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CANTOR'S THEOREM IN TVS-CONE METRIC SPACES

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Abstract

In 2010, Du [3] introduced the TVS-cone metric spaces, in 2011, Lahiri et al. [7] showed the Cantor's theorem in a complete 2-metric space and proved some of its applications to fixed point problems. In this paper, based on these ideas, we prove Cantor's theorem in complete cone metric spaces for orders generated by cone in a real Hausdorff locally convex topological vector spaces.

1. Introduction

Recently, there are some ideas for the development of the metric spaces. For example, the concept of a 2-metric space has been initiated by Gahler [4] in 1963. In 2007, Huang and Zhang [5] introduced the concept of cone metric spaces by replacing the set of real numbers with an ordered Banach space and afterwards, a series of articles in this field have been dedicated to

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the improvement of fixed point theory. In 2010, Du [3] introduced the TVS-cone metric spaces by replacing Banach spaces with topological vector spaces in the definition of cone metric spaces and these spaces have aroused many mathematical scholar's interests and some interesting results have been obtained in the past years. In 2011, Lahiri et al. [7] showed the Cantor's theorem in a 2-metric space and proved some fixed point theorem as a Cantor's theorem application. Our idea is studying some adding properties of TVS-cone metric spaces based on above information. In fact, we prove analogues of Cantor's theorem in TVS-cone metric spaces and prove Baire category theorem as a Cantor's theorem application.

2. TVS-cone Metric Spaces

In this section, we define cone metric spaces and show some properties of these spaces. Let E always be real Hausdorff locally convex topological vector spaces and C is a subset of E. We say that C is a *cone* in E if

(i) C is closed, $\operatorname{int} C$ is nonempty and $C \neq \{0\}$;

(ii) $a\sigma + b\tau \in C$ for all $\sigma, \tau \in C$ and non-negative real numbers a, b;

(iii) $C \cap (-C) = \{0\}$.

For a given cone C in E, we can define a partial ordering $\preceq$ with respect to C by $\sigma \preceq \tau$ if and only if $\tau - \sigma \in C$, while $\sigma \prec \tau$ stands for $\tau - \sigma \in \operatorname{int} C$, where $\operatorname{int} C$ denotes the interior of C. In this paper, we always suppose E is a real Hausdorff locally convex topological vector spaces, C a cone in E with $\operatorname{int} C \neq \varnothing$ and $\preceq$ is partial ordering with respect to C.

Definition 2.1. Let C be a cone in E. Then C is said to have *neighborhood properties* if for any neighborhood U of origin in E, there is a neighborhood V of origin in E such that $C \cap (V - C) \subset U$.

Remark 2.2. If C has a closed convex bounded base, then C has neighborhood properties (see Proposition 1.8 in [8]).

Proposition 2.3. *Assume that C has neighborhood properties. Then, for any neighborhood U of origin in E, there is $e \in E$, $0 \prec e$ such that*

$$C \cap (e - C) \subset U.$$

Proof. Assume that C has neighborhood properties. Let U be an arbitrary neighborhood of origin in E. Then there is a neighborhood V of origin in E such that

$$C \cap (V - C) \subset U.$$

Since $\operatorname{int} C \neq \varnothing$, we can choose $a \in E$, $0 \prec a$. From $\frac{1}{n} a \to 0$, there is n_0 such that

$$\frac{1}{n} a \in V \text{ for all } n \geq n_0.$$

Set $e = \frac{1}{n_0} a$. Hence $e \in E$, $0 \prec e$ and

$$C \cap (e - C) \subset C \cap (V - C) \subset U.$$

Definition 2.4 (See [5])**.** Let C be a cone in a normed space E. Then we say that C is *normal* if there is a number $M > 0$ such that for all $\sigma, \tau \in E$,

$$0 \preceq \sigma \preceq \tau \text{ implies } \| \sigma \| \leq M \| \tau \|.$$

