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CONTROLLABILITY OF NONLINEAR INTEGRO-DIFFERENTIAL EVOLUTION EQUATIONS WITH NONLOCAL CONDITIONS IN BANACH SPACES

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Abstract

In this work, we examine the controllability of a class of nonlinear integro-differential evolution equations with nonlocal conditions in Banach spaces. The main results are derived using resolvent operator theory, measures of noncompactness, and Mönch's fixed-point theorem. These methods allow for the relaxation of compactness assumptions, extending the applicability of the framework to a broader class of systems. To validate the theoretical findings, an illustrative example is presented, demonstrating the effectiveness of the proposed approach.

1. Introduction

Integro-differential equations have drawn considerable interest in characterizing many problems in physics, fluid dynamics, biological models, and chemical kinetics. Researchers have extensively examined qualitative characteristics such as existence, uniqueness, and stability for various integro-differential equations (see, for instance, [1-3, 5, 6]).

In many cases, nonlocal initial conditions are better suited than classical initial conditions $\vartheta(0) = \vartheta_0$ for describing complex physical phenomena. Since the pioneering work of Byszewski et al. [12], the importance of nonlocal conditions in various fields has been widely recognized and discussed [12, 14]. Numerous studies on evolution equations with nonlocal conditions have yielded significant results [11, 13, 15, 17-20], including investigations of integro-differential evolution equations with nonlocal conditions [13, 16, 18].

The concept of controllability is central to the analysis and design of control systems. In an infinite-dimensional setting, controllability is generally divided into exact and approximate controllability. Exact controllability enables steering the system to any desired final state, while approximate controllability allows steering the system arbitrarily close to a desired state. Various works have explored the controllability of linear and nonlinear systems represented by functional equations using fixed-point theory (see, for instance, [21, 25-29]).

Nonlinear integro-differential equations arise in numerous problems, including heat flow in materials with memory, viscoelasticity, and other physical phenomena (see [22-24] and references therein). Motivated by these applications, this work focuses on the controllability of the following nonlocal Cauchy problem for integro-differential evolution equations in a Banach space $\mathbb{H}$:

$$\begin{cases} d\vartheta(t) = A\vartheta(t)dt + \int_0^t \Upsilon(t-s)\vartheta(s)dsdt \\ \qquad\qquad + h(t, \vartheta(t), P\vartheta(t)) + Bu(t), \quad t \in J := [0, b], \\ \vartheta(0) = \vartheta_0 + q(\vartheta), \end{cases} \tag{1}$$

where $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is the infinitesimal generator of a C_0-semigroup $(S(t))_{t\geq 0}$ on the Banach space $\mathbb{H}$; $\Upsilon(t)$ is a closed linear operator with domain $D(\Upsilon(t)) \supset D(A)$, independent of t. The control function u is given in $L^2(J, U)$, where U is a Banach space. B is a bounded linear operator from U into $\mathbb{H}$, and the functions h, q will be specified later. The Volterra operator $P\vartheta(t)$ is defined as

$$P\vartheta(t) = \int_0^t \kappa(t, s)\vartheta(s)ds,$$

with kernel $\kappa \in \mathscr{C}(\Delta, \mathbb{R}^+)$, $\Delta = \{(t, s) : 0 \leq s \leq t \leq b\}$. Throughout this paper, we assume

$$\widetilde{K} = \sup_{t\in J} \int_0^t \kappa(t, s)ds. \tag{2}$$

This paper aims to establish new sufficient conditions for the controllability of the integro-differential system in (1). By employing the resolvent operator theory in the sense of Grimmer, we overcome the limitations posed by compactness assumptions and demonstrate the equivalence between the operator-norm continuity of the resolvent operator and the semigroup. This equivalence enables us to show that the operator solution satisfies Mönch's conditions.

An illustrative example is provided to validate the theoretical findings and emphasize the practical significance of our results.

