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SINGULAR FACTORS OF GENERALIZED THUE-MORSE WORD

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Abstract

In this paper, we study the singular factors of Thue-Morse word over an alphabet of size q. At first, we determine the ancestors of factors of

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word $\mathbf{t}_q$. Then, we describe explicitly the singular factors by using ancestors. At last, we characterize the bispecial factors of $\mathbf{t}_q$ by using the singular factors.

1. Introduction

Among the factors of an infinite word, singular factors play a particular role. A factor w of an infinite word u is singular if $|w| = 1$ or there exist a factor v of u and some letters x, x', y, y' such that $w = xvy$, $x \neq x'$, $y \neq y'$ and $x'vy$, xvy' are factors of u. In this paper, we present singular factors of the generalized Thue-Morse word.

The Thue-Morse word $\mathbf{t}_2$ is the infinite word generated by the morphism μ is defined by $\mu(0) = 01$ and $\mu(1) = 10$. It was discovered and used in an article published in 1912 by Thue [13]. In 1921, Morse reuses this word to give an example of a non periodic recurrent sequence solving a problem of differential geometry [11]. The combinatorial properties of the Thue-Morse word have been intensively studied by some authors [1, 6].

The Thue-Morse word can be generalized over an alphabet

$$\mathcal{A}_q = \{0, 1, ..., q-1\}.$$

It is the infinite word $\mathbf{t}_{p,q}$ generated by the morphism $\mu_{p,q}$ is defined by

$$\mu_{p,q}(k) = k(k+1)\cdots(k+p-1),$$

where $p \geq 2$ and the letters are expressed modulo q. A study of the word $\mathbf{t}_q (p = q)$ has been made in [12].

The singular factors have been introduced in [14] to study the overlap properties and the local isomorphism of Fibonacci word. They permit also to determine recurrence functions of infinite words [2, 4, 5]. In [2, 8], the authors are interested in the study of singular factors using the recurrence functions of the binary and ternary Thue-Morse words.

In this paper, we study the singular factors of the word $\mathbf{t}_q$. More precisely, it is the generalization of results established in [8].

This paper is organized as follows. After a few definitions and notations in Section 2, we introduce ancestors of factors of $\mathbf{t}_q$. In Section 3, we study the singular factors of the word $\mathbf{t}_q$, then we use these factors to give the nature of bispecial factors.

2. Definitions

An alphabet $\mathcal{A}$ is a finite set of symbols. The set of the finite words over $\mathcal{A}$ is denoted by $\mathcal{A}^*$ and the empty word ε. For all $u \in \mathcal{A}^*$, $|u|$ denotes the length of u. An infinite word is an infinite sequence of letters on $\mathcal{A}$. The set of the infinite word over $\mathcal{A}$ is denoted by $\mathcal{A}^\omega$. A finite word w is a factor of u if there exist some words $u' \in \mathcal{A}^*$, $u'' \in \mathcal{A}^\omega$ such that $u = u'wu''$. If $u' = \varepsilon$ (resp. $u'' = \varepsilon$), then w is a prefix (resp. a suffix) of u.

On $\mathcal{A}_q$, we define the mapping E_q by

$$E_q(i) = \begin{cases} i+1 & \text{if } i \in \{0, 1, ..., q-2\} \\ 0 & \text{if } i = q-1. \end{cases}$$

A mapping Φ defined on a monoid $\mathcal{A}^*$ is a morphism if for all $u, v \in \mathcal{A}^*$ one has $\Phi(uv) = \Phi(u)\Phi(v)$.

A morphism is non-erasing if $\Phi(a) \neq \varepsilon$ for all $a \in \mathcal{A}$. It is prolongable on x_0, $x_0 \in \mathcal{A}$ if there exists $w \in \mathcal{A}^+$ such that $\Phi(x_0) = x_0 w$.

Definition 1. A substitution is a *non-erasing endomorphism* for the concatenation of free monoid $\mathcal{A}^*$.

The following definition is inspired by [10].

Definition 2. Let w be a *factor* of a fixed point u of a substitution φ. Then the word $v_0 v_1 v_2 \cdots v_n \in F_{n+1}(u)$ is said to be an *ancestor* of w if

(i) w is a factor of $\varphi(v_0 v_1 v_2 \cdots v_n)$,

(ii) w is neither a factor of $\varphi(v_1 v_2 \cdots v_n)$ nor $\varphi(v_0 v_1 v_2 \cdots v_{n-1})$.

Definition 3. Let v be a *bispecial factor* of $\mathbf{t}_q$. Then v is said to generate singular factors if there exist letters $x, x', y, y' \in \mathcal{A}_q$ such that $x \neq x'$, $y \neq y'$ and $xvy, x'vy, xvy', x'vy'$ are factors of $\mathbf{t}_q$.

v is said to generate a unique singular factor if there exist letters $x, x', y, y' \in \mathcal{A}_q$ such that $x \neq x', y \neq y'$ and $xvy, x'vy, xvy'$ are factors of $\mathbf{t}_q$.

Definition 4 [3]**.** Let u be an *infinite word* on $\mathcal{A}$ and v be a *bispecial factor* of u. Then

(i) v is called *strong bispecial* if ava, avb, bva and bvb are factors of u.

(ii) v is called *weak bispecial* if uniquely ava and bvb or avb and bva are factors of u.

(iii) v is called *ordinary bispecial* if v is neither strong nor weak.

We make the following remark.

Remark 1. Let u be a bispecial factor of $\mathbf{t}_q$. Then we have

• u is strong bispecial if and only if u generates singular factors.

• u is ordinary bispecial if and only if u generates a unique singular factor.

• u is weak bispecial if and only if u does not generate any singular factor.

3. Ancestor of Factors of Word $\mathbf{t}_q$

3.1. Ancestor of short bispecial *q*-prolongable factors of $\mathbf{t}_q$

We denote by $BSQ(\mathbf{t}_q)$, the set of the factors of $\mathbf{t}_q$ which are both right *q*-prolongable and left *q*-prolongable.