Proposition 2.5. *Let C be a normal cone in a normed space E. Then C has neighborhood properties.*

Proof. Assume that C does not have neighborhood properties. Then there exists $\varepsilon > 0$ such that for any $n \geq 1$, we have

$$C \cap \left[B\left(0, \frac{1}{n}\right) - C \right] \not\subset B(0, \varepsilon),$$

where $B(0, \delta) = \{\sigma \in E : \| \sigma \| < \delta\}$. Thus, for any $n \geq 1$, there exists $\sigma_n \in C$, $\tau_n \in B\left(0, \frac{1}{n}\right)$ such that $\sigma_n \preceq \tau_n$ and $\sigma_n \notin B(0, \varepsilon)$. Since C is a normal cone and $\tau_n \to 0$, $\sigma_n \to 0$. Hence $0 \notin B(0, \varepsilon)$. This is a contradiction.

Definition 2.6 (See [3])**.** Let H be a nonempty set. Then the mapping $d : H \times H \to E$ is called a *cone metric* on H if

(1) $0 \preceq d(\sigma, \tau)$ for all $\sigma, \tau \in H$ and $d(\sigma, \tau) = 0$ if and only if $\sigma = \tau$;

(2) $d(\sigma, \tau) = d(\tau, \sigma)$ for all $\sigma, \tau \in H$;

(3) $d(\sigma, \tau) \preceq d(\sigma, \varsigma) + d(\varsigma, \tau)$ for all $\sigma, \tau, \varsigma \in H$.

The pair (H, d) is called a *TVS-cone metric space*.

Definition 2.7 (See [3])**.** Let (H, d) be a TVS-cone metric space, and $\{\sigma_n\}$ be a sequence in H. Then

(1) σ is the *limit* of $\{\sigma_n\}$ if for every $e \in E$ with $0 \prec e$, there is n_0 such that $d(\sigma_n, \sigma) \prec e$ for all $n \geq n_0$. We denote this by $\sigma_n \to \sigma$.

(2) $\{\sigma_n\}$ is a *Cauchy sequence* if for every $e \in E$ with $0 \prec e$, there is n_0 such that $d(\sigma_n, \sigma_m) \prec e$ for all $n, m \geq n_0$.

Definition 2.8 (See [3])**.** Let (H, d) be a TVS-cone metric space. If every Cauchy sequence is convergent in H, then (H, d) is called a *complete TVS-cone metric space*.

Now, we prove some results in [5] for orders generated by the cone in real Hausdorff locally convex topological vector spaces.

Lemma 2.9. *Let (H, d) be a TVS-cone metric space and $\{\sigma_n\}$ be a sequence in H. Then*

(i) *If $\{\sigma_n\}$ converges to $\sigma \in H$, then $\{\sigma_n\}$ is a Cauchy sequence.*

(ii) *If $\{\sigma_n\}$ converges to $\sigma \in H$ and $\{\sigma_n\}$ converges to $\tau \in H$, then $\sigma = \tau$.*

Proof. (i) Let $e \in E$ be arbitrary with $0 \prec e$ (i.e., $e \in \operatorname{int} C$). Since $\operatorname{int} C$ is nonempty, there exists $0 < \lambda < \frac{1}{2}$ such that

$$\delta := \lambda e \in \operatorname{int} C \quad \text{and} \quad \delta + \delta = 2\lambda e \prec e.$$

Because $\sigma_n \to \sigma \in H$, there is n_0 such that

$$d(\sigma_n, \sigma) \prec \delta \text{ for all } n \geq n_0.$$

Hence for any $n, m \geq n_0$, by the triangle inequality,

$$d(\sigma_n, \sigma_m) \preceq d(\sigma_n, \sigma) + d(\sigma_m, \sigma) \prec \delta + \delta \prec e \text{ for all } n, m \geq n_0.$$

Thus $d(\sigma_n, \sigma_m) \prec e$ for all $n, m \geq n_0$. Since $0 \prec e$ was arbitrary, $\{\sigma_n\}$ is a Cauchy sequence.