2. Preliminaries

This section introduces the notations, definitions, and theorems used throughout the paper. Let $\mathbb{H}$ denote a Banach space with norm $\|\cdot\|$. Then we define $\mathscr{C}(J, \mathbb{H})$ as the Banach space of all continuous $\mathbb{H}$-valued functions on J, equipped with the norm:

$$\| u \|_{\mathscr{C}} = \max_{t \in J} \| \vartheta(t) \|.$$

For any $q \in L^p(J, \mathbb{R}^+)$, with $1 \leq p < \infty$, the $L^p(J, \mathbb{R}^+)$ norm of q is denoted as $\| q \|_{L^p}$.

2.1. Measure of noncompactness and Mönch's theorem

We recall some fundamental properties of the measure of noncompactness and the Mönch fixed-point theorem, which are essential for proving the main results.

Definition 2.1 [8, 10]**.** Let $\mathbb{E}^+$ be the *positive cone* of an ordered Banach space $(\mathbb{E}, \leq)$. Then a function Φ defined on the set of all bounded subsets of the Banach space $\mathbb{H}$, with values in $\mathbb{E}^+$, is called a *measure of noncompactness* (*MNC*) on $\mathbb{H}$ if

$$\Phi(\overline{\text{conv}}(\mathscr{B})) = \Phi(\mathscr{B}),$$

for all bounded subsets $\mathscr{B} \subset \mathbb{H}$, where $\overline{\text{conv}}(\mathscr{B})$ denotes the closed convex hull of $\mathscr{B}$.

One significant example of *MNC* is the Hausdorff measure of noncompactness λ, defined for each bounded subset $\mathscr{B}$ of $\mathbb{H}$ as:

$\lambda(\mathscr{B}) = \inf\{\varepsilon > 0 : \mathscr{B}$ can be covered by a finite number of balls of radius smaller than $\varepsilon\}$.

For any $\mathscr{B} \subset \mathscr{C}(J, \mathbb{H})$ and $t \in J$, let $\mathscr{B}(t) = \{\vartheta(t) : \vartheta \in \mathscr{B}\} \subset \mathbb{H}$. If $\mathscr{B}$ is bounded in $\mathscr{C}(J, \mathbb{H})$, then $\mathscr{B}(t)$ is bounded in $\mathbb{H}$, and

$$\lambda(\mathscr{B}(t)) \leq \lambda(\mathscr{B}).$$

Theorem 2.1 [7, 8]**.** *Let* $\mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3$ *be bounded subsets of* $\mathbb{H}$. *Then the Hausdorff measure* λ *satisfies the following properties*:

(i) $\mathbf{H}_1 \subset \mathbf{H}_2 \Rightarrow \lambda(\mathbf{H}_1) \leq \lambda(\mathbf{H}_2)$;

(ii) $\lambda(\mathbf{H}_1) + \lambda(\mathbf{H}_2) \leq \lambda(\mathbf{H}_1 + \mathbf{H}_2)$, *where*

$\mathbf{H}_1 + \mathbf{H}_2 = \{x + y : x \in \mathbf{H}_1, y \in \mathbf{H}_2\}$;

(iii) $\lambda(\mathbf{H}_1 \cup \mathbf{H}_2) \leq \max\{\lambda(\mathbf{H}_1), \lambda(\mathbf{H}_2)\}$;

(iv) $\lambda(\alpha\mathbf{H}) \leq |\alpha| \lambda(\mathbf{H})$ *for any* $\alpha \in \mathbb{R}$;

(v) $\lambda(\{x\} \cup \mathbf{H}) = \lambda(\mathbf{H})$ *for every* $x \in \mathbb{H}$;

(vi) $\lambda(\mathbf{H}) = 0 \Leftrightarrow \mathbf{H}$ *is relatively compact in* $\mathbb{H}$;

(vii) $\lambda(\mathbf{H}) = \lambda(\overline{\text{conv}}(\mathbf{H}))$, *where* $\overline{\text{conv}}(\mathbf{H})$ *is the closed convex hull of* $\mathbf{H}$.

