$, $f \in L^2(\Omega)$, $v \in D(\widetilde{A}_k)$, is equivalent to $A_k v = -\Delta f$, where $-\Delta f \in H^2(\Omega)'$, it follows from the elliptic regularity results of [1] and [2] that $v \in H^{2k-2}(\Omega)$, so that $D(\widetilde{A}_k) = H^{2k-2}(\Omega) \cap H_0^{k-1}(\Omega)$.

Noting then that $\widetilde{A}_k^{-1}$ maps $L^2(\Omega)$ onto $H^{2k-2}(\Omega)$ and recalling that $k \geq 2$, we deduce that $\widetilde{A}_k$ has compact inverse.

We now note that, considering the spectral properties of $-\Delta$ and A_k (see, e.g., [36]) and recalling that these two operators commute, $-\Delta$ and A_k

have a spectral basis formed of common eigenvectors. This yields that, $\forall s_1, s_2 \in \mathbb{R}$, $(-\Delta)^{s_1}$ and $A_k^{s_2}$ commute.

In view of this, we see that $\widetilde{A}_k^{\frac{1}{2}} = (-\Delta)^{-\frac{1}{2}} A_k^{\frac{1}{2}}$, so that $D(\widetilde{A}_k^{\frac{1}{2}}) = H_0^{k-1}(\Omega)$, and, for $(v, w) \in D(\widetilde{A}_k^{\frac{1}{2}})^2$,

$$((\widetilde{A}_k^{\frac{1}{2}} v, \widetilde{A}_k^{\frac{1}{2}} w)) = \sum_{|\alpha|=k} a_\alpha ((\mathcal{D}^\alpha (-\Delta)^{-\frac{1}{2}} v, \mathcal{D}^\alpha (-\Delta)^{-\frac{1}{2}} w)).$$

Finally, as far as the equivalences of norms are concerned, we can note that, for instance, the norm $\| \widetilde{A}_k^{\frac{1}{2}} . \|$ is equivalent to the norm $\| (-\Delta)^{-\frac{1}{2}} . \|_{H^k(\Omega)}$ and, thus, to the norm $\| (-\Delta)^{\frac{k-1}{2}} . \|$. □

We actually rewrite (2.1) in the form

$$\frac{\partial u}{\partial t} - \Delta A_k u - \Delta B_k u - \Delta f(u) + \lambda \Delta u = 0, \tag{2.6}$$

where

$$B_k v = \sum_{i=1}^{k-1} (-1)^i \sum_{|\alpha|=i} a_\alpha \mathcal{D}^{2\alpha} v.$$

As far as the nonlinear term f is concerned, we assume that

$$f \in C^2(-1, 1), \quad f(0) = 0, \tag{2.7}$$

$$\lim_{s \to \pm 1} f(s) = \pm\infty, \tag{2.8}$$

$$\lim_{s \to \pm 1} f'(s) = +\infty, \tag{2.9}$$

$$f' \geq 0, \tag{2.10}$$

$$f''(s)\,\mathrm{sgn}(s) \geq 0, \quad s \in (-1, 1). \tag{2.11}$$

We then set, for $s \in (-1, 1)$,

$$\widetilde{f}(s) = f(s) - \lambda s.$$

It follows from the above that

$$\widetilde{f}' \geq -\lambda, \tag{2.12}$$

$$-c_1 \leq \widetilde{F}(s) \leq \widetilde{f}(s)s + c_2,\ c_1,\ c_2 \geq 0,\ s \in (-1, 1), \tag{2.13}$$

where $\widetilde{F}(s) = \int_0^s \widetilde{f}(\tau)d\tau.$

Remark 2.1. In particular, the thermodynamically relevant logarithmic function (1.2) satisfies the above assumptions.

We next define, for $N \in \mathbb{N}$, the approximated function f_N by:

$$f_N(s) = \begin{cases} f\left(-1 + \frac{1}{N}\right) + f'\left(-1 + \frac{1}{N}\right)\left(s + 1 - \frac{1}{N}\right),\ s < -1 + \frac{1}{N}, \\ f(s),\ |s| \leq 1 - \frac{1}{N}, \\ f\left(1 - \frac{1}{N}\right) + f'\left(1 - \frac{1}{N}\right)\left(s - 1 + \frac{1}{N}\right),\ s > 1 - \frac{1}{N}. \end{cases}$$

Setting $F_N(s) = \int_0^s f_N(\tau)d\tau,\quad \widetilde{f}_N(s) = f_N(s) - \lambda s,\quad \widetilde{F}_N(s) = \int_0^s \widetilde{f}_N(\tau)d\tau,$ $s \in (-1, 1)$, it is easy to see that (see also [10] and [18])

$$f_N(s)s \geq F_N(s) \geq 0, \quad s \in \mathbb{R}, \tag{2.14}$$

$$F_N(s) \geq c_3 s^4 - c_4,\ c_3 > 0,\ c_4 \geq 0, \quad s \in \mathbb{R}, \tag{2.15}$$

