Far East Journal of Mathematical Sciences (FJMS)
© 2026 Pushpa Publishing House, Prayagraj, India
https://pphmjopenaccess.com
https://doi.org/10.17654/0972087126026 **P-ISSN: 0972-0871**
Volume 143, Number 2, February 2026, Pages 433-465 **E-ISSN: 2584-1246**

FIXED POINT THEOREM FOR A GENERALIZED $\mathcal{H}$-$\mathcal{F}$-CONTRACTIVE MAPPING IN COMPLETE FUZZY METRIC SPACES

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Received: June 19, 2025; Revised: September 10, 2025; Accepted: October 6, 2025
2020 Mathematics Subject Classification: 30Lxx, 54Exx, 37Cxx, 46Sxx.
Keywords and phrases: fixed point, fuzzy metric space, generalized $\mathcal{H}$-$\mathcal{F}$-contractive mapping, generalized contraction.
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How to cite this article: J. Ravinder, A. Bernick Raj, A. Velankanni and C. D. Nandakumar, Fixed point theorem for a generalized $\mathcal{H}$-$\mathcal{F}$-contractive mapping in complete fuzzy metric spaces, Far East Journal of Mathematical Sciences (FJMS) 143(2) (2026), 433-465.
https://doi.org/10.17654/0972087126026

Published Online: December 12, 2025

Abstract

In this paper, we introduce a new class of mappings, called generalized $\mathcal{H}$-$\mathcal{F}$-contractive mappings, within the framework of complete fuzzy metric spaces. Our approach generalizes existing fixed point results and establishes conditions under which a unique fixed point exists. Some examples are provided to demonstrate the applicability of our theorem. This work contributes to the ongoing development of fixed point theory in fuzzy metric spaces, particularly in the context of non-classical contractions.

1. Introduction

Fixed point theory in fuzzy metric spaces has garnered significant attention because of its potential in modeling real-world problems involving uncertainty. Since the pioneering work of Kramosil and Michálek, numerous researchers have extended classical fixed point results to fuzzy settings. Several contractive conditions, including Banach, Kannan [29], Chatterjea [30], and Ciric types [31], have been adapted to fuzzy metric spaces. Recently, more generalized notions such as F-contraction, ψ-contraction, and simulation functions have been studied.

2. Preliminaries

We recall essential definitions related to fuzzy metric spaces and introduce the concepts required for our main result.

In 1906, Maurice Fréchet, in his doctoral dissertation, introduced the notation of the distance function as: $d : X \times X \to \mathbb{R}$ is a distance function between any two general objects x, y of a non-empty set X such that satisfy four postulates:

(i) $d(x, y) \geq 0$,

(ii) $d(x, y) = 0$ if and only if $x = y$,

(iii) $d(x, y) = d(y, x)$,

(iv) $d(x, z) \leq d(x, y) + d(y, z)$,

for all $x, y, z \in X$.

Later in the year 1914, Felix Hausdorff named it as Metric Space. After that many researchers generalized this concept and investigated the unique fixed point mappings in metric spaces. In this context, a very basic result is known as the Banach contraction principle, introduced by Banach [2] in 1922. Later, many researchers generalized this principle to different spaces.

In fact, classical theory is a pure concept oriented and it does not deal with the concepts where the uncertainty is involved. To overcome this, in 1965, Zadeh [28] introduced the idea of a fuzzy set. In 1975, Kramosil and Michálek [13] expanded upon the idea of probabilistic metric spaces, introducing and developing the concept of fuzzy metric space, which is also known as KM-fuzzy metric space.

In 1981, Heilpern [10] generalized the concept of contraction type mappings to the fuzzy sets defined on a complete metric linear space, called *contraction fuzzy mapping* and investigated the fixed point theorem for these mappings. This pioneering work has since inspired numerous researchers to explore various types of contraction and contractive type fuzzy mappings to establish a unique fixed point in various metric spaces [1, 3, 4, 14, 16].

On the other hand, Grabiec [8] initiated the study of fuzzy fixed point theorems in fuzzy metric spaces by formulating a fuzzy version of the Banach contraction principle. Subsequently, many researchers have extended this line of work by exploring contraction and contractive mappings in fuzzy metric spaces, thereby establishing conditions for the existence of unique fixed points [9, 11, 12, 15, 18, 20-27].

In the sequel, unless there is a special explanation, the set of natural numbers is denoted by $\mathbb{N}$, non-negative integers by $\mathbb{N}_0$, real numbers by $\mathbb{R}$, and positive real numbers by $\mathbb{R}^+ = (0, \infty)$. Moreover, $\mathcal{H}((0, 1])$ is a set of all strictly decreasing functions defined as $\eta : (0, 1] \to [0, 1]$ satisfies the conditions:

$(\eta - 1)$: for any $p, q \in (0, 1]$ with $p < q$, $\eta(p) > \eta(q)$ and $\eta(p) < p$.

$(\eta - 1)$: $\{\eta(p) : \forall p \in (0, 1]\}$ is bounded.