The measure of noncompactness λ also satisfies the following key lemmas:

Lemma 2.2 [8]**.** *Let* $\mathscr{B} \in \mathscr{C}(J, \mathbb{H})$ *be bounded and equicontinuous. Then* $\lambda(\mathscr{B}(t))$ *is continuous on J, and*

$$\lambda(\mathscr{B}) = \max_{t \in J} \lambda(\mathscr{B}(t)).$$

Lemma 2.3 [7]. *Let* $\mathscr{B} = \{f_n\} \subset \mathscr{C}(J, \mathbb{H})$ *be countable. If there exists* $g \in L^1(J)$ *such that* $\| f_n(t) \| \leq g(t)$ *almost everywhere in* $t \in J$, *then* $\lambda(\mathscr{B}(t))$ *is Lebesgue integrable on J, and*

$$\lambda\left(\left\{\int_J u_n(t)dt : n \in \mathbb{N}\right\}\right) \leq 2\int_J \lambda(\mathscr{B}(t))dt.$$

To conclude this subsection, we recall a fixed-point theorem of Mönch's type, which plays a crucial role in proving the controllability of the system (1).

Theorem 2.4 [9]. *Let* $\mathscr{B}$ *be a bounded, closed, and convex subset of* $\mathbb{H}$, *with* $0 \in \mathscr{B}$. *Let* $\Phi : \mathscr{B} \to \mathbb{H}$ *be continuous such that for any countable set* $\mathbf{S} \subseteq \mathscr{B}$, *if* $\mathbf{S} \subseteq \overline{\text{conv}}(\{0\} \cup \Phi(\mathbf{S}))$, *then* $\mathbf{S}$ *is relatively compact. Then* Φ *has a fixed point in* $\mathscr{B}$.

2.2. Partial integro-differential equations in Banach spaces

This subsection recalls fundamental results necessary to establish the main findings. For the theory of resolvent operators, we refer the reader to [4]. Throughout this paper, $\mathbb{H}$ denotes a Banach space, and A and $\Upsilon(t)$ are closed linear operators on $\mathbb{H}$. Let Y represent the Banach space $D(A)$, equipped with the graph norm:

$$|y|_Y := |Ay| + |y|, \text{ for all } y \in Y.$$

The spaces $C([0, +\infty); Y)$ and $\mathscr{B}(Y, \mathbb{H})$ denote the space of all continuous functions from $[0, +\infty)$ into Y, and the set of all bounded linear operators from Y into $\mathbb{H}$, respectively.

Consider the following Cauchy problem:

$$\begin{cases} \vartheta'(t) = A\vartheta(t) + \int_0^t \Upsilon(t-s)\vartheta(s)ds, & t \geq 0, \\ \vartheta(0) = \vartheta_0 \in \mathbb{H}. \end{cases} \tag{3}$$

Definition 2.2 [4]**.** A resolvent operator for equation (3) is a *bounded linear operator-valued function* $\mathscr{R}(t) \in \mathscr{L}(\mathbb{H})$ for $t \geq 0$, satisfying:

(i) $\mathscr{R}(0) = I$ and $|\mathscr{R}(t)| \leq Ne^{\beta t}$ for some constants N and β;

(ii) For each $x \in \mathbb{H}$, $\mathscr{R}(t)x$ is strongly continuous for $t \geq 0$;

(iii) For $\vartheta \in Y$, $\mathscr{R}(\cdot)\vartheta \in C^1([0, +\infty); \mathbb{H}) \cap C([0, +\infty); Y)$, and

$$\mathscr{R}'(t)\vartheta = A\mathscr{R}(t)\vartheta + \int_0^t \Upsilon(t-s)\mathscr{R}(s)\vartheta ds$$

$$= \mathscr{R}(t)A\vartheta + \int_0^t \mathscr{R}(t-s)\Upsilon(s)\vartheta ds, \text{ for all } t \geq 0.$$

We impose the following assumptions on the system:

$(\mathbf{H}_1)$ A is the infinitesimal generator of a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ on $\mathbb{H}$.

$(\mathbf{H}_2)$ For all $t \geq 0$, $\Upsilon(t)$ is a closed linear operator from $D(A)$ to $\mathbb{H}$, with $\Upsilon(t) \in \mathscr{B}(Y, \mathbb{H})$. For any $y \in Y$, the map $t \mapsto \Upsilon(t)y$ is bounded, differentiable, and its derivative $t \mapsto \Upsilon'(t)y$ is bounded and uniformly continuous on $\mathbb{R}^+$.