(ii) Suppose $\sigma_n \to \sigma$ and $\sigma_n \to \tau$. Let $e \in E$ with $0 \prec e$. Then for each $k \geq 1$, choose $\delta_k \in \operatorname{int} C$ with $\delta_k + \delta_k \prec \frac{e}{k}$. By convergence, there exists n_k such that for all $n \geq n_k$,

$$d(\sigma_n, \sigma) \prec \delta_k \quad \text{and} \quad d(\sigma_n, \tau) \prec \delta_k.$$

Therefore $n \geq n_k$,

$$d(\sigma, \tau) \preceq d(\sigma_n, \sigma) + d(\sigma_n, \tau) \prec \delta_k + \delta_k \prec \frac{e}{k}.$$

Hence $d(\sigma, \tau) \prec \frac{e}{k}$ for every $k \geq 1$, which implies

$$\frac{e}{k} - d(\sigma, \tau) \in \operatorname{int} C \subset C \text{ for all } k \geq 1.$$

Letting $k \to \infty$ and by closedness of C, $-d(\sigma, \tau) \in C$. But also $d(\sigma, \tau) \in C$. Thus both $-d(\sigma, \tau)$ and $d(\sigma, \tau)$ lie in C, so by $C \cap (-C) = \{0\}$, we get $d(\sigma, \tau) = 0$. This proves that $\sigma = \tau$.

Lemma 2.10. *Let (H, d) be a TVS-cone metric space, C has neighborhood properties and $\{\sigma_n\}$ be a sequence in H. Then the following* *hold*:

(i) $\sigma_n \to \sigma \in H$ *if and only if* $d(\sigma_n, \sigma) \to 0$.

(ii) $\{\sigma_n\}$ *is a Cauchy sequence if and only if* $d(\sigma_n, \sigma_m) \to 0$.

Proof. (i) Suppose that $\sigma_n \to \sigma \in H$. Let U be an arbitrary neighborhood of origin in E. Since C has neighborhood properties, there is $e \in E, 0 \prec e$ such that $C \bigcap (e - C) \subset U$. By $\sigma_n \to \sigma \in H$, there is n_0 such that $d(\sigma_n, \sigma) \prec e$ for all $n \geq n_0$. This implies $d(\sigma_n, \sigma) \in e - \operatorname{int} C \subset e - C$ for all $n \geq n_0$. Hence $d(\sigma_n, \sigma) \in C \bigcap (e - C)$ for all $n \geq n_0$. Thus, $d(\sigma_n, \sigma) \in U$ for all $n \geq n_0$. This means $d(\sigma_n, \sigma) \to 0$.

Conversely, suppose that $d(\sigma_n, \sigma) \to 0$. For $e \in E, 0 \prec e$, there is a neighborhood U of the origin in E such that $e - U \subset \operatorname{int} C$. Since $d(\sigma_n, \sigma) \to 0$, there is n_0 such that $d(\sigma_n, \sigma) \in U$ for all $n \geq n_0$. This implies $e - d(\sigma_n, \sigma) \in \operatorname{int} C$ for all $n \geq n_0$. This means that $d(\sigma_n, \sigma) \prec e$ for all $n \geq n_0$. Hence $\sigma_n \to \sigma \in H$.

(ii) Suppose that $\{\sigma_n\}$ is a Cauchy sequence. Let U be an arbitrary neighborhood of origin in E. Because C has neighborhood properties, there is $e \in E$, $0 \prec e$ such that $C \bigcap (e - C) \subset U$. Since $\{\sigma_n\}$ is a Cauchy sequence, there is n_0 such that $d(\sigma_m, \sigma_n) \prec e$ for all $n, m \geq n_0$. This implies $d(\sigma_n, \sigma_m) \in e - \operatorname{int} C \subset e - C$ for all $n, m \geq n_0$. Hence $d(\sigma_n, \sigma_m) \in C \bigcap (e - C)$ for all $n, m \geq n_0$. Thus, $d(\sigma_m, \sigma_n) \in U$ for all $n, m \geq n_0$. This means $d(\sigma_n, \sigma_m) \to 0$.