Recall the following theorem on bispecial *q*-prolongable factors of $\mathbf{t}_q$.

Theorem 1 [9]. *The set* $BSQ(\mathbf{t}_q)$ *is given by*

$$BSQ(\mathbf{t}_q) = \bigcup_{n\geq 0}\{\mu_q^n(i(i+1)\cdots(i+k-2)) : i \in \mathcal{A}_q,\ 2 \leq k \leq q\} \cup \{\varepsilon\}.$$

For $n = 0$, we obtain the set $BSQ_m(\mathbf{t}_q)$ of short bispecial *q*-prolongable factors of $\mathbf{t}_q$. It is given by

$$BSQ_m(\mathbf{t}_q) = \{i(i+1)\cdots(i+k-2) : i \in \mathcal{A}_q,\ 2 \leq k \leq q\} \cup \{\varepsilon\}.$$

Hereinafter, we describe the ancestors of short bispecial *q*-prolongable factors of $\mathbf{t}_q$. For $j \in \mathcal{A}_q$, consider

$$w_k^{(j)} = j(j+1)\cdots(j+k-2),\ \ 2 \leq k \leq q.$$

Also, consider the following set

$$C_{jk} = \{j+1,\ j+2,\ \ldots,\ j+k-2\},$$

where $3 \leq k \leq q$. Then we have the following result:

Proposition 1. *Let* $i,\ j \in \mathcal{A}_q$ *and* $k \in [\![2, q]\!]$. *Then*

$$Anc(w_k^{(j)}) = \begin{cases} \{i : i \in \mathcal{A}_q\} & \text{if } k = 2 \\ \{i : i \in \mathcal{A}_q \backslash C_{jk}\} \cup \{ii : i \in C_{jk}\} & \text{if } k \in [\![3, q]\!]. \end{cases}$$

Proof. We distinguish two cases.

Case 1: $k = 2$. Then $w_2^{(j)} = j$. If the image of any letter by μ_q contains all letters of the alphabet, then $w_2^{(j)}$ appears in $\mu_q(i)$ for all $i \in \mathcal{A}_q$. Therefore,

$$Anc(w_2^{(j)}) = \{i : i \in \mathcal{A}_q\}.$$

Case 2: $k \in [\![3, q]\!]$. We distinguish two sub-cases.

• $i = j$. In this case, $w_k^{(j)}$ is prefix of $\mu_q(j)$. Thus, $w_k^{(j)}$ is factor of $\mu_q(j)$.

• $i = j + k - 2$. In this case, $k \in [\![3, q]\!]$, $i \neq j$ because $k - 2 \geq 1$. Therefore, $w_k^{(j)}$ does not appear in $\mu_q(i)$ for all $i \in C_{jk}$. Otherwise there would be two occurrences of i in $\mu_q(i)$ because i is factor of $w_k^{(j)}$ and $j \neq i$. That is absurd because each letter of $\mathcal{A}_q$ appears only one time in $\mu_q(i)$. As a result, $w_k^{(j)}$ appears in $\mu_q(i)$ for all $i \in \mathcal{A}_q \backslash C_{jk}$.

Moreover, $w_k^{(j)}$ appears in $\mu_q(ii)$ for all $i \in C_{jk}$. Indeed, $i = j + k - 2$ implies that $j = i - k + 2$. So,

$$\begin{aligned}\mu_q(ii) &= \mu_q(i)\mu_q(i) \\ &= i(i+1)\cdots\underbrace{(i-k+2)(i-k+3)\cdots(i+q-1)i}i^{-1}\mu_q(i) \\ &= i(i+1)\cdots\underbrace{j(j+1)\cdots(j+k-3)(j+k-2)}_{w_k^{(j)}}i^{-1}\mu_q(i).\end{aligned}$$

Then, $w_k^{(j)}$ is factor of $\mu_q(ii)$ for all $i \in C_{jk}$, $3 \leq k \leq q$. Thus, it follows from two sub-cases that

$$Anc(w_k^{(j)}) = \{i : i \in \mathcal{A}_q \backslash C_{jk}\} \cup \{ii : i \in C_{jk}\}, \forall k \in [\![3, q]\!]. \qquad \square$$

Corollary 1. *Let* $j \in \mathcal{A}_q$. *Then*

$$Anc(\mu_q(j)) = \{j\} \cup \{ii : i \in \mathcal{A}_q \backslash \{j\}\}.$$

Proof. For $k = q + 1$, $w_{q+1}^{(j)} = \mu_q(j)$. Thus, one has

$$Anc(w_{q+1}^{(j)}) = \{i : i \in \mathcal{A}_q \backslash C_{j(q+1)}\} \cup \{ii : i \in C_{j(q+1)}\}.$$

But

$$C_{j(q+1)} = \{j + 1, j + 2, ..., j + q - 1\}$$
$$= \mathcal{A}_q \backslash \{j\}.$$

Then

$$Anc(w_{q+1}^{(j)}) = \{i : i \in \mathcal{A}_q \backslash (\mathcal{A}_q \backslash \{j\})\} \cup \{ii : i \in \mathcal{A}_q \backslash \{j\}\}.$$

Moreover, $\mathcal{A}_q = (\mathcal{A}_q \backslash \{j\}) \cup \{j\}$. Therefore $\mathcal{A}_q \backslash (\mathcal{A}_q \backslash \{j\}) = \{j\}$. Hence

$$Anc(w_{q+1}^{(j)}) = \{j\} \cup \{ii : i \in \mathcal{A}_q \backslash \{j\}\}. \qquad \square$$

3.2. Ancestor of short bispecial biprolongable factors of $\mathbf{t}_q$

The set of the factors of $\mathbf{t}_q$ which is both right biprolongable and left biprolongable is denoted by $BSB(\mathbf{t}_q)$.