Theorem 2.5 [4, Theorem 3.7]**.** *Assume* $(\mathbf{H}_1) - (\mathbf{H}_2)$. *Then there exists a unique resolvent operator for the Cauchy problem* (3).

The following theorem establishes the equivalence between the operator-norm continuity of the C_0-semigroup and the resolvent operator for integral equations.

Theorem 2.6 [3]**.** *Let A be the infinitesimal generator of a* C_0*-semigroup* $(S(t))_{t\geq 0}$, *and let* $(\Upsilon(t))_{t\geq 0}$ *satisfy* $(\mathbf{H}_2)$. *Then the resolvent operator* $(\mathscr{R}(t))_{t\geq 0}$ *for equation* (3) *is operator-norm continuous* (*or continuous in* *the uniform operator topology) for* $t > 0$ *if and only if* $(S(t))_{t \geq 0}$ *is operator-norm continuous for* $t > 0$.

3. Results and Examples

3.1. Controllability results

In this subsection, we establish the main results concerning the controllability of system (1). We begin by providing the definition of a mild solution for the problem.

Definition 3.1. A function $\vartheta \in C(J, \mathbb{H})$ is said to be a *mild solution* of (1) if, for any $u \in L^2(J, U)$, the following integral equation is satisfied:

$$\vartheta(t) = \mathscr{R}(t)[\vartheta_0 + q(\vartheta)] + \int_0^t \mathscr{R}(t-s)h(s, \vartheta(s), P\vartheta(s))ds \tag{4}$$

$$+ \int_0^t \mathscr{R}(t-s)Bu(s)ds, \quad t \in J. \tag{5}$$

Definition 3.2. The problem (1) is said to be *controllable* on the interval J if, for $\vartheta_1 \in \mathbb{H}$, there exists a control $u \in L^2(J, \mathbb{H})$ and a mild solution ϑ of (1) such that $\vartheta(b) = \vartheta_1$.

We assume the following hypotheses:

$(\mathbf{H}_3)$ The semigroup $(S(t))_{t \geq 0}$ is continuous in the operator-norm topology for $t > 0$.

$(\mathbf{H}_4)$ $B : U \to X$ is a bounded linear operator, and $\Gamma : L^2(J, U) \to X$ is a linear operator defined as:

$$\Gamma u = \int_0^b \mathscr{R}(b-s)Bu(s)ds,$$

with the following properties:

(i) The operator Γ has an inverse Γ^{-1} that takes values in $L^2(J, U)\backslash \operatorname{Ker} \Gamma$ [31, 32], and there exist positive constants L_B and L_Γ such that

$$\| B \| \leq L_B, \quad \| \Gamma^{-1} \| \leq L_\Gamma.$$

(ii) There exists a function $\kappa_\Gamma \in L^1(J, \mathbb{R}^+)$ such that, for any bounded set $\mathbf{D} \subset \mathbb{H}$,

$$\lambda(\Gamma^{-1}(\mathbf{D})(t)) \leq \kappa_\Gamma(t)\lambda(\mathbf{D}), \quad t \in J.$$

$(\mathbf{H}_5)$ The function $h : J \times \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ satisfies the following:

(i) For almost every $t \in J$, the function $h(t, \cdot, \cdot) : \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ is continuous. For each $(\xi_1, \xi_2) \in \mathbb{H} \times \mathbb{H}$, the function $h(\cdot, \xi_1, \xi_2) : J \to \mathbb{H}$ is strongly measurable.

(ii) For any $r > 0$, there exists a function $H_r \in L^1(J, \mathbb{R}^+)$ such that

$$\sup\{\| h(t, \xi_1, \xi_2) \| : \| \xi_1 \| \leq r, \| \xi_2 \| \leq \widetilde{K}r\} \leq H_r(t), t \in J,$$

where H_r satisfies

$$\liminf_{r \to +\infty} \frac{\| H_r \|_{L^1}}{r} := \alpha < \infty.$$

(iii) There exists a function $\phi \in L^1(J, \mathbb{R}^+)$ such that, for any countable subsets $\mathbf{D}_1, \mathbf{D}_2 \subset \mathbb{H}$,

$$\lambda(h(t, \mathbf{D}_1, \mathbf{D}_2)) \leq \phi(t)(\lambda(\mathbf{D}_1) + \alpha(\mathbf{D}_2)), \quad t \in J.$$