Conversely, suppose that $d(\sigma_n, \sigma_m) \to 0$. Let $e \in E$, $0 \prec e$. Then there is a neighborhood U of the origin in E such that $e - U \subset \operatorname{int} C$. Since $d(\sigma_n, \sigma_m) \to 0$, there is n_0 such that $d(\sigma_m, \sigma_n) \in U$ for all $m, n \geq n_0$. This implies $e - d(\sigma_m, \sigma_n) \in \operatorname{int} C$ for all $n, m \geq n_0$. This means that $d(\sigma_n, \sigma_m) \prec e$ for all $n, m \geq n_0$. Hence $\{\sigma_n\}$ is Cauchy.

Lemma 2.11. *Let* (H, d) *be a TVS-cone metric space, the cone C has neighborhood properties. Let* $\{\sigma_n\}$, $\{\tau_n\}$ *be two sequences in* H *and* $\sigma_n \to \sigma \in H$, $\tau_n \to \tau \in H$. *Then* $d(\sigma_n, \tau_n) \to d(\sigma, \tau)$.

Proof. Let U be an arbitrary balanced neighborhood of origin in E. Because C has neighborhood properties, there is $e \in E$, $0 \prec e$ such that $C \bigcap (e - C) \subset U$. Since $\sigma_n \to \sigma \in H$, $\tau_n \to \tau \in H$, there is n_0 such that

$$d(\sigma_n, \sigma) \prec \frac{e}{4} \quad \text{and} \quad d(\tau_n, \tau) \prec \frac{e}{4} \text{ for all } n \geq n_0.$$

On the other hand, we have

$$d(\sigma_n, \tau_n) \preceq d(\sigma_n, \sigma) + d(\sigma, \tau) + d(\tau, \tau_n) \prec \frac{e}{2} + d(\sigma, \tau),$$

$$d(\sigma, \tau) \preceq d(\sigma_n, \sigma) + d(\sigma_n, \tau_n) + d(\tau_n, \tau) \prec \frac{e}{2} + d(\sigma_n, \tau_n)$$

for all $n \geq n_0$. This implies

$$0 \preceq d(\sigma, \tau) - d(\sigma_n, \tau_n) + \frac{e}{2} \prec e \text{ for all } n \geq n_0.$$

Hence

$$d(\sigma, \tau) - d(\sigma_n, \tau_n) + \frac{e}{2} \in C \bigcap (e - C) \subset U \text{ for all } n \geq n_0.$$

Remarking $\frac{e}{2} \in U$, we have

$$d(\sigma, \tau) - d(\sigma_n, \tau_n) \subset U \text{ for all } n \geq n_0.$$

Thus, $d(\sigma_n, \tau_n) \to d(\sigma, \tau)$.

3. Cantor's Theorem

Definition 3.1. Let (H, d) be a TVS-cone metric space. For $\sigma_0 \in H$ as the center and $e \in E$, $0 \prec e$ as the radius, we define:

(i) The Open Ball $B(\sigma_0, e)$ is the subset of H given by

$$B(\sigma_0, e) := \{\sigma \in H : d(\sigma_0, \sigma) \prec e\}.$$

(ii) The Closed Ball $\overline{B}(\sigma_0, e)$ is the subset of H given by

$$\overline{B}(\sigma_0, e) := \{\sigma \in H : d(\sigma_0, \sigma) \preceq e\}.$$

Definition 3.2. Let (H, d) be a TVS-cone metric space, $A \subset H$. We say that A is an *open set* if for any $\sigma \in A$, there is $e_\sigma \in E$, $0 \prec e_\sigma$ such that $B(\sigma, e_\sigma) \subset A$. A set $B \subset H$ is closed if its complement is open.