and, for $N \geq N_0 = N_0(\lambda)$,

$$\widetilde{f}_N(s)s \geq c_5 f_N(s)s - c_6 \geq \frac{c_5}{2}|f_N(s)| - c_7,\ c_5 > 0,\ c_6,\ c_7 \geq 0, \quad s \in \mathbb{R} \tag{2.16}$$

$$2F_N(s) + c_8 \geq \widetilde{F}_N(s) \geq \frac{1}{2} F_N(s) - c_8,\ c_8 \geq 0, \quad s \in \mathbb{R}, \tag{2.17}$$

where the constants c_i, $i = 5, ..., 8$, only depend on λ. Furthermore, it follows from (2.10), (2.12) and the explicit expression of f_N that

$$f_N' \geq 0, \quad \widetilde{f}_N' \geq -\lambda. \tag{2.18}$$

We finally introduce the approximated problems

$$\frac{\partial u_N}{\partial t} - \Delta A_k u_N - \Delta B_k u_N - \Delta \widetilde{f}_N(u_N) = 0 \tag{2.19}$$

$$\mathcal{D}^\alpha u_N = 0 \text{ on } \Gamma, |\alpha| \leq k, \tag{2.20}$$

$$u_N|_{t=0} = u_0. \tag{2.21}$$

Throughout the article, the same letters c, c' and c'' denote (generally positive) constants which may vary from line to line and are independent of N. Similarly, the same letter Q denotes (positive) monotone increasing and continuous functions which may vary from line to line and are independent of N.

3. A Priori Estimates

In this section, we derive uniform (with respect to N) a priori estimates on u_N which will allow us and we assume from now on that $-1 < u_0(x) < 1$ a.e. $x \in \Omega$.

For a more general singular nonlinear terms f, we would need a stronger separation property from the singular value ± 1, namely $\| u_0 \|_{L^\infty(\Omega)} < 1$.

We multiply (2.19) by $(-\Delta)^{-1} \frac{\partial u_N}{\partial t}$ and integrate over Ω by parts. This gives

$$\frac{d}{dt}\left(\| A_k^{\frac{1}{2}} u_N \|^2 + B_k^{\frac{1}{2}}[u_N] + 2\int_\Omega \widetilde{F}_N(u_N)\,dx \right) + 2\left\| \frac{\partial u_N}{\partial t} \right\|_{-1}^2 = 0, \tag{3.1}$$

where

$$B_1^{\frac{1}{2}}[u_N] = 0$$

and, for $k \geq 2$,

$$B_k^{\frac{1}{2}}[u_N] = \sum_{i=1}^{k-1} \sum_{|\alpha|=i} a_\alpha \| \mathcal{D}^\alpha u_N \|^2 \tag{3.2}$$

(note that $B_k^{\frac{1}{2}}[u_N]$ is not necessarily nonnegative). We note that, owing to the interpolation inequality

$$\| v \|_{H^i(\Omega)} \leq c(i) \| v \|_{H^m(\Omega)}^{\frac{i}{m}} \| v \|^{1-\frac{i}{m}}, \tag{3.3}$$

$$v \in H^m(\Omega),\ i \in \{1, \dots, m-1\},\ m \in \mathbb{N},\ m \geq 2,$$

there holds

$$| B_k^{\frac{1}{2}}[u_N] | \leq \frac{1}{2} \| A_k^{\frac{1}{2}} u_N \|^2 + c \| u_N \|^2. \tag{3.4}$$

This yields, employing (2.15),

$$\| A_k^{\frac{1}{2}} u_N \|^2 + B_k^{\frac{1}{2}}[u_N] + 2\int_\Omega F_N(u_N)dx$$

$$\geq \frac{1}{2} \| A_k^{\frac{1}{2}} u_N \|^2 + \int_\Omega F_N(u_N)dx + c\| u_N \|_{L^4(\Omega)}^4 - c' \| u_N \|^2 - c'',$$

whence

$$\| A_k^{\frac{1}{2}} u_N \|^2 + B_k^{\frac{1}{2}}[u_N] + 2\int_\Omega F_N(u_N)dx$$

$$\geq c\left(\| u_N \|_{H^k(\Omega)}^2 + \int_\Omega F_N(u_N)dx \right) - c', \quad c > 0, \tag{3.5}$$

noting that, owing to Young's inequality,

$$\| u_N \|^2 \leq \varepsilon \| u_N \|^4_{L^4(\Omega)} + c(\varepsilon), \quad \forall \varepsilon > 0. \tag{3.6}$$

We then multiply (2.19) by $(-\Delta)^{-1} u_N$, integrate over Ω by part and have

$$\frac{1}{2} \frac{d}{dt} \| u_N \|^2_{-1} + \| A_k^{\frac{1}{2}} u_N \|^2 + B_k^{\frac{1}{2}}[u_N] + ((\widetilde{f}_N(u_N), u_N)) = 0.$$