Further, $\mathcal{F}([0, 1])$ is a set of all strictly increasing functions defined as $F : [0, 1] \to [0, \infty)$ such that $F(p) < F(q)$ for all $p, q \in [0, 1]$ with $p < q$. Similarly, $\Phi(\mathbb{R}^+)$ is a set of all right continuous functions defined as $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(r) < r$ for all $r > 0$.

Definition 2.1 [19]**.** A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is said to be *continuous t-norm* (*triangular norm*) if $([0, 1], *)$ is an abelian topological monoid. That is, for $a, b, c, d \in [0, 1]$, the following conditions hold:

$(t\text{-}1)$: $(a * b) * c = a * (b * c)$ (i.e., $*$ is associative).

$(t\text{-}2)$: $a * b = b * a$ (i.e., $*$ is commutative).

$(t\text{-}3)$: $a * 1 = a$ for all $a \in [0, 1]$.

$(t\text{-}4)$: $a * b \leq c * d$, whenever $a \leq c$ and $b \leq d$.

$(t\text{-}5)$: $*$ is continuous.

For example, $a * b = \min\{a, b\}$, $a * b = \max\{a + b - 1, 0\}$, $a * b = ab$, $a * b = \dfrac{ab}{a + b - ab}$ and $a * b = \dfrac{ab}{\max\{a, b, \lambda\}}$ for some $\lambda \in (0, 1)$, etc. are continuous *t*-norms.

Definition 2.2 [13]**.** Let X be any nonempty arbitrary set, $*$ be the continuous *t*-norm and M be a fuzzy set on $X^2 \times [0, \infty)$. Then the 3-tuple $(X, M, *)$ is said to be a *fuzzy metric space*, if for each $x, y, z \in X$ and $t, s > 0$, (KM-1): $M(x, y, 0) = 0.$

(KM-2): $M(x, y, t) = 1$ for all $t > 0,$ if and only if $x = y.$

(KM-3): $M(x, y, t) = M(y, x, t).$

(KM-4): $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s).$

(KM-5): $M(x, y, \bullet) : [0, \infty) \to [0, 1]$ is left continuous.

Definition 2.3 [8]**.** A sequence $\{x_n\}$ in *X* is said to *converge* to *x* in *X*, if $\lim_{n \to \infty} M(x_n, x, t) = 1$ for each $t > 0,$ and $\{x_n\}$ is said to be *Cauchy* in *X* if $\lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1$ for each $t > 0$ and $p > 0.$

As the way of obtaining Hausdorff topology on these spaces, George and Veeramani [6] imposed some excessive conditions on the fuzzy metric and since then it was treated as a modified definition of fuzzy metric space (it is known as GV-Fuzzy Metric Space). In this paper, fuzzy metric space refers to the GV-Fuzzy Metric Space.

Definition 2.4 [6]**.** The 3-tuple $(X, M, *)$ is said to be a *fuzzy metric space*, if for each $x, y, z \in X$ and $t, s > 0,$

(GV-1): $M(x, y, t) > 0,$

(GV-2): $M(x, y, t) = 1$ for all $t > 0,$ if and only if $x = y,$

(GV-3): $M(x, y, t) = M(y, x, t),$

(GV-4): $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s),$

(GV-5): $M(x, y, \bullet) : (0, \infty) \to [0, 1]$ is continuous,

where *X* is an arbitrary nonempty set, "$*$" is a continuous *t*-norm and *M* is a fuzzy set on $X^2 \times (0, \infty).$

Lemma 2.5 [8]**.** *For all x, y in X,* $M(x, y, \bullet)$ *is non-decreasing.*

In addition, we recall some definitions and concepts for our use.

Definition 2.6 [6]**.** Let $(X, M, *)$ be a GV-fuzzy metric space.

(i) For every $t > 0,$ an *open ball* $B(x, r, t)$ centered at $x \in X$ with radius $r \in (0, 1)$ is defined as $B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$

(ii) A subset A of X is said to be *F-bounded* if for every $t > 0,$ there exists $r \in (0, 1)$ such that $M(x, y, t) > 1 - r$ for all $x, y \in A.$

(iii) A sequence $\{x_n\}$ converges to $x \in X,$ if for each $\delta \in (0, 1)$ and $t > 0,$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1 - \delta$ for all $n \geq n_0.$ Moreover, $\lim_{n \to \infty} M(x_n, x, t) = 1.$

(iv) A sequence $\{x_n\}$ is said to be *Cauchy* in X if and only if for each $\varepsilon > 0$ and $t > 0,$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $m, n \geq n_0.$

(v) A fuzzy metric space is complete iff every Cauchy sequence converges in it.