Theorem 2 [9]**.** *The set* $BSB(\mathbf{t}_q)$ *is given by*

$$BSB(\mathbf{t}_q) = \bigcup_{n \geq 0} \{\mu_q^n(\mu_q(i) i (i + 1) \cdots (i + k - 2)) : i \in \mathcal{A}_q, 2 \leq k \leq q\}.$$

For $n = 0$, we obtain the set $BSB_m(\mathbf{t}_q)$ of short bispecial biprolongable factors of $\mathbf{t}_q$. It is given by

$$BSB_m(\mathbf{t}_q) = \{\mu_q(i) i (i + 1) \cdots (i + k - 2) : i \in \mathcal{A}_q, 2 \leq k \leq q\}.$$

Let $j \in \mathcal{A}_q$. Then we designate by

$$v_k^{(j)} = \mu_q(j)\, j(j+1)\cdots(j+k-2),\ 2 \le k \le q$$

the short bispecial biprolongable factors of $\mathbf{t}_q$.

Proposition 2. *Let* $j \in \mathcal{A}_q$ *and* $k \in [\![2, q]\!]$. *Then one has*:

$$Anc(v_k^{(j)}) = \begin{cases} \{ii : i \in \mathcal{A}_q\} & \text{if } k = 2 \\ \{ii : i \in \mathcal{A}_q \backslash C_{jk}\} & \text{if } k \in [\![3, q]\!]. \end{cases}$$

Proof. We distinguish two cases:

Case 1: $k = 2$. Since j appears in $\mu_q(i) = i(i+1)\cdots(i+q-1)$, $j = i + l$ with $i \in [\![0, q-1]\!]$,

$$\mu_q(ii) = i(i+1)\cdots\underbrace{(i+l)(i+l+1)\cdots(i+q-1)i(i+1)\cdots(i+l)}(i+l+1)\cdots(i+q-1)$$

$$= i(i+1)\cdots\underbrace{j(j+1)\cdots(j+q-l)\cdots(j+q-1)\,j}_{v_2^{(j)}}(i+l+1)\cdots(i+q-1).$$

Thus, $v_2^{(j)}$ is a factor of $\mu_q(ii)$ for all $i \in \mathcal{A}_q$. Therefore,

$$Anc(v_2^{(j)}) = \{ii : i \in \mathcal{A}_q\}.$$

Case 2: $k \in [\![3, q]\!]$. We distinguish two sub-cases.

• Let $i \in C_{jk}$.

Then there exists $k \in [\![3, q]\!]$ such that $i = j + k - 2$. That implies that $i \ne j$ because $k - 2 \ge 1$ for all $k \in [\![3, q]\!]$. As a result, $v_k^{(j)}$ is not factor of $\mu_q(ii)$. Otherwise it would be an occurrence of i in $v_k^{(j)}$ which is not a prefix of $\mu_q(i)$. Therefore, $\mu_q(ii)$ would contain 3 occurrences of i. That is absurd because there exist two occurrences of each letter in $\mu_q(ii)$.

• Let $i \in \mathcal{A}_q \backslash C_{jk} = \{j\} \cup \{j + k - 1,\ j + k, \ldots,\ j + q - 1\}$. Then one has:

$*$ $i = j$. Then $v_k^{(j)}$ is proper prefix of $\mu_q(ii)$. So $v_k^{(j)}$ is a factor of $\mu_q(ii)$.

$*$ $i \in [\![j + k - 1,\ j + q - 1]\!]$. Therefore, $i = j + l - 1$ with $k \leq l \leq q$. By proceeding similarly as in Case 1 one shows that $v_k^{(j)}$ is factor of $\mu_q(ii)$.

Hence

$$v_k^{(j)} \in F(\mu_q(ii)),\ \forall i \in \mathcal{A}_q \backslash C_{jk}.$$

So,

$$Anc(v_k^{(j)}) = \{ii : i \in \mathcal{A}_q \backslash C_{jk}\}.$$

□

3.3. Ancestor of non bispecial factors of $\mathbf{t}_q$

Proposition 3. *Let w be a factor of* $\mathbf{t}_q$ *and v be an ancestor of w. Then,* $\mu_q(v) = \delta_1 w \delta_2$ *with* $|\delta_1|, |\delta_2| \leq q - 1$.

Proof. Suppose without loss of generality that $|\delta_1| = |\delta_2| = q$. Since δ_1 (resp. δ_2) starts (resp. ends) with the image of a letter, there exist $a, b \in \mathcal{A}_q$ such that $\delta_1 = \mu_q(a)$ and $\delta_2 = \mu_q(b)$. As a result, $\mu_q(v) = \mu_q(a) w \mu_q(b)$. Thus, there exists $v_1 \in F(\mathbf{t}_q)$ of length inferior than v such that $w = \mu_q(v_1)$. That contradicts the minimality of v. □

Proposition 4. *Any non bispecial factor of* $\mathbf{t}_q$ *admits a unique ancestor.*

Proof. Let w be a non bispecial factor of $\mathbf{t}_q$. Suppose that w admits 2 distinct ancestors v_1 and v_2. Then one has $\mu_q(v_1) = \delta_1 w \delta_2$ and $\mu_q(v_2) = \delta_3 w \delta_4$. Necessarily, $\delta_1, \delta_2, \delta_3, \delta_4$ are all of length inferior or equal to $q - 1$ by Proposition 3. As a result,

$$w = \delta_1^{-1}\mu_q(v_1)\delta_2^{-1} = \delta_3^{-1}\mu_q(v_2)\delta_4^{-1}$$

and

$$\mu_q(v_1) = \delta_1\delta_3^{-1}\mu_q(v_2)\delta_4^{-1}\delta_2.$$

We distinguish the following cases:

Case 1: w is not special. Then $\delta_1 = \delta_3$ and $\delta_2 = \delta_4$. Consequently, $\mu_q(v_1) = \mu_q(v_2)$ and $v_1 = v_2$, since μ_q is injective.