Noting that it follows from (2.14) and (2.16) that

$$((\widetilde{f}_N(u_N), u_N)) \geq c_5((f_N(u_N), u_N)) - c \geq c_5 \int_\Omega F_N(u_N)dx - c'$$

and

$$((\widetilde{f}_N(u_N), u_N)) \geq \frac{c_5}{2} \| f_N(u_N) \|_{L^1(\Omega)} - c,$$

this yields

$$\frac{d}{dt} \| u_N \|^2_{-1} + c\left(\| u_N \|^2_{H^k(\Omega)} + \int_\Omega F_N(u_N)dx + \| f_N(u_N) \|_{L^1(\Omega)} \right) \leq c', \; c > 0. \tag{3.7}$$

Summing (3.1) and (3.7), we obtain a differential inequality of the form

$$\frac{dE_{1,N}}{dt} + c\left(E_{1,N} + \| f_N(u_N) \|_{L^1(\Omega)} + \left\| \frac{\partial u_N}{\partial t} \right\|^2_{-1} \right) \leq c', \; c > 0, \tag{3.8}$$

where

$$E_{1,N} = \| u_N \|^2_{-1} + \| A_k^{\frac{1}{2}} u_N \|^2 + B_k^{\frac{1}{2}}[u_N] + 2\int_\Omega \widetilde{F}_N(u_N)dx$$

satisfies, owing to (2.17) and (3.5),

$$E_{1,N} \geq c\left(\| u_N \|^2_{H^k(\Omega)} + \int_\Omega F_N(u_N)dx \right) - c', \quad c > 0. \tag{3.9}$$

In particular, it follows from (3.8)-(3.9) and Gronwall's lemma that

$$\| u_N(t) \|^2_{H^k(\Omega)} \le ce^{-c't}\left(\| u_0 \|^2_{H^k(\Omega)} + \int_\Omega F(u_0)dx \right) + c'', \; c' > 0, \; t \ge 0 \tag{3.10}$$

and

$$\int_t^{t+r} \left\| \frac{\partial u_N}{\partial t} \right\|^2_{-1} ds$$

$$\le ce^{-c't}\left(\| u_0 \|^2_{H^k(\Omega)} + \int_\Omega F(u_0)dx \right) + c''(r), \; c' > 0, \; t \ge 0, \; r > 0 \text{ given.} \tag{3.11}$$

Actually, noting that $F_N(u_0)$ is bounded (independently of N and u_0), there holds

$$\| u_N(t) \|^2_{H^k(\Omega)} \le ce^{-c't} \| u_0 \|^2_{H^k(\Omega)} + c'', \; c' > 0, \; t \ge 0 \tag{3.12}$$

and

$$\int_t^{t+r} \left\| \frac{\partial u_N}{\partial t} \right\|^2_{-1} ds \le ce^{-c't} \| u_0 \|^2_{H^k(\Omega)} + c''(r), \; c' > 0, \; t \ge 0, \; r > 0 \text{ given.} \tag{3.13}$$

Next, multiplying (2.19) by $\widetilde{A}_k u_N$, we obtain

$$\frac{d}{dt} \| \widetilde{A}_k^{\frac{1}{2}} u_N \|^2 + c\| u_N \|^2_{H^{2k}(\Omega)} \le c'(\| u_N \|^2 + \| \widetilde{f}_N(u_N) \|^2), \; c > 0. \tag{3.14}$$

It follows from the continuity of f_N, F_N and $\widetilde{f}_N$, the continuous embedding $H^k(\Omega) \subset C(\overline{\Omega})$ (recall that $k \ge 2$) and (3.12) that

$$\| u_N \|^2 + \| \widetilde{f}_N \|^2 \le Q(\| u_N \|_{H^k(\Omega)})$$

$$\le e^{-ct} Q(\| u_0 \|_{H^k(\Omega)}) + c', \; c > 0, \; t \ge 0, \tag{3.15}$$

so that

$$\frac{d}{dt}\| \widetilde{A}_k^{\frac{1}{2}} u_N \|^2 + c\| u_N \|_{H^{2k}(\Omega)}^2 \leq e^{-c't} Q(\| u_0 \|_{H^k(\Omega)}) + c'', \; c, \; c' > 0, \; t \geq 0. \tag{3.16}$$

Summing (3.8) and (3.16), we obtain a differential inequality of the form

$$\frac{dE_{2,N}}{dt} + c\left(E_{2,N} + \| u_N \|_{H^{2k}(\Omega)}^2 + \| f_N(u_N) \|_{L^1(\Omega)} + \left\| \frac{\partial u_N}{\partial t} \right\|_{-1}^2 \right)$$

$$\leq e^{-c't} Q(\| u_0 \|_{H^k(\Omega)}) + c'', \; c, \; c' > 0, \tag{3.17}$$

where

$$E_{2,N} = E_{1,N} + \| \widetilde{A}_k^{\frac{1}{2}} u_N \|^2$$

satisfies

$$E_{2,N} \geq c\left(\| u_N \|_{H^k(\Omega)}^2 + \int_\Omega F_N(u_N) dx \right) - c', \quad c > 0. \tag{3.18}$$

It follows from (3.17)-(3.18) that

$$\int_t^{t+r} \| u_N \|_{H^{2k}(\Omega)}^2 ds \leq e^{-ct} Q(\| u_0 \|_{H^k(\Omega)}) + c', \; c' > 0, \; t \geq 0, \; r > 0 \text{ given.} \tag{3.19}$$