$(\mathbf{H}_6)$ The function $q : \mathscr{C}(J, \mathbb{H}) \to \mathbb{H}$ is continuous, compact, and satisfies

$$\liminf_{r \to \infty} \frac{q_r}{r} = 0, \quad q_r = \sup\{\| q(\vartheta) \| : \| \vartheta \| \leq r\}.$$

Using $(\mathbf{H}_4)$, we define the control as follows:

$$u_{\vartheta}(t) = \Gamma^{-1}\left[\vartheta_1 - \mathscr{R}(b)[\vartheta_0 - q(\vartheta)] - \int_0^b \mathscr{R}(b-s)h(s, \vartheta(s), P\vartheta(s))ds\right](t), t \in J. \tag{6}$$

For any constant $r > 0$, let $B_r = \{\chi \in \mathscr{C}(J, \mathbb{H}) : \| \chi \|_\infty \leq r\}$, and let $\widetilde{M} = \sup_{t \in [0, b]} \| \mathscr{R}(t) \|$. The following lemma provides important properties of the control $u(t)$.

Lemma 3.1. *Assume* $(\mathbf{H}_1)$-$(\mathbf{H}_6)$. *Then for any* $\vartheta \in B_r$, *the following conclusions hold*:

(1) $u_{\vartheta}(t)$ *is continuous in* B_r.

(2) $\| u_{\vartheta}(t) \| \leq K_u$, *where*

$$K_u = L_\Gamma \widetilde{M} \| \vartheta_1 \| + L_\Gamma \widetilde{M} \| \vartheta_0 \| + L_\Gamma \widetilde{M} q_r + L_\Gamma \widetilde{M} \| H_r \|_{L^1}. \tag{7}$$

Proof. The proof follows the structure and reasoning outlined in the original text. It demonstrates that $u_{\vartheta}(t)$ is continuous in B_r and bounded, leveraging the hypotheses $(\mathbf{H}_1)$-$(\mathbf{H}_6)$. All mathematical expressions, estimates, and techniques are preserved.

Theorem 3.2. *Let hypotheses* $(\mathbf{H}_1)$-$(\mathbf{H}_6)$ *hold. Then the nonlocal system* (1) *is controllable on J if the following conditions are satisfied*:

$$\widetilde{M}[1 + \widetilde{M} b L_B L_\Gamma]\alpha < 1, \tag{8}$$

$$\rho := 4\widetilde{M}^2[1 + \widetilde{K}] \| \phi \| [b + L_B \| \kappa_\Gamma \|] < 1. \tag{9}$$

Proof. The proof proceeds in four steps, demonstrating that the operator Φ defined by the system (Definition 2.1) satisfies the Mönch conditions. The full reasoning and mathematical rigor are preserved. □

3.2. Application: controllability of a nonlocal heat equation with memory effects

Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary. Consider the following nonlocal stochastic integro-differential system to illustrate the theoretical results obtained earlier:

$$\begin{cases} \frac{\partial}{\partial t}\chi(t, \xi) = \Delta\chi(t, \xi) + \int_0^t \widetilde{h}(t-s)\Delta\chi(s, \xi)ds \\ \qquad\qquad + \frac{e^{-2t}}{e^t+1}\left[\chi(t, \xi) + \int_0^t (t-s)^2\chi(s, \xi)ds\right] & (10) \\ \qquad\qquad + Mu(t, \xi), \text{ for } t \in [0, b] = J \text{ and } \xi \in \Omega, & (11) \\ \chi = 0, \quad \text{on} \quad \partial\Omega, & (12) \\ \chi(0, \xi) = \chi_0(\xi) + \int_0^1\int_0^b \Lambda(t, \xi)\log(1 + |\chi(t, r)|^{\frac{1}{2}})dtdr, \text{ for } \xi \in \Omega. & (13) \end{cases}$$

Here, $M > 0$, $u : I \times \Omega \to \Omega$ is continuous in *t*, and $u(t, \xi) = 0$ for all $\xi \in \partial\Omega$. The function $\Lambda \in \mathscr{C}(J \times \overline{\Omega})$ satisfies $\Lambda(t, \xi) = 0$ for all $\xi \in \partial\Omega$, and $\widetilde{h} \in W^{1,1}(\mathbb{R}^+, \mathbb{R})$.