Case 2: w is right special. Then $\delta_1 = \delta_3$, since w is not left special. As a result, $\mu_q(v_1) = \mu_q(v_2)\delta_4^{-1}\delta_2$. We have the following equalities:

$$|\mu_q(v_1)| = |\mu_q(v_2)\delta_4^{-1}\delta_2|$$

$$= |\mu_q(v_2)| + |\delta_4^{-1}\delta_2|.$$

Then

$$|\mu_q(v_1)| = |\mu_q(v_2)| = |\delta_4^{-1}\delta_2|.$$

So, $q(|v_1| - |v_2|) = |\delta_4^{-1}\delta_2|$. Therefore, $|\delta_4^{-1}\delta_2| = 0$ because $|\delta_4^{-1}\delta_2|$ is a multiple of q and $-q+1 \leq |\delta_4^{-1}\delta_2| \leq q-1$ (Proposition 3). Hence, $\delta_2 = \delta_4$. Consequently, $\mu_q(v_1) = \mu_q(v_2)$ and by injectivity of μ_q, $v_1 = v_2$. □

4. Singular Factors of $\mathbf{t}_q$

4.1. Study of singular factors of $\mathbf{t}_q$

Let i, j, l and m be letters of $\mathcal{A}_q$ such that $j = E_q(i)$, $m = E_q^{q-1}(i)$ and $l = E_q^{q-2}(i)$.

We define respectively the sequences $(m_n)_{n\geq 1}$, $(l_n)_{n\geq 1}$ by

$$\begin{cases} m_1 = m \\ m_n = E_q^{q-1}(m_{n-1}) \end{cases}$$

and

$$\begin{cases} l_1 = l \\ l_n = E_q^{q-1}(l_{n-1}). \end{cases}$$

We have the following lemma.

Lemma 1. *For all* $n \geq 1$ *and for all* $q \geq 3$,

$$m_n = E_q^{(q-1)(n-1)}(m_1) \text{ and } l_n = E_q^{(q-1)(n-1)}(l_1).$$

Proof. Let us proceed by induction. Then:

• For $n = 1$ one has $m_1 = E_q^0(m_1) = E_q^{(q-1)(1-1)}(m_1)$.

• Assume for $n \geq 2$ that $m_n = E_q^{(q-1)(n-1)}(m_1)$ and show that $m_{n+1} = E_q^{(q-1)n}(m_1)$,

$$\begin{aligned} m_{n+1} &= E_q^{q-1}(m_n) \\ &= E_q^{q-1}(E_q^{(q-1)(n-1)}(m_1)) \\ &= E_q^{(q-1)(n-1)+(q-1)}(m_1) \\ &= E_q^{(q-1)(n-1+1)}(m_1) \\ &= E_q^{(q-1)n}(m_1). \end{aligned}$$

Thus, for all $n \geq 1$,

$$m_n = E_q^{(q-1)(n-1)}(m_1).$$

We proceed similarly to show that $l_n = E_q^{(q-1)(n-1)}(l_1)$ for all $n \geq 1$. □

Remark 2. For all $i \in \mathcal{A}_q$, $n \in \mathbb{N}$ we have $E_q^n(i) = (i+n) \bmod q$.

Proposition 5 [9]. *For all* $n \geq 0$, $E_q(F_n(\mathbf{t}_q)) = F_n(\mathbf{t}_q)$.

The following proposition describes the singular factors generated by short bispecial q-prolongable factors.

Proposition 6. *For all* $k \in [\![2, q]\!]$, $i \in \mathcal{A}_q$, $w_k^{(i)}$ *generates a singular factor given by* $mw_k^{(i)} j_k^{(i)}$, *where* $j_k^{(i)} = E_q^{k-1}(i)$.

Proof. Since $E_q(F_n(\mathbf{t}_q)) = F_n(\mathbf{t}_q)$, it suffices to take $i = 0$. The other cases can be obtained by successive application of E_q. So, $w_k^{(0)} = 01 \cdots (k-2)$. We distinguish two cases:

Case 1: $k \in [\![3, q]\!]$.

By Proposition 1, the set of ancestors $w_k^{(0)}$ is given by

$$Anc(w_k^{(0)}) = \{i : i \in \mathcal{A}_q \backslash \{1, ..., (k-2)\}\} \cup \{ii : i \in \{1, ..., (k-2)\}\}, \forall k \geq 3.$$

• For $i = 0$, $w_k^{(0)}$ is prefix of $\mu_q(0)$. Since 0 is left prolongable by $q-1$,

$$\mu_q((q-1)0) = (q-1)01 \cdots (q-2)\underbrace{01 \cdots (k-2)}_{w_k^{(0)}}(k-1) \cdots (q-1).$$

It follows that $(q-2)w_k^{(0)}(k-1)$ is a factor of $\mathbf{t}_q$.

Moreover, $w_k^{(0)}$ is suffix of $\mu_q(k-1)$. Since $k-1$ is right prolongable by k, it follows that

$$\mu_q((k-1)k) = (k-1)k \cdots (q-1)\underbrace{01 \cdots (k-2)}_{w_k^{(0)}}k(k+1) \cdots (k-1).$$

So, $(q-1)w_k^{(0)}k$ is a factor of $\mathbf{t}_q$.

• For $i \in \mathcal{A}_q \setminus \{0, 1, ..., k-2, k-1\}$, $w_k^{(0)}$ is a factor of $\mu_q(i)$ and is neither suffix nor prefix of this factor. Consequently, $(q-1)w_k^{(0)}(k-1)$ is a factor of $\mathbf{t}_q$ because the letter which precedes 0 in the image of a letter is $q-1$ and the letter which follows $k-2$ is $k-1$. Therefore

$$(q-2)w_k^{(0)}(k-1),\ (q-1)w_k^{(0)}(k-1),\ (q-1)w_k^{(0)}k \in F(\mathbf{t}_q).$$

Hence $w_k^{(0)}$ generates a singular factor and is given by $(q-1)w_k^{(0)}(k-1) = mw_k^{(0)}j_k^{(0)}$.

• For all $i \in \{0, 1, ..., k-2, k-1\}$,

$$\mu_q(ii) = i(i+1)\cdots(q-1)\underbrace{01\cdots(i-1)i\cdots(k-2)}_{w_k^{(0)}}(k-1)\cdots(i+q-1).$$

Then $(q-1)w_k^{(0)}(k-1)$ is a factor of $\mathbf{t}_q$.