We now multiply (2.19) by u_N and obtain, employing (2.18) and the interpolation inequality (3.3),

$$\frac{d}{dt}\| u_N \|^2 + c\| u_N \|_{H^{k+1}(\Omega)}^2 \leq \lambda \| u_N \|_{H^1(\Omega)}^2, \quad c > 0,$$

whence, proceeding as above,

$$\frac{d}{dt}\| u_N \|^2 + c\| u_N \|_{H^{k+1}(\Omega)}^2 \leq e^{-c't} Q(\| u_0 \|_{H^k(\Omega)}) + c'', \; c, \; c' > 0. \tag{3.20}$$

We also multiply (2.19) by $\frac{\partial u_N}{\partial t}$ and obtain, noting that the continuous embedding $H^k(\Omega) \subset \mathcal{C}(\overline{\Omega})$, so that

$$\| \Delta \widetilde{f}_N(u_N) \|^2 \leq Q(\| u_N \|_{H^k(\Omega)}),$$

and proceeding as above,

$$\frac{d}{dt}(\| \overline{A}_k^{\frac{1}{2}} u_N \|^2 + \overline{B}_k^{\frac{1}{2}}[u_N]) + c\left\| \frac{\partial u_N}{\partial t} \right\| \leq e^{-c't} Q(\| u_0 \|_{H^k(\Omega)}) + c'', \ c, \ c' > 0, \tag{3.21}$$

where

$$\overline{B}_k^{\frac{1}{2}}[u_N] = \sum_{i=1}^{k-1} \sum_{|\alpha|} a_\alpha \| \nabla \mathcal{D}^\alpha u_N \|^2.$$

Summing finally (3.17), (3.20) and (3.21), we find a differential inequality of the form

$$\frac{dE_{3,N}}{dt} + c\left(E_{3,N} + \| u_N \|^2_{H^{2k}(\Omega)} + \| f_N(u_N) \|_{L^1(\Omega)} + \left\| \frac{\partial u_N}{\partial t} \right\|^2 \right)$$

$$\leq e^{-c't} Q(\| u_0 \|_{H^k(\Omega)}) + c'', \ c, \ c' > 0, \tag{3.22}$$

where

$$E_{3,N} = E_{2,N} + \| u_N \|^2 + \| \overline{A}_k^{\frac{1}{2}} u_N \|^2 + \overline{B}_k^{\frac{1}{2}}[u_N]$$

satisfies, proceeding as above,

$$E_{3,N} \geq c\left(\| u_N \|^2_{H^{k+1}(\Omega)} + \int_\Omega F_N(u_N) dx \right) - c', \ c > 0. \tag{3.23}$$

In particular, it follows from (3.22)-(3.23) that

$$\| u_N(t) \|_{H^{k+1}(\Omega)} \leq e^{-ct} Q(\| u_0 \|_{H^{k+1}(\Omega)}) + c', \ c > 0, \ t \geq 0 \tag{3.24}$$

and

$$\int_t^{t+r} \left\| \frac{\partial u_N}{\partial t} \right\|^2 ds \leq e^{-ct} Q(\| u_0 \|_{H^{k+1}(\Omega)}) + c',\ c > 0,\ t \geq 0,\ r > 0 \text{ given.}$$

(3.25)

We then differentiate (2.19) with respect to time and find

$$\frac{\partial}{\partial t} \frac{\partial u_N}{\partial t} - \Delta A_k \frac{\partial u_N}{\partial t} - \Delta B_k \frac{\partial u_N}{\partial t} - \Delta\left(\tilde{f}'_N(u_N) \frac{\partial u_N}{\partial t} \right) = 0, \tag{3.26}$$

$$\mathcal{D}^\alpha \frac{\partial u_N}{\partial t} = 0 \text{ on } \Gamma,\ |\alpha| \leq k. \tag{3.27}$$

Multiplying (3.26) by $(-\Delta)^{-1} \frac{\partial u_N}{\partial t}$, we have, owing to (2.18) and the interpolation inequalities (3.3),

$$\frac{d}{dt} \left\| \frac{\partial u_N}{\partial t} \right\|_{-1}^2 + c \left\| \frac{\partial u_N}{\partial t} \right\|_{H^k(\Omega)}^2 \leq \lambda \left\| \frac{\partial u_N}{\partial t} \right\|^2,$$

hence, employing the interpolation inequality

$$\| v \|^2 \leq c \| v \|_{-1} \| \nabla v \|,\ \forall v \in H_0^1(\Omega), \tag{3.28}$$

the differential inequality

$$\frac{d}{dt} \left\| \frac{\partial u_N}{\partial t} \right\|_{-1}^2 \leq c \left\| \frac{\partial u_N}{\partial t} \right\|_{-1}^2. \tag{3.29}$$

It then follows from (3.13), say, for $r = 1$ and the uniform Gronwall's lemma that

$$\left\| \frac{\partial u_N}{\partial t}(t) \right\|_{-1}^2 \leq ce^{-c't} \| u_0 \|_{H^k(\Omega)}^2 + c'',\ c > 0,\ t \geq 1. \tag{3.30}$$

Remark 3.1. Actually, it follows from the uniform Gronwall's lemma that

$$\left\| \frac{\partial u_N}{\partial t}(t+r) \right\|_{-1}^2 \leq \frac{c(r)}{r} e^{-c't} \| u_0 \|_{H^k(\Omega)}^2 + c''(r),\ c > 0,\ t \geq 0, \quad (3.31)$$

$r > 0$ given.