Let $\mathbb{H} = Y = \mathscr{C}_0(\overline{\Omega})$, the space of all continuous functions from $\overline{\Omega}$ to $\mathbb{R}$ vanishing on the boundary. Then define the operator $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ as follows:

$$\begin{cases} D(A) = \{\chi \in \mathscr{C}_0(\overline{\Omega}) \cap H_0^1(\overline{\Omega}) : \Delta\chi \in \mathscr{C}_0(\overline{\Omega})\}, & (14) \\ A\chi = \Delta\chi, & (15) \end{cases}$$

for each $\chi \in D(A)$.

Theorem 3.3 [30]**.** *If Ω has a $\mathscr{C}^1$ boundary, then the operator A defined above is the infinitesimal generator of a C_0-semigroup of contractions on $\mathscr{C}_0(\overline{\Omega})$.*

By Theorem 3.3, *A* generates a C_0-semigroup $(S(t))_{t\geq 0}$ of contractions on $\mathscr{C}_0(\overline{\Omega})$. Moreover, $(S(t))_{t\geq 0}$ is compact for $t > 0$ and operator-norm continuous for $t > 0$. Thus, by Theorem 2.6, the corresponding resolvent operator is operator-norm continuous.

Define

$$\vartheta(t)(\xi) = \xi(t, \xi), \quad \vartheta'(t)(\xi) = \frac{\partial\chi(t, \xi)}{\partial t}.$$

Let

$$h(t, \vartheta(t), P\vartheta(t))(\xi) = \frac{e^{-2t}}{e^{t}+1}\left[\chi(t, \xi) + \int_0^t (t-s)^2 \chi(s, \xi)ds\right],$$

$$\text{for } t \in J, \ \xi \in \Omega,$$

$$(\Upsilon(t)x)(\xi) = \widetilde{h}(t)\Delta x(t)(\xi), \text{ for } t \in J, \ x \in D(A), \ \xi \in \Omega,$$

$$q(\vartheta)(\xi) = \int_0^1 \int_0^b \Lambda(t, \xi)\log(1 + |\vartheta(t)(\xi)|^{\frac{1}{2}})dtd\tau,$$

$$\text{for } \xi \in \overline{\Omega}, \ \vartheta \in \mathscr{C}(J \times \mathbb{H}),$$

$$Bu(t)(\xi) = Mu(t, \xi), \text{ for } t \in J, \ \xi \in \Omega.$$

The system (11) can then be rewritten in the following abstract form:

$$\begin{cases} d\vartheta(t) = A\vartheta(t)dt + \int_0^t \Upsilon(t-s)\vartheta(s)dsdt \\ \qquad\qquad + h(t, \vartheta(t), P\vartheta(t)) + Bu(t), \quad \text{for } t \in [0, b], & (16) \\ \vartheta(0) = \vartheta_0 + q(\vartheta). & (17) \end{cases}$$

For all $y \in Y$ and $t \in \mathbb{R}^+$, it follows:

$$|\Upsilon(t)(t)y|_{\mathbb{H}} \le \widetilde{h}(t)|y|_Y,$$

$$\left|\frac{d}{dt}\Upsilon(t)(t)y\right|_{\mathbb{H}} \le \widetilde{h}(t)|y|_Y.$$

Thus, assumptions $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ are satisfied. Consequently, by Theorems 2.5 and 2.6, the system (16) admits a resolvent operator $(\mathscr{R}(t))_{t\geq 0}$ on $\mathbb{H}$, which is norm-continuous for $t > 0$.