It follows that

$$(q-2)w_k^{(0)}(k-1), (q-1)w_k^{(0)}(k-1), (q-1)w_k^{(0)}k \in F(\mathbf{t}_q). \tag{1}$$

Hence, $w_k^{(0)}$ generates a singular factor given by

$$(q-1)w_k^{(0)}(k-1) = mw_k^{(0)}j_k^{(0)}.$$

Case 2: In case $k = 2$, $w_2^{(0)} = 0$.

Observe that the set of ancestors of 0 is given by

$$Anc(0) = \{i : i \in \mathcal{A}_q\}.$$

By proceeding similarly as in Case 1, we verify that

$$(q-2)w_2^{(0)}1, (q-1)w_2^{(0)}1, (q-1)w_2^{(0)}2 \in F(\mathbf{t}_q).$$

Hence, $w_2^{(0)}$ generates a singular factor given by $(q-1)w_2^{(0)}1 = mw_2^{(0)}j_2^{(0)}$. □

In the following, we designate by $Anc(w)$ the set of ancestors of w with $w \in F(\mathbf{t}_q)$. Like short bispecial q-prolongable factors, the short bispecial biprolongable factors generate singular factors. They are described in the following proposition.

Proposition 7. *Let* $i \in \mathcal{A}_q$. *Then*

(i) *For all* $k \in [\![2, q-1]\!]$, *the bispecial biprolongable factor* $v_k^{(i)}$ *generates a singular factor given by* $mv_k^{(i)} j_k^{(i)}$ *with* $j_k^{(i)} = E_q^{k-1}(i)$.

(ii) *For* $k = q$, $v_q^{(i)}$ *does not generate a singular factor*.

Proof. Since $E_q(F_n(\mathbf{t}_q)) = F_n(\mathbf{t}_q)$, it suffices to take the case $i = 0$ because the others can be obtained by successive application of E_q. So,

$$v_k^{(0)} = \mu_q(0)01\cdots(k-2).$$

(i) Let us show that $v_k^{(0)}$ generates a singular factor for all $k \in [\![2, q-1]\!]$. Then we distinguish two cases:

Case 1: In case $k = 2$, $v_2^{(0)} = \mu_q(0)0$.

According to Proposition 2, the set of ancestors of $v_2^{(0)}$ is given by

$$Anc(v_2^{(0)}) = \{ii : i \in \mathcal{A}_q\}.$$

• For $i = 0$, $\mu_q(0)0$ is prefix of $\mu_q(00)$. Since 00 is left prolongable by $q-1$,

$$\mu_q((q-1)00) = (q-1)01\cdots(q-2)\underbrace{012\cdots(q-1)0}_{v_2^{(0)}}12\cdots(q-1).$$

Thus, $(q-2)v_2^{(0)}1 = (q-2)v_2^{(0)}j_2^{(0)}$ is a factor of $\mathbf{t}_q$.

• For $i = 1$, $\mu_q(0)0$ is suffix of $\mu_q(11)$. As 11 is right prolongable by 2,

$$\mu_q(112) = 12\cdots(q-1)\underbrace{012\cdots(q-1)0}_{v_2^{(0)}}23\cdots1.$$

Thus, $(q-1)\mu_q(0)02 = mv_2^{(0)}2$ is a factor of $\mathbf{t}_q$.

• For all $i \in \mathcal{A}_q\backslash\{0, 1\}$, $v_2^{(0)}$ is factor of $\mu_q(ii)$ and is neither prefix nor suffix of this factor. As a result, $v_2^{(0)}$ is left prolongable by $(q-1)$ and right prolongable by 1. Thus, $(q-1)v_2^{(0)}1 = mv_2^{(0)}j_2^{(0)}$ is a factor of $\mathbf{t}_q$.

It follows that $(q-2)v_2^{(0)}1, (q-1)v_2^{(0)}1$ and $(q-1)v_2^{(0)}2$ are factors of $\mathbf{t}_q$. So, $v_2^{(0)}$ generates a singular factor given by $(q-1)v_2^{(0)}1 = mv_2^{(0)}j_2^{(0)}$.

Case 2: $k \in [\![3, q-1]\!]$. One has $v_k^{(0)} = \mu_q(0)012\cdots(k-2)$.

By Proposition 2, the set of ancestors of $v_k^{(0)}$ is given by

$$Anc(v_k^{(0)}) = \{ii : i \in \mathcal{A}_q\backslash\{1, ..., k-2\}\}.$$

• For $i = 0$, $v_k^{(0)}$ is prefix of $\mu_q(00)$. Since 00 is left prolongable by $q-1$, $(q-1)00$ is a factor of $\mathbf{t}_q$. By applying μ_q to this factor, we show that $(q-2)v_k^{(0)}(k-1)$ is a factor of $\mathbf{t}_q$.

• For $i = k-1$, one remarks that $v_k^{(0)}$ is a suffix of $\mu_q((k-1)(k-1))$. Observe that $(k-1)(k-1)$ is right prolongable by k in $\mathbf{t}_q$, by applying μ_q to $(k-1)(k-1)k$, we show that $(q-1)v_k^{(0)}k \in F(\mathbf{t}_q)$.

• For $i \in \mathcal{A}_q\backslash\{0, 1, ..., k-2, k-1\}$, $v_k^{(0)}$ is neither prefix nor suffix of $\mu_q(ii)$. Consequently, $v_k^{(0)}$ is left prolongable by $(q-1)$ and right prolongable by $k-1$. Thus, $(q-1)v_k^{(0)}(k-1) \in F(\mathbf{t}_q)$. It follows that $(q-2)v_k^{(0)}(k-1)$, $(q-1)v_k^{(0)}k$ and $(q-1)v_k^{(0)}(k-1)$ are factors of $\mathbf{t}_q$. As a result, $v_k^{(0)}$ generates a singular factor which is $(q-1)v_k^{(0)}(k-1)$ $= mv_k^{(0)} j_k^{(0)}$.