Remark 3.2. We assume that $\| u_0 \|_{L^\infty(\Omega)} < 1$. We can note that, if $u_0 \in H^{2k}(\Omega)$, then $(-\Delta)^{-\frac{1}{2}} \frac{\partial u_N}{\partial t} \in L^2(\Omega)$ and it follows from the continuity of f and the continuous embedding $H^{2k}(\Omega) \subset \mathcal{C}(\overline{\Omega})$ that, for N large enough $\left(\text{note that } \widetilde{f}_N \text{ coincides with } \widetilde{f} = \widetilde{F}' \text{ when } |s| < 1 - \frac{1}{N} \right)$,

$$\left\| \frac{\partial u_N}{\partial t}(0) \right\|_{-1} \leq Q(\| u_0 \|_{H^{2k}(\Omega)}). \quad (3.32)$$

It then follows from (3.29) and Gronwall's lemma that

$$\left\| \frac{\partial u_N}{\partial t}(t) \right\|_{-1}^2 \leq e^{ct} Q(\| u_0 \|_{H^{2k}(\Omega)}),\ c > 0,\ t \geq 0. \quad (3.33)$$

Collecting (3.30) and (3.33) $(\text{for } t \in [0, 1])$, we finally deduce that

$$\left\| \frac{\partial u_N}{\partial t}(t) \right\|_{-1}^2 \leq e^{-ct} Q(\| u_0 \|_{H^{2k}(\Omega)}) + c',\ c > 0,\ t \geq 0. \quad (3.34)$$

We finally rewrite (2.19) as an elliptic equation, for $t > 0$ fixed,

$$A_k u_N = -(-\Delta)^{-1} \frac{\partial u_N}{\partial t} - B_k u_N - \widetilde{f}_N(u_N),\ \mathcal{D}^\beta u = 0 \text{ on } \Gamma,\ |\beta| \leq k-1. \quad (3.35)$$

Multiplying (3.35) by $A_k u_N$, we have, owing to (2.18) and the interpolation inequality (3.3),

$$\| A_k u_N \|^2 \leq c\left(\| u_N \|^2 + \| \widetilde{f}_N(u_N) \|^2 + \left\| \frac{\partial u_N}{\partial t} \right\|_{-1}^2 \right), \quad (3.36)$$

hence, proceeding as above (employing, in particular, (3.15)),

$$\| u_N(t) \|^2_{H^{2k}(\Omega)} \le e^{-ct} Q(\| u_0 \|_{H^k(\Omega)}) + \left\| \frac{\partial u_N}{\partial t} \right\|^2_{-1} + c',\ c > 0. \qquad (3.37)$$

Employing (3.34),

$$\| u_N(t) \|_{H^{2k}(\Omega)} \le e^{-ct} Q(\| u_0 \|_{H^{2k}(\Omega)}) + c',\ c > 0,\ t \ge 0. \qquad (3.38)$$

Remark 3.3. Let u_N^1 and u_N^2 be two solutions to (2.19)-(2.20) with initial conditions u_0^1 and u_0^2, respectively. Then, we have, setting $u_N = u_N^1 - u_N^2$ and $u_0 = u_0^1 - u_0^2$,

$$\frac{\partial u_N}{\partial t} - \Delta A_k u_N - \Delta B_k u_N - \Delta(\widetilde{f}_N(u_N^1) - \widetilde{f}_N(u_N^2)) = 0, \qquad (3.39)$$

$$\mathcal{D}^\alpha u_N = 0 \text{ on } \Gamma, |\alpha| \le k, \qquad (3.40)$$

$$u_N |_{t=0} = u_0. \qquad (3.41)$$

Multiplying (3.39) by $(-\Delta)^{-1} u_N$, we obtain, in view of (2.18),

$$\frac{1}{2} \frac{d}{dt} \| u_N \|^2_{-1} + \| A_k^{\frac{1}{2}} u_N \|^2 + B_k^{\frac{1}{2}}[u_N] \le \lambda \| u_N \|^2. \qquad (3.42)$$

Employing the interpolation inequalities (3.3) and (3.42), we deduce that

$$\frac{d}{dt} \| u_N \|^2_{-1} + c \| u_N \|^2_{H^k(\Omega)} \le c' \| u_N \|^2_{-1},\ c > 0. \qquad (3.43)$$

It follows from (3.35) and Gronwall's lemma that

$$\| u_N^1(t) - u_N^2(t) \|_{-1} \le e^{ct} \| u_0^1 - u_0^2 \|_{-1}, \quad t \ge 0, \qquad (3.44)$$

hence the uniqueness of solutions to (2.19) and (2.21), as well as the continuous dependence with respect to the initial data in the H^{-1}-norm.