Remark 3.3. If (H, d) is a TVS-cone metric space, then the following statements are true:

(i) The empty set $\varnothing$ and the space H are open and closed.

(ii) The open ball $B(\sigma_0, e)$ is open, closed ball $\overline{B}(\sigma_0, e)$ is closed, where

$$\overline{B}(\sigma_0, e) := \{\sigma \in H : d(\sigma_0, \sigma) \preceq e\}, \quad e \in E, \quad \sigma_0 \in H.$$

(iii) The union of open sets is open, the intersection of an arbitrary number of closed sets is closed.

(iv) The intersection of a finite number of open sets is open, the union of a finite number of closed sets is closed.

Proof. (iv) Let $U_1, U_2, ..., U_k$, $k \in \mathbb{N}^*$ be open sets, and set $U = \bigcap_{i=1}^{k} U_i$. Take $\sigma \in U$. For each i, there exist $e_i \in E$, $0 \prec e_i$ such that $B(\sigma, e_i) \subset U_i$. Choose $e \in E$, $0 \prec e$ with $e \preceq e_i$ for all i. Then $B(\sigma, e) \subset \bigcap_{i=1}^{k} B(\sigma, e_i) \subset U$. So U is open.

The union of a finite number of closed sets is closed. Indeed, let $V_1, V_2, ..., V_k,\ k \in \mathbb{N}^*$ be closed sets. Then the complement of $\bigcup_{i=1}^{k} V_i$ is

$$H \setminus \bigcup_{i-1}^{k} V_i = \bigcap_{i=1}^{k} (H \setminus V_i).$$

Since each $H \setminus V_i$ is open and the finite intersection of open sets is open, the complement is open. Hence $\bigcup_{i=1}^{k} V_i$ is closed.

Let D be all family open sets in a cone metric space (H, d). Then (H, D) is a topological space called a *topological TVS-cone metric space*. Clearly, (H, D) is a Hausdorff topological space.

Definition 3.4. Let (H, D) be a topological TVS-cone metric space and $A \in H$. Then we say that

(i) The *closure* of *A*, denoted by $\overline{A}$, is defined to be the intersection of closed sets containing *A*.

(ii) The *interior* of *A*, denoted by $\operatorname{int} A$, is defined to be the union of open sets contained in *A*.

(iii) *A* is said to be *dense* in *H* if $\overline{A} = H$.

(iv) *A* is said to be *no-where dense* in *X* if $\operatorname{int} \operatorname{int} \overline{A} = \varnothing$.

Lemma 3.5. *A sequence* $\{\sigma_n\}$ *is convergent to* σ *in* (H, d) *if and only if for any open set W containing* σ, *there exists* σ_0 *such that* $\sigma_n \in W$ *for all* $n \geq n_0$.

Proof. Suppose that $\sigma_n \to \sigma$ as $n \to \infty$. Let W be an open set containing σ. Then there is $e \in E,\ 0 \prec e$ such that $B(\sigma, e) \subset W$. Since $\sigma_n \to \infty$ as $n \to \infty$, there exists n_0 such that $d(\sigma_n, \sigma) \prec e$ for all $n \geq n_0$. Hence $\sigma_n \in W$ for all $n \geq n_0$.

Conversely, let any $e \in E$, $0 \prec e$. Then $B(\sigma, e)$ is an open set containing σ. By hypothesis, there is n_0 such that $\sigma_n \in B(\sigma, e)$ for all $n \geq n_0$. Hence $d(\sigma_n, \sigma) \prec e$ for all $n \geq n_0$. Thus, $\sigma_n \to \infty$ as $n \to \infty$.