(ii) For $k = q$, $v_q^{(0)} = \mu_q(0)01\cdots(q-2)$. Then by Proposition 2,

$$Anc(v_q^{(0)}) = \{ii : i \in \mathcal{A}_q \backslash \{1, ..., q-2\}\}$$
$$= \{00, (q-1)(q-1)\}.$$

Observe that 00 (resp. $(q-1)(q-1)$) is uniquely left (resp. right) prolongable by $q-1$ (resp. 0). By applying μ_q to $(q-1)00$ and $(q-1)(q-1)0$, we show that $(q-2)v_q^{(0)}(q-1)$ and $(q-1)v_q^{(0)}0$ are the only extensions of $v_q^{(0)}$ in $\mathbf{t}_q$. Thus, $v_q^{(0)}$ does not generate any singular factor. □

Proposition 8. *Let* $k \in [\![2, q]\!]$ *and* $n \geq 1$. *Then*

(i) $\mu_q^n(w_k^{(i)})$ *generates a singular factor if and only if* $\mu_q^{n-1}(w_k^{(i)})$ *generates a singular factor*.

(ii) $\mu_q^n(v_k^{(i)})$ *generates a singular factor if and only if* $\mu_q^{n-1}(v_k^{(i)})$ *generates a singular factor*.

(iii) $\mu_q^n(i)$ *generates singular factors if and only if* $\mu_q^{n-1}(i)$ *generates singular factors for all* $n \geq 2$ *and* $i \in \mathcal{A}_q$.

Proof. (i) $\Rightarrow$ Suppose that $\mu_q^n(w_k^{(i)})$ generates a singular factor. Then there exist $x, x', y, y' \in \mathcal{A}_q$ such that $x \neq x'$, $y \neq y'$ and

$$x\mu_q^n(w_k^{(i)})y,\ x'\mu_q^n(w_k^{(i)})y,\ x\mu_q^n(w_k^{(i)})y' \in F(\mathbf{t}_q).$$

Observe that *x* is the end of the image of $x - q + 1$. As a result,

$$\mu_q(x - q + 1)\mu_q^n(w_k^{(i)})\mu_q(y) = \mu_q((x - q + 1)\mu_q^{n-1}(w_k^{(i)})y) \in F(\mathbf{t}_q).$$

Hence

$$(x - q + 1)\mu_q^{n-1}(w_k^{(i)})y \in F(\mathbf{t}_q).$$

Similarly, we show that

$$(x' - q + 1)\mu_q^{n-1}(w_k^{(i)})y,\ (x - q + 1)\mu_q^{n-1}(w_k^{(i)})y' \in F(\mathbf{t}_q).$$

Moreover, $x - q + 1 \neq x' - q + 1$ because $x \neq x'$. Consequently, $\mu_q^{n-1}(w_k^{(i)})$ generates singular factor.

$\Leftarrow$ Suppose that $\mu_q^{n-1}(w_k^{(i)})$ generates a singular factor. Then, there exist letters $x, x', y, y' \in \mathcal{A}_q$ such that $x \neq x'$, $y \neq y'$ and

$$x\mu_q^{n-1}(w_k^{(i)})y,\ x'\mu_q^{n-1}(w_k^{(i)})y,\ x\mu_q^{n-1}(w_k^{(i)})y' \in F(\mathbf{t}_q).$$

By applying μ_q to each of these factors, we obtain

$$(x + q - 1)\mu_q^n(w_k^{(i)})y,\ (x' + q - 1)\mu_q^n(w_k^{(i)})y,\ (x + q - 1)\mu_q^n(w_k^{(i)})y' \in F(\mathbf{t}_q).$$

Hence, $\mu_q^n(w_k^{(i)})$ generates a singular factor.

(ii) $\Rightarrow$ We proceed as in (i).

(iii) $\Rightarrow$ Suppose $\mu_q^n(i)$ generates singular factors. Then there exist letters $x, x', y, y' \in \mathcal{A}_q$ with $x \neq x'$, $y \neq y'$ such that

$$x\mu_q^n(i)y,\ x'\mu_q^n(i)y,\ x\mu_q^n(i)y',\ x'\mu_q^n(i)y' \in F(\mathbf{t}_q).$$

Since *x* is suffix of $\mu_q(x - q + 1)$ and *y* is prefix of $\mu_q(y)$,

$$\mu_q(x - q + 1)\mu_q^n(i)\mu_q(y) = \mu_q((x - q + 1)\mu_q^{n-1}(i)y) \in F(\mathbf{t}_q).$$

As a result, $(x - q + 1)\mu_q^{n-1}(i)y \in F(\mathbf{t}_q)$. Similarly, we show that

$$(x' - q + 1)\mu_q^{n-1}(i)y,\ (x - q + 1)\mu_q^{n-1}(i)y',\ (x' - q + 1)\mu_q^{n-1}(i)y' \in F(\mathbf{t}_q).$$

Thus, $\mu_q^{n-1}(i)$ generates singular factors.

$\Leftarrow$ Suppose that $\mu_q^{n-1}(i)$ generates singular factors. Then, there exist letters $x, x', y, y' \in \mathcal{A}_q$ such that $x \neq x'$, $y \neq y'$ and

$$x\mu_q^{n-1}(i)y,\ x'\mu_q^{n-1}(i)y,\ x\mu_q^{n-1}(i)y',\ x'\mu_q^{n-1}(i)y' \in F(\mathbf{t}_q).$$

By applying μ_q to each of these factors, it follows that

$$(x + q - 1)\mu_q^{n}(i)y,\ (x' + q - 1)\mu_q^{n}(i)y,$$

$$(x + q - 1)\mu_q^{n}(i)y',\ (x' + q - 1)\mu_q^{n}(i)y' \in F(\mathbf{t}_q).$$

Therefore, $\mu_q^n(i)$ generates singular factors for all $n \geq 1$. □

Proposition 9. *Let* $k \in [\![2, q]\!]$, $i \in \mathcal{A}_q$ *and* $n \in \mathbb{N}$. *Then*

(i) $\mu_q^n(w_k^{(i)})$ *generates a singular factor for all* $k \in [\![3, q]\!]$.