4. The Dissipative Semigroup

In this section, to pass to the limit $N \to \infty$ and prove the existence of a solution to the original singular problem, in a suitable setting (i.e., as mentioned in the introduction, based on a proper variational inequality), we proceed as follows.

First of all, we derive a variational inequality from (2.1). To do so, we multiply this equation by $(-\Delta)^{-1}(u - v)$, where $v = v(x)$ is smooth and satisfies $v = 0$ on Γ. We then have

$$\left(\left((-\Delta)^{-1}\frac{\partial u}{\partial t}, u - v\right)\right) + ((A_k^{\frac{1}{2}}u, A_k^{\frac{1}{2}}(u - v))) + ((B_k^{\frac{1}{2}}u, B_k^{\frac{1}{2}}(u - v)))$$
$$+ ((f(u), u - v)) - \lambda((u, u - v)) = 0.$$

Noting that, owing to (2.10), f is monotone increasing, this yields the variational inequality

$$\left(\left((-\Delta)^{-1}\frac{\partial u}{\partial t}, u - v\right)\right) + ((A_k^{\frac{1}{2}}u, A_k^{\frac{1}{2}}(u - v))) + ((B_k^{\frac{1}{2}}u, B_k^{\frac{1}{2}}(u - v)))$$
$$+ ((f(v), u - v)) - \lambda((u, u - v)) \le 0, \tag{4.1}$$

i.e., the nonlinear term now acts on the test functions and not on the solution.

Based on this, we give the following definition (see also [32]):

Definition 4.1. We assume that $u_0 \in H_0^k(\Omega)$, with $\| u_0 \|_{L^\infty(\Omega)} < 1$. Then, $u = u(t, x)$ is a variational solution to (2.1) and (2.3) if, for any given $T > 0$,

(i) $-1 < u(t, x) < 1$ a.e. (t, x),

(ii) $u \in C([0, T]; L^2(\Omega)) \cap L^\infty(0, T; H_0^k(\Omega))$
$\cap L^2(0, T; H^{2k}(\Omega) \cap H_0^k(\Omega))$,

(iii) $\frac{\partial u}{\partial t} \in L^2((0, T), H^{-1}(\Omega))$,

(iv) $f(u) \in L^1((0, T) \times \Omega)$,

(v) $u(0) = u_0$,

(vi) the variational inequality (4.1) is satisfied for every $t > 0$ and every test function $v = v(x)$ such that $v \in H_0^k(\Omega)$, with $f(v) \in L^1(\Omega)$.

First of all, we prove the uniqueness of variational solutions. To do so, we need to define the admissible test functions to be the solutions themselves, i.e. we need to define admissible time-dependent test functions. More precisely, we call a function $v = v(t, x)$ admissible if $v \in C([0, T]; L^2(\Omega)) \cap L^\infty(0, T; H_0^k(\Omega)) \cap L^2(0, T; H^{2k}(\Omega) \cap H_0^k(\Omega))$ and $f(v) \in L^1(\Omega)$ and $\frac{\partial v}{\partial t} \in L^2((0, T), H^{-1}(\Omega)), \forall T > 0.$ Then, we write (4.1) for $v = v(t, \cdot)$, for almost every $t > 0$. Noting that, owing to the regularity assumptions on u and v, all terms are L^1 with respect to time, we can integrate with respect to time to obtain

$$\int_s^t \Bigg[\left(\left((-\Delta)^{-1} \frac{\partial u}{\partial t}, u - v \right) \right) + ((A_k^{\frac{1}{2}} u, A_k^{\frac{1}{2}} (u - v))) + ((B_k^{\frac{1}{2}} u, B_k^{\frac{1}{2}} (u - v)))$$

$$+ ((f(v), u - v)) - \lambda((u, u - v)) \Bigg] d\xi \leq 0, \; 0 < s < t \tag{4.2}$$

and for every admissible test function $v = v(t, x)$. In particular, since $H^k(\Omega) \subset \mathcal{C}(\overline{\Omega}), k \geq 2,$ it follows from the above regularity that $((f(u), u - v)) \in L^1(0, T), \forall T > 0.$

Remark 4.1. We can replace (4.1) by (4.2) in Definition 4.1 (vi).

We will actually need a second variational inequality. To do so, let $w = w(t, x)$ be a test function satisfying the above assumptions and set

$$v_\eta = (1 - \eta)u + \eta w, \quad \eta \in (0, 1].$$

Since, owing to (2.11), the function $|f|$ is convex, there holds

$$|f(v_\eta)| \leq |f(u)| + |f(w)|. \tag{4.3}$$

This yields that $f(v_\eta) \in L^1((0, T) \times \Omega)$, hence v_η is an admissible test function. Taking $v = v_\eta$ in (4.2) and dividing by η, we find

$$\int_s^t \Bigg[\left(\left((-\Delta)^{-1} \frac{\partial u}{\partial t}, u - w \right)\right) + ((A_k^{\frac{1}{2}} u, A_k^{\frac{1}{2}}(u - w))) + ((B_k^{\frac{1}{2}} u, B_k^{\frac{1}{2}}(u - w)))$$