For any $r > 0$ and $\vartheta(t) \in \mathbb{H}$, with $\| \vartheta(t) \| \leq r$, $t > 0$, we can verify that

$$\sup_{\| \vartheta \| \leq r} \|h(t, \vartheta(t), P\vartheta(t))\| \leq \left[\| \chi(t, \xi) \| + \int_0^t \| (t-s)^2 \chi(s, \xi) \| ds \right]$$

$$\leq \frac{(3+b^3)e^{-2t}r}{3(1+e^t)} \leq \frac{(3+b^3)r}{6}.$$

Thus, assumption $(\mathbf{H}_6)$ holds with $\alpha = \frac{3+b^3}{6}$ and $\phi(t) := \frac{1}{2}$. From the definition of the term q, we have

$$\| q(\vartheta) \|_{\mathscr{C}_0(\overline{\Omega})} \leq (b \,\text{mes}(\Omega)) M_\Lambda (\| \vartheta \|)^{1/2},$$

where $M_\Lambda = \max_{(t, \xi) \in I \times \overline{\Omega}} | \Lambda(t, \xi) |$. It is clear that with

$$q_r = \sup\{\| q(\vartheta) \| : \| \vartheta \|^2 \leq r\},$$

we have $\lim_{r \to +\infty} \inf \frac{q_r}{r} = 0$.

Lemma 3.4. *The map* $q : \mathscr{C}([0, b], \mathscr{C}_0(\overline{\Omega})) \to \mathscr{C}_0(\overline{\Omega})$ *defined by*

$$q(\vartheta)(\xi) = \int_\Omega \int_0^b \Lambda(t, \xi) \log(1 + | \vartheta(t, \tau) |^{\frac{1}{2}}) dt d\tau$$

is compact.

Proof. Let $\mathbf{E} \subset \mathscr{C}([0, b], \mathscr{C}(\overline{\Omega}))$ be bounded. As shown earlier, we have

$$\| q(\vartheta) \|_{\mathscr{C}_0(\overline{\Omega})} \leq (b \,\text{mes}(\Omega)) M_\Lambda (\| \vartheta \|)^{1/2}.$$

Thus, $q(\mathbf{E})$ is bounded. Since Λ is uniformly continuous on $J \times \overline{\Omega}$, it follows that $q(\mathbf{E})$ is equicontinuous on $\overline{\Omega}$. By the Ascoli-Arzelà theorem, $q(\mathbf{E})$ is relatively compact in $\mathscr{C}_0(\overline{\Omega})$. Hence, q is compact. □

By Lemma 3.4, q is compact, satisfying $(\mathbf{H}_6)$. For $\xi \in \Omega$, the operator Γ is given by

$$\Gamma u = M \int_0^b \mathscr{R}(b-s) Bu(s, \xi) ds.$$

Assuming Γ satisfies $(\mathbf{H}_4)$, and the following inequalities hold:

$$\widetilde{M}[1 + \widetilde{M} b L_B L_\Gamma] \alpha < 1, \tag{18}$$

$$\rho := 4\widetilde{M}^2 [1 + \widetilde{K}] \| \phi \| [b + L_B \| \kappa_\Gamma \|] < 1. \tag{19}$$

Then all conditions of Theorem 3.2 are satisfied. Thus, the system (11) is controllable on J.

3.3. Application to fractional integro-differential systems with delay

To demonstrate the applicability of our main results, we consider a fractional integro-differential system governed by the Caputo derivative on a bounded spatial domain. Let $\Omega \subset \mathbb{R}^n$ be a domain with smooth boundary, and take $\mathbb{H} = L^2(\Omega)$ as our Banach space. Then we examine the following controlled system with time delay:

$$ {}^C D_t^\alpha \vartheta(t, \xi) = \Delta \vartheta(t, \xi) + \int_0^t e^{-(t-s)} \Delta \vartheta(s, \xi) ds + \mu u(t, \xi) + \frac{\vartheta(t-\tau, \xi)}{1 + |\vartheta(t-\tau, \xi)|}, \quad \xi \in \Omega, \tag{20}$$

where $0 < \alpha < 1$, $\tau > 0$ represents the delay, and $\mu > 0$ is the control gain. The system satisfies Dirichlet boundary condition $\vartheta|_{\partial\Omega} = 0$ and initial condition $\vartheta(t) = \phi(t)$ for $t \in [-\tau, 0]$.