(ii) $\mu_q^n(v_k^{(i)})$ *generates a singular factor for all* $k \in [\![2, q-1]\!]$.

(iii) $\mu_q^n(i)$ *generates singular factors for all* $n \geq 1$.

Proof. • By Proposition 8, to show that $\mu_q^n(w_k^{(i)})$ generates a singular factor it suffices to show that $w_k^{(i)}$ generates a singular factor. By Proposition 7, $w_k^{(i)}$ generates a singular factor. So, $\mu_q^n(w_k^{(i)})$ generates a singular factor.

• Similarly, we show that $\mu_q^n(v_k^{(i)})$ generates a singular factor.

• By Proposition 8, to show that $\mu_q^n(i)$ generates singular factors it suffices to show that $\mu_q(i)$ generates singular factors. Since the letter plays a symmetrical role, without loss of generality we take $i = 0$. Observe that $w_{q+1}^{(0)} = \mu_q(0)$. Consider the relation (1) and take $k = q + 1$, it follows that

$$(q-2)\mu_q(0)0,\ (q-1)\mu_q(0)0,\ (q-1)\mu_q(0)1 \in F(\mathbf{t}_q). \tag{2}$$

Moreover, by Corollary 1

$$Anc(\mu_q(0)) = \{0\} \cup \{ii : i \in \mathcal{A}_q \backslash \{0\}\}.$$

∗ For all $i \in \mathcal{A}_q \backslash \{0\}$, $(q-1)\mu_q(0)0$ is a factor $\mu_q(ii)$. Thus, $(q-1)\mu_q(0)0$ is a singular factor since

$$(q-1)\mu_q(0)1,\ (q-2)\mu_q(0)0 \in F(\mathbf{t}_q).$$

∗ To find the others singular factors it suffices to consider the bilateral extensions of 0 to which we apply μ_q. The bilateral extensions of 0 in $\mathbf{t}_q$ are in the form $i'0j$ with i', j some letters of $\mathcal{A}_q$ such that $(i', j) \in \{q-1\} \times \mathcal{A}_q$ or $(i', j) \in \mathcal{A}_q \times \{1\}$. The bilateral extensions of 0 for $(i', j) = (0, 1)$ and $(i', j) = (q-1, 0)$ give us 001 and $(q-1)00$. By applying μ_q to these factors we obtain $(q-1)\mu_q(0)1$, $(q-2)\mu_q(0)0$.

For $i' = q-1$, $(q-1)0j$ is a factor of $\mathbf{t}_q$ for all $j \in \mathcal{A}_q$. In particular for $j = 1$, we observe that $(q-2)\mu_q(0)1$ appears in the image of $(q-1)01$ by μ_q. Furthermore, $(q-2)\mu_q(0)1$ is a singular factor because $(q-1)\mu_q(0)1, (q-2)\mu_q(0)0 \in F(\mathbf{t}_q)$. Thus $(q-2)\mu_q(0)j$ is not a singular factor of $\mathbf{t}_q$ for all $j \in \mathcal{A}_q \backslash \{0, 1\}$. Suppose that $k\mu_q(0)j$ is a factor of $\mathbf{t}_q$ with $k \neq q-2$. Since $k\mu_q(0)j$ appears in $\mu_q((k+1)0j)$, $(k+1)0j \in F(\mathbf{t}_q)$. That is impossible because $k+1 \neq q-1$ and $j \neq 1$ by hypothesis.

For all $i' \in \mathcal{A}_q \backslash \{0, q-1\}$, $i'0$ extends uniquely to the right by 1. By applying μ_q to $i'01$, it follows that $(i'+q-1)\mu_q(0)1$ is a factor of $\mathbf{t}_q$.

Let us show that $(i'+q-1)\mu_q(0)1$ is not a singular factor of $\mathbf{t}_q$.

Suppose that $(i'+q-1)\mu_q(0)k \in F(\mathbf{t}_q)$ with $k \neq 1$. We observe that $(i'+q-1)\mu_q(0)k$ appears in $\mu_q(i'0k)$, then $i'0k$ is in $\mathbf{t}_q$. That is absurd since $i' \neq q-1$ and $k \neq 1$. Therefore, $\mu_q(0)$ generates singular factors given by

$$(q-2)\mu_q(0)0,\ (q-2)\mu_q(0)1,\ (q-1)\mu_q(0)0,\ (q-1)\mu_q(0)1. \qquad \square$$

Let $i, j \in \mathcal{A}_q$ such that $j = E_q(i)$. Then we have the following result:

Theorem 3. *The set of singular factors of* $\mathbf{t}_q$ *generated by* $\mu_q^n(i)$, $n \geq 1$ *is given by*

$$S_n = \{m_n\mu_q^n(i)i,\ m_n\mu_q^n(i)j,\ l_n\mu_q^n(i)i,\ l_n\mu_q^n(i)j : j = E_q(i)\}, \quad \forall n \geq 1.$$

Proof. We proceed by induction on *n*.