$$+ ((f(v_\eta), u - w)) - \lambda((u, u - w)) \Bigg] d\xi \leq 0. \tag{4.4}$$

Finally, passing to the limit $\eta \to 0$ and employing Lebesgue's dominated convergence theorem (see (4.3)), we have

$$\int_s^t \Bigg[\left(\left((-\Delta)^{-1} \frac{\partial u}{\partial t}, u - w \right)\right) + ((A_k^{\frac{1}{2}} u, A_k^{\frac{1}{2}}(u - w))) + ((B_k^{\frac{1}{2}} u, B_k^{\frac{1}{2}}(u - w)))$$

$$+ ((f(u), u - w)) - \lambda((u, u - w)) \Bigg] d\xi \leq 0, \; 0 < s < t, \tag{4.5}$$

and for every admissible test function $w = w(t, x)$.

Now, let u_1 and u_2 be two variational solutions with initial data $u_{1,0}$ and $u_{2,0}$, respectively. Taking $u = u_1$ and $v = u_2$ in (4.2) and $u = u_2$ and $w = u_1$ in (4.5) and summing the two resulting inequalities, we find, after simplifications and noting that all terms are absolutely continuous from $[0, T]$ onto $H^{-1}(\Omega)$,

$$\frac{1}{2}\| u_1(t) - u_2(t) \|_{-1}^2 - \frac{1}{2}\| u_1(s) - u_2(s) \|_{-1}^2$$

$$+ \int_s^t (\| A_k^{\frac{1}{2}}(u - v) \|^2 + B_k^{\frac{1}{2}}[u_1 - u_2] - \lambda \| u_1 - u_2 \|^2) d\xi \leq 0. \quad (4.6)$$

Proceeding exactly as in the previous section, i.e., employing once more the interpolation inequalities (3.3) and (3.28), we deduce that

$$\frac{1}{2}\| u_1(t) - u_2(t) \|_{-1}^2 - \frac{1}{2}\| u_1(s) - u_2(s) \|_{-1}^2 \leq \int_s^t \| u_1 - u_2 \|_{-1}^2 d\xi,$$

which yields, employing Gronwall's lemma,

$$\| u_1(t) - u_2(t) \|_{-1} \leq e^{c(t-s)} \| u_1(s) - u_2(s) \|_{-1},$$

where the constant c is independent of t, s, u_1 and u_2. Passing finally to the limit $s \to 0$, we find

$$\| u_1(t) - u_2(t) \|_{-1} \leq e^{ct} \| u_{1,0} - u_{2,0} \|_{-1}, \quad t \geq 0, \quad (4.7)$$

hence the uniqueness, as well as the continuous dependence with respect to the initial data in the H^{-1}-norm.

We have the

Theorem 4.1. *We assume that* $u_0 \in H_0^k(\Omega)$, *with* $\| u_0 \|_{L^\infty(\Omega)} < 1$. *Then*, (2.1)-(2.3) *possesses a unique variational solution u*.

Proof. We consider the solution u_N to the approximated problem (2.19)-(2.21); the existence and uniqueness, as well as the regularity, of u_N can be obtained in a standard way (see also [34]). Furthermore, proceeding as above, it is easy to see that u_N satisfies a variational inequality which is analogous to (4.2), namely,

$$\int_s^t \Bigg[\left(\left((-\Delta)^{-1} \frac{\partial u_N}{\partial t}, u_N - v \right) \right) + ((A_k^{\frac{1}{2}} u_N, A_k^{\frac{1}{2}} (u_N - v)))$$

$$+ ((B_k^{\frac{1}{2}} u, B_k^{\frac{1}{2}} (u_N - v)))$$

$$+ ((f_N(v), u_N - v)) - \lambda((u_N, u_N - v)) \Bigg] d\xi \leq 0,\ 0 < s < t, \tag{4.8}$$

and for every admissible test function $v = v(t, x)$.

Noting that the a priori estimates derived in the previous section are now fully justified, we deduce from (3.17), (3.22) and (3.37) that, up to a subsequence, u_N converges to a limit function u in the following sense:

$u_N \to u$ in $L^{\infty}(0, T; H^k(\Omega))$ weak $- \star$ and $L^2(0, T; H^{2k}(\Omega))$ weak;

$\frac{\partial u_N}{\partial t} \to \frac{\partial u}{\partial t}$ in $L^2(0, T; H^{-1}(\Omega))$ weak

$u_N \to u$ in $\mathcal{C}([0, T]; H^{k+1-\varepsilon}(\Omega))$,

$L^2(0, T; H^{2k-\varepsilon}(\Omega))$ and a.e. in $(0, T) \times \Omega$, $\varepsilon > 0$.

We now pass to the limit in (4.8) as $N \to +\infty$. We note that the above convergences allow us to pass to the limit in all terms in (4.8), except in the nonlinear term $\int_s^t ((f_N(v), u_N - v)) d\xi$. To pass to the limit in the nonlinear term, we note that, by construction,

$$| f_N(v) | \leq | f(v) |$$

and we use Lebesgue's dominated convergence theorem (recall that, since v is an admissible test function, $f(v) \in L^1((0, T) \times \Omega)$; also note that u and v belong to $L^{\infty}((0, T) \times \Omega)$.