Lemma 3.6. *Let A be a subset of a TVS-cone metric space (H, d) and C has neighborhood properties. Then A is a closed subset if and only if for any $\{\sigma_n\} \subset A$ convergent to $\sigma \in H$ implies $\sigma \in A$.*

Proof. Suppose that A is a closed subset in TVS-cone metric space (H, d). Assume that $\{\sigma_n\} \subset A$, $\sigma_n \to \sigma \in H$. We prove that $\sigma \in A$. Indeed, assume that $\sigma \in H \backslash A$. Since $H \backslash A$ is open in (H, d), there is n_0 such that $\sigma_n \in H \backslash A$ for all $n \geq n_0$, which is a contradiction with $\{\sigma_n\} \subset A$.

Conversely, suppose that if $\{\sigma_n\} \subset A$ converges to $\sigma \in H$, then $\sigma \in A$. If A is not closed, then $H \backslash A$ is not open. This implies that there exist $\sigma \in H \backslash A$ and $e \in E$, $0 \prec e$ such that $B\left(\sigma, \frac{1}{n} e\right) \not\subset H \backslash A$ for all $n \geq 1$. For each $n \geq 1$, we choose $\sigma_n \in B\left(\sigma, \frac{1}{n} e\right)$ such that $\sigma_n \notin H \backslash A$. Hence, we have $\{\sigma_n\} \subset A$. Now, we prove $\sigma_n \to \sigma$. Indeed, let U be an arbitrary neighborhood of the origin in E. Since C has neighborhood properties, there is a neighborhood V of the origin in E such that $C \bigcap (V - C) \subset U$. Since $\sigma_n \in B\left(\sigma, \frac{1}{n} e\right)$ for $n \geq 1$, $\frac{1}{n} e - d(\sigma_n, \sigma) \in C$ for all $n \geq 1$. By $\frac{1}{n} e \to 0$, there is n_0 such that $\frac{1}{n} e \in V$ for all $n \geq n_0$. This implies

$$\frac{1}{n} e - d(\sigma_n, \sigma) \in C \bigcap (V - C)$$

for all $n \geq n_0$. Hence $\frac{1}{n} e - d(\sigma_n, \sigma) \in U$ for all $n \geq n_0$. Thus,

$\frac{1}{n}e - d(\sigma_n, \sigma) \to 0$ as $n \to \infty$. It follows that $d(\sigma_n, \sigma) \to 0$ as $n \to \infty$. This implies $\sigma_n \to \sigma$ as $n \to \infty$. By hypothesis, we have $\sigma_n \in A$, which is a contradiction with $\sigma \in H \setminus A$.

Now, we prove Cantor's theorem in a TVS-cone metric space with the cone C having neighborhood properties.

Theorem 3.7. *Let* (H, d) *be a complete TVS-cone metric space and* C *has neighborhood properties. If* $\{\overline{B}_n(\sigma_n, e_n)\}$ *is any decreasing sequence of closed ball with* $e_n \to 0$ *as* $n \to \infty$, *then* $\bigcap_{n=1}^{\infty} \overline{B}_n(\sigma_n, e_n)$ *contains only one point.*

Proof. Since $\overline{B}_m(\sigma_m, e_m) \subset \overline{B}_n(\sigma_n, e_n)$ for all $m \geq n$, we have

$$\sigma_m \in \overline{B}_n(\sigma_n, e_n) \text{ for all } m \geq n.$$

This implies

$$d(\sigma_n, \sigma_m) \preceq e_n \text{ for all } m \geq n.$$

Hence $e_n - d(\sigma_n, \sigma_m) \in C$ for all $m \geq n$. Let U be a neighborhood of the origin in E. Since C has neighborhood properties, there exists a neighborhood V of origin in E such that

$$C \cap (V - C) \subset U.$$

Since $e_n \to 0$ as $n \to \infty$, there is n_0 such that $e_n \in V$ for all $n \geq n_0$. This implies