Since $E_q(F_n(\mathbf{t}_q)) = F_n(\mathbf{t}_q)$, take $i = 0$. We know that the set of singular factors generated by $\mu_q(0)$ is given by

$$(q-1)\mu_q(0)0,\ (q-1)\mu_q(0)1,\ (q-2)\mu_q(0)0 \text{ and } (q-2)\mu_q(0)1.$$

But $q-1 = E_q^{q-1}(0) = m_1$ and $q-2 = E_q^{q-2}(0) = l_1$. Then,

$$(q-1)\mu_q(0)0 = m_1\mu_q(0)0,\ (q-1)\mu_q(0)1 = m_1\mu_q(0)1,$$

$$(q-2)\mu_q(0)0 = l_1\mu_q(0)0,\ (q-2)\mu_q(0)1 = l_1\mu_q(0)1.$$

Assume that

$$S_n = \{m_n\mu_q^n(i)i,\ m_n\mu_q^n(i)j,\ l_n\mu_q^n(i)i,\ l_n\mu_q^n(i)j\}$$

and show that

$$S_{n+1} = \{m_{n+1}\mu_q^{n+1}(i)i,\ m_{n+1}\mu_q^{n+1}(i)\,j,\ l_{n+1}\mu_q^{n+1}(i)i,\ l_{n+1}\mu_q^{n+1}(i)\,j : j = E_q(i)\}.$$

By applying μ_q to elements of S_n we show that

$$(m_n + q - 1)\mu_q^{n+1}(i)i,\ (m_n + q - 1)\mu_q^{n+1}(i)\,j,$$

$$(l_n + q - 1)\mu_q^{n+1}(i)i,\ (l_n + q - 1)\mu_q^{n+1}(i)\,j,$$

are factors of $\mathbf{t}_q$. But, $m_n + q - 1 = E_q^{q-1}(m_n) = m_{n+1}$ and $l_n + q - 1 = E_q^{q-1}(l_n) = l_{n+1}$. Then,

$$(m_n + q - 1)\mu_q^{n+1}(i)i = m_{n+1}\mu_q^{n+1}(i)i,\ (m_n + q - 1)\mu_q^{n+1}(i)\,j = m_{n+1}\mu_q^{n+1}(i)\,j,$$

$$(l_n + q - 1)\mu_q^{n+1}(i)i = l_{n+1}\mu_q^{n+1}(i)i,\ (l_n + q - 1)\mu_q^{n+1}(i)\,j = l_{n+1}\mu_q^{n+1}(i)\,j.$$

As a result,

$$S_{n+1} = \{m_{n+1}\mu_q^{n+1}(i)i,\ m_{n+1}\mu_q^{n+1}(i)\,j,\ l_{n+1}\mu_q^{n+1}(i)i,\ l_{n+1}\mu_q^{n+1}(i)\,j : j = E_q(i)\}.$$

□

Theorem 4. *Let* $k \in [\![2, q]\!]$, $i \in \mathcal{A}_q$ *and* $n \in \mathbb{N}$. *Then*

(i) *The unique singular factor* s'_n *generated by* $\mu_q^n(w_k^{(i)})$ *is given by*

$$s'_n = m_{n+1}\mu_q^n(w_k^{(i)})\,j_k^{(i)},\ \ \forall n \geq 0,\ k \in [\![3, q]\!].$$

(ii) *For all* $k \neq q$. *The unique singular factor* s''_n *generated by* $\mu_q^n(v_k^{(i)})$ *is given by*

$$s''_n = m_{n+1}\mu_q^n(v_k^{(i)})\,j_k^{(i)},\ \ \forall n \geq 0,\ k \in [\![2, q-1]\!].$$

Proof. • Let us proceed by induction on *n*. Then:

For $n = 0$, by Proposition 6, $w_k^{(i)}$ generates a singular factor and is given by $m_1 w_k^{(i)} j_k^{(i)}$.

Assuming that the singular factor generated by $\mu_q^n(w_k^{(i)})$ is $m_{n+1}\mu_q^n(w_k^{(i)})\, j_k^{(i)}$, we show that the singular factor generated by $\mu_q^{n+1}(w_k^{(i)})$ is $m_{n+2}\mu_q^{n+1}(w_k^{(i)})\, j_k^{(i)}$.

By applying μ_q to $m_{n+1}\mu_q^n(w_k^{(i)})\, j_k^{(i)}$, we show that

$$(m_{n+1} + q - 1)\mu_q^{n+1}(w_k^{(i)})\, j_k^{(i)}$$

is a factor of $\mathbf{t}_q$. But $m_{n+1} + q - 1 = E_q^{q-1}(m_{n+1}) = m_{n+2}$. Then, the singular factor generated by $\mu_q^{n+1}(w_k^{(i)})$ is $m_{n+2}\mu_q^{n+1}(w_k^{(i)})\, j_k^{(i)}$. Thus for all $n \geq 0$, the singular factor generated by $\mu_q^n(w_k^{(i)})$ is $m_{n+1}\mu_q^n(w_k^{(i)})\, j_k^{(i)}$.

• We proceed similarly to show that the singular factor generated by $\mu_q^n(v_k^{(i)})$ is $m_{n+1}\mu_q^n(v_k^{(i)})\, j_k^{(i)}$. □

The singular factors can be used to give the set of differents forms of bispecial factors of $\mathbf{t}_q$. So, we have the following result:

Proposition 10. *Let* $n \in \mathbb{N}^*$ *and* $k \in [\![2, q]\!]$. *Then*

(i) *The set* $SB(\mathbf{t}_q)$ *of strong bispecial factors of* $\mathbf{t}_q$ *is given by*

$$SB(\mathbf{t}_q) = \bigcup_{n\geq 1}\{\mu_q^n(i) : i \in \mathcal{A}_q\} \cup \{\varepsilon\}.$$

(ii) *The set* $OB(\mathbf{t}_q)$ *of ordinary bispecial factors of* $\mathbf{t}_q$ *is given by*

$$OB(\mathbf{t}_q) = \bigcup_{n\geq 0}(\{\mu_q^n(w_k^{(i)}) : i \in \mathcal{A}_q \text{ and } k \in [\![3, q]\!]\}$$

$$\cup \{\mu_q^n(v_k^{(i)}) : i \in \mathcal{A}_q \text{ and } k \in [\![2, q-1]\!]\}).$$

(iii) *The set* $WB(\mathbf{t}_q)$ *of weak bispecial factor of* $\mathbf{t}_q$ *is given by*

$$WB(\mathbf{t}_q) = \bigcup_{n \geq 0} \{\mu_q^n(v_q^{(i)}) : i \in \mathcal{A}_q\}.$$

Proof. The proof follows from Proposition 7, Proposition 9 and Remark 1. □

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