Noting that $f_N(u_N)$ is bounded, uniformly with respect to N, in $L^1((0, T) \times \Omega)$, it follows from the explicit expression of f_N that

$$\text{meas}(E_{N,M}) \leq \pi\left(\frac{1}{N}\right), \; M \geq N, \tag{4.9}$$

where

$$E_{N,M} = \left\{(t, x) \in (0, T) \times \Omega, \, | u^M(t, x) | > 1 - \frac{1}{N}\right\}$$

and

$$\pi(s) = \frac{c}{\max(| f(1-s)|, | f(s-1)|)},$$

for some positive constant c which is independent of N and M. Note indeed that there holds

$$\int_0^T \int_\Omega | f_M(u_M)| dxdt \geq \int_{E_{N,M}} | f_M(u_M)| dxdt \geq c' \text{meas}(E_{N,M}) \frac{1}{\pi\left(\frac{1}{N}\right)}, \tag{4.10}$$

where the constant c' is independent of N and M. We can pass to the limit $M \to +\infty$ (employing Fatou's lemma, see (4.10)) and then $N \to +\infty$ (noting that $\pi(s) \to 0$ as $s \to 0$) to find

$$\text{meas}\{(t, x) \in (0, T) \times \Omega, | u(t, x) | \geq 1\} = 0,$$

so that

$$-1 < u(t, x) < 1 \text{ a.e. } (t, x). \tag{4.11}$$

Next, it follows from the above almost everywhere convergence of u_N to u, (4.11), and again the explicit expression of f_N that

$$f_N(u_N) \to f(u) \text{ a.e. in } (0, T) \times \Omega. \tag{4.12}$$

Then, we deduce from Fatou's lemma that

$$\| f(u) \|_{L^1((0,T)\times\Omega)} \leq \liminf \| f(u_N) \|_{L^1((0,T)\times\Omega)} < +\infty,$$

hence the proof of the existence. □

It follows from Theorem 4.1 that we can define the family of operators

$$S(t) : \Phi \to \Phi,\ u_0 \mapsto u(t),\ t \geq 0,$$

where

$$\Phi = \{v \in H_0^k(\Omega),\ -1 < v(x) < 1 \text{ a.e. } x \in \Omega\}.$$

This family of operators forms a semigroup (i.e., $S(0) = I$ (identity operator) and $S(t+\tau) = S(t) \circ S(\tau),\ t,\ \tau \geq 0$), which is, owing to (4.7), continuous in the H^{-1}-topology. Furthermore, it follows from (3.12) (which also holds in the limit $N \to +\infty$) that this semigroup is dissipative, in the sense that it possesses a bounded absorbing set $\mathcal{B}_0 \subset \Phi$ (i.e., $\forall \mathcal{B} \subset \Phi$ bounded, $\exists t_0 = t_0(\mathcal{B}) \geq 0$ such that $t \geq t_0 \Rightarrow S(t)\mathcal{B} \subset \mathcal{B}_0$).

It then follows from (4.7) that we can actually extend (in a unique way and by continuity) $S(t)$ to the closure of Φ in the H^{-1}-topology, namely,

$$S(t) : \Phi_1 \to \Phi_1, \quad t \geq 0,$$

where

$$\Phi_1 = \{v \in L^\infty(\Omega),\ \| v \|_{L^\infty(\Omega)}\}.$$

It also follows from the a priori estimates derived in the previous section that $S(t)$ instantaneously regularizes, i.e.,

$$S(t) : \Phi_1 \to \Phi, \quad t > 0,$$

and that it possesses a bounded absorbing set $\mathcal{B}_1$ which is compact in $H^{-1}(\Omega)$ and bounded in $H^{2k}(\Omega)$. We thus deduce from standard results (see, e.g., [19] and [36]) that we have the

Theorem 4.2. *The semigroup* $S(t)$ *possesses the global attractor* $\mathcal{A}$ *which is compact in* $H^{-1}(\Omega)$ *and bounded in* $H^{2k}(\Omega)$.

Remark 4.2. One recalls that the global attractor $\mathcal{A}$ is the smallest (for the inclusion) compact set of the phase space which is invariant by the flow (i.e., $S(t)\mathcal{A} = \mathcal{A}$, for $t \geq 0$) and attracts all bounded sets of initial data as time goes to infinity; it thus appears as a suitable object in view of the study of the asymptotic behavior of the system. An important question is whether the global attractor $\mathcal{A}$ has finite dimension, in the sense of covering dimensions such as the Hausdorff and the fractal dimensions. The finite-dimensionality means, very roughly speaking, that, even though the initial phase space has infinite dimension, the reduced dynamics can be described by a finite number of parameters (we refer the interested reader to, e.g., [19] and [36] for discussions on this subject). When $k = 1$, i.e., for the classical Cahn-Hilliard equation, this can easily be established, owing again to the strict separation from the singular values ± 1 (see, e.g., [9]). However, when $k \geq 2$, the situation is much more involved and one idea could be to proceed as in [18]. This will be addressed elsewhere.

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