$$e_n - d(\sigma_n, \sigma_m) \in C \cap (V - C) \text{ for all } m \geq n.$$

It follows that

$$e_n - d(\sigma_n, \sigma_m) \in U \text{ for all } m \geq n.$$

Thus, $e_n - d(\sigma_n, \sigma_m) \to 0$ as $m, n \to \infty$. Since $e_n \to 0$ as $n \to \infty$, we have $d(\sigma_n, \sigma_m) \to 0$ as $m, n \to \infty$. Hence $\{\sigma_n\}$ is a Cauchy sequence in H. By the completeness of H, there is $\sigma^* \in H$ such that $\sigma_n \to \sigma^*$ as $n \to \infty$. On the other hand, for $n \geq 1$, we have

$$\sigma_{n+k} \in \overline{B}_n(\sigma_n, e_n) \text{ for all } k \geq 1.$$

By closedness of $\overline{B}_n(\sigma_n, e_n)$, $\sigma^* \in \overline{B}_n(\sigma_n, e_n)$ for all $k \geq 1$.

Suppose by contradiction that $\tau^* \in H$ such that $\tau^* \in \bigcap_{n=1}^{\infty} \overline{B}_n(\sigma_n, e_n)$. Then for all $n \geq 1$, we have

$$d(\sigma^*, \tau^*) \preceq d(\sigma^*, \sigma_n) + d(\sigma_n, \tau^*) \preceq 2e_n.$$

This implies $2e_n - d(\sigma^*, \tau^*) \in C$ for all $n \geq 1$. Hence $-d(\sigma^*, \tau^*) \in C$. Therefore, $\sigma^* = \tau^*$.

Corollary 3.8 (Baire category theorem). *Let (H, d) be a complete TVS-cone metric space and C has neighborhood properties. Then X cannot be written as a countable union of no-where dense sets.*

Proof. If possible, assume that

$$H = \bigcup_{n=1}^{\infty} H_n = \bigcup_{n=1}^{\infty} \overline{H}_n,$$

where $\operatorname{int} \overline{H}_n = \varnothing$ for all $n \geq 1$. Let $\sigma \in H$ and $e \in E$, $0 \prec e$. Since $\operatorname{int} \overline{H}_1 = \varnothing$, there exists $\sigma_1 \in B(\sigma, e) \backslash \overline{H}_1$. This implies that there is $e_1 \in E$, $0 \prec e_1$ such that $B_1(\sigma_1, e_1) \subset B(\sigma, e) \backslash \overline{H}_1$. We can choose e_1 such that $e_1 \prec \frac{e}{2}$ and $B_1(\sigma_1, e_1) \cap \overline{H}_1 = \varnothing$. Moreover, since $\operatorname{int} \overline{H}_2 = \varnothing$, $\sigma_2 \in B_1(\sigma_1, e_1) \backslash \overline{H}_2$. Hence, there is $e_2 \in E$, $0 \prec e_2 \prec \frac{e_1}{2}$ such that

$B_2(\sigma_2, e_2) \cap \overline{H}_2 = \varnothing$. Continuing in this way, we obtain a sequence of closed balls $\{\overline{B}_n(\sigma_n, e_n)\}$ such that

(i) $\overline{B}_{n+1}(\sigma_{n+1}, e_{n+1}) \subset \overline{B}_n(\sigma_n, e_n)$ for all $n \geq 1$.

(ii) $0 \prec e_n \prec \dfrac{e}{2^n}$ for all $n \geq 1$.

Since $\dfrac{e}{2^n} \to 0$ as $n \to \infty$, $e_n \to 0$ as $n \to \infty$. By Theorem 3.7, there exists unique $\tau^* \in H$ such that $\tau^* \in \overline{B}_n(\sigma_n, e_n)$ for all $n \geq 1$. Since $\overline{B}_n(\sigma_n, e_n) \cap \overline{H}_n = \varnothing$ for all $n \geq 1$, $\sigma^* \notin \overline{H}_n$ for all $n \geq 1$. Hence $\sigma^* \notin \bigcup_{n=1}^{\infty} \overline{H}_n = H$. This is a contradiction.

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