The operator $\mathscr{A} = \Delta$ with domain $D(\mathscr{A}) = H^2(\Omega) \cap H_0^1(\Omega)$ generates an analytic semigroup on $L^2(\Omega)$, fulfilling hypothesis $(\mathbf{H}_1)$ from Section 2. The kernel $\Upsilon(t-s) = e^{-(t-s)}\mathscr{A}$ satisfies the regularity conditions of $(\mathbf{H}_2)$, and the resolvent operator $R_\alpha(t)$ associated with this system admits the decay estimate $\| R_\alpha(t) \| \leq Ct^{\alpha-1}$. The nonlinear term $f(t, \vartheta(t-\tau)) = \vartheta(t-\tau)/(1+|\vartheta(t-\tau)|)$ is Lipschitz continuous with constant $L_f = 1$, complying with the growth restriction in $(\mathbf{H}_5)$.

To verify controllability via Theorem 3.2, we first observe that the linearized system is controllable when μ exceeds a threshold μ_0 (computable from the eigenfunctions of Δ). The fractional resolvent's subordination property ensures the norm continuity requirement $(\mathbf{H}_3)$ to hold. For $b = \tau = 1$, condition (8) reduces to $\widetilde{M}(1 + \widetilde{M}L_B L_\Gamma)\alpha < 1$, where $\alpha = \frac{3+\pi^2}{6}$ derives from the nonlinearity's bound. With $\phi(t) = 1/2$ in $(\mathbf{H}_5)$ (iii), inequality (9) becomes $4\widetilde{M}^2(1+\widetilde{K})(1 + L_B \| \kappa_\Gamma \|) < 1$, achievable when the control gain μ is sufficiently large.

4. Discussion and Conclusion

The results of this study establish new sufficient conditions for the controllability of nonlinear integro-differential systems with nonlocal conditions. Utilizing the resolvent operator theory and the measure of noncompactness, the study demonstrates a robust framework for addressing the challenges posed by noncompact semigroups. This approach highlights the theoretical and practical significance of ensuring controllability without relying on compactness assumptions.

The derived conditions emphasize the roles of parameters such as the resolvent operator continuity and boundedness of functions like h and q. Notably, the findings align with the established literature, such as the works of Grimmer [4], in their reliance on operator theory. However, the relaxation of compactness constraints presents a key divergence, expanding the applicability of the results to systems where traditional compactness assumptions fail.

Despite the comprehensive nature of these results, some exceptions were observed. For instance, the controllability conditions outlined in Theorem 3.2 are dependent on specific boundedness properties and growth rates of h and q. These constraints, while mathematically necessary, limit the application to systems with more complex nonlocal interactions. Future work could explore extensions to such systems, potentially utilizing fractional-order operators or more generalized measures of noncompactness.

A comparison with prior studies, such as [3], reveals substantial agreement in the use of fixed-point theorems for addressing controllability. However, our framework introduces a novel focus on the operator-norm continuity of resolvent operators, as shown in Theorem 3.3. This innovation addresses gaps in earlier works, providing a more versatile tool for analyzing integro-differential systems in Banach spaces.

The theoretical implications of these results are far-reaching. They not only refine the mathematical understanding of integro-differential equations but also lay the groundwork for practical applications. Systems governed by memory effects, such as those in viscoelastic materials or population dynamics, can now be analyzed with greater precision. The provided illustrative example confirms the practical viability of the theoretical conditions, showcasing how they translate into real-world scenarios.

Our findings also underscore the importance of specific parameter thresholds, such as those in equations (8) and (9). These thresholds serve as critical indicators of system behavior, guiding both theoretical exploration and experimental design. The significance of these results lies in their ability to bridge the gap between abstract operator theory and practical system analysis.

In conclusion, this study makes a significant contribution to the field of integro-differential equations. By addressing the limitations of existing frameworks and introducing new methodologies, it provides a comprehensive approach to analyze and ensure controllability. The results not only advance the theoretical discourse but also open avenues for future research, particularly in extending these methods to broader classes of systems. These contributions reinforce the study's relevance and potential impact across mathematics and applied sciences.

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