(vi) [9] A sequence $\{t_n\}$ is said to be an *s-increasing sequence* if there exists $n_0 \in \mathbb{N}$ such that $t_n + 1 \leq t_{n+1}$ for all $n \geq n_0.$

Theorem 2.7 [9]**.** *For every* $\varepsilon > 0$ *and an s-increasing sequence* $\{t_n\}$ *in a complete fuzzy metric space* $(X, M, *),$ *there exists* $N \in \mathbb{N}$ *such that*

$$\prod_{n \geq N} M(x, y, t_n) \geq 1 - \varepsilon.$$

Moreover, $\lim_{N \to \infty} \prod_{n=1}^{N} M(x, y, t_n) = 1.$

Definition 2.8. Let $(X, M, *)$ be a fuzzy metric space and $T : X \to X$ be a function. Then

(i) T is said to be *continuous*, if for any sequence $x_n \to x$ as $n \to \infty$ in X and given $\varepsilon \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $M(Tx_n, Tx, t) \geq 1 - \varepsilon$ for all $n \geq n_0$ and $t > 0$ [9].

(ii) T is said to be *t-uniformly continuous*, if for every $\varepsilon \in (0, 1)$, there exists $r \in (0, 1)$ such that $M(Tx, Ty, t) \geq 1 - \varepsilon$ whenever $M(x, y, t) \geq 1 - r$, for all $x, y \in X$ and $t > 0$ [9].

(iii) T is said to be α-*admissible*, if there exists a function $\alpha : X^2 \times \mathbb{R}^+ \to [0, \infty)$ such that, for all $x, y \in X$ and $t > 0$,

(a) $\alpha(x, y, t) \geq 1 \Rightarrow \alpha(Tx, Ty, t) \geq 1$,

(b) for any $z \in X$, $\alpha(x, z, t) \geq 1$ and $\alpha(y, z, t) \geq 1$ [7].

(iv) [7] T is said to be β-*admissible*, if there exists a function $\beta : X^2 \times \mathbb{R}^+ \to \mathbb{R}^+$ such that, for all $x, y \in X$ and $t > 0$,

(a) $\beta(x, y, t) \leq 1 \Rightarrow \beta(Tx, Ty, t) \leq 1$,

(b) for any $z \in X$, $\beta(x, z, t) \leq 1$ and $\beta(y, z, t) \leq 1$.

This work provided an important basis for the construction of fixed point theory in fuzzy metric spaces [8, 9, 21, 22, 25]. A number of fixed point theorems have been obtained by various authors in a fuzzy metric space by using the concept of contractive map, iteration contraction map, F-contraction map, ψ-contractive map, ε-contractive map [5, 9, 17, 18, 20], etc.

However, the existence of a fixed point in any fuzzy metric space X depends on a contraction map, which is a self mapping defined on the fuzzy metric space X. Since the introduction of the fuzzy fixed point theory, many contraction type mappings were defined on complete fuzzy metric spaces to establish a unique fixed point. The aim of this paper is to list the contraction principles and compare the multitude of definitions.

3. Definitions of Fuzzy Contraction and Contractive Mappings

Let X be a fuzzy metric space under the fuzzy metric M, the continuous t-norm "$*$" and T be a self mapping defined on X.

(3.1) Fuzzy Banach contraction [8]**.** T is said to be *fuzzy contraction*, if $M(Tx, Ty, kt) \geq M(x, y, t)$, where $0 < k < 1$, for all $x, y \in X$ and $t > 0$.

(3.2) Fuzzy contractive mapping [9]**.** T is said to be a *fuzzy contractive mapping*, if there exists $k \in (0, 1)$ such that

$$\frac{1}{M(Tx, Ty, t)} - 1 \leq k\left(\frac{1}{M(x, y, t)} - 1\right)$$

for all $x, y \in X$ and $t > 0$.

(3.3) Fuzzy ψ-contractive mapping [15]**.** T is said to be a *fuzzy ψ-contractive mapping*, if $M(Tx, Ty, t) \geq \psi(M(x, y, t))$ for all $x, y \in X$ and $t > 0$, where ψ is a non-decreasing continuous function defined as $\psi : [0, 1] \to [0, 1]$ such that $\psi(s) > s$ for every $s \in (0, 1)$.

(3.4) Fuzzy $\mathcal{H}$-contractive mapping [26]**.** T is said to be a *fuzzy $\mathcal{H}$-contractive*, if there exists $\eta \in \mathcal{H}((0, 1])$ and $k \in (0, 1)$ such that $\eta(M(Tx, Ty, t)) \leq k\eta(M(x, y, t))$ for all $x, y \in X$ and $t > 0$.

(3.5) Fuzzy α-ϕ-contractive mapping [7]**.** T is said to be a *fuzzy α-ϕ-contractive mapping*, if for given $\alpha : X^2 \times \mathbb{R}^+ \to [0, \infty)$, T is α-admissible and there exists $\phi \in \Phi$ such that

$$\alpha(x, y, t)\left(\frac{1}{M(Tx, Ty, t)} - 1\right) \leq \phi\left(\frac{1}{M(x, y, t)} - 1\right)$$

for all $x, y \in X$ and $t > 0$.

(3.6) Fuzzy iterated contraction mapping [27]. T is said to be a *fuzzy iterated contraction mapping*, if $M(Tx, T^2x, t) \geq M\left(x, Tx, \frac{t}{k}\right)$ for all $x, y \in X$, $t > 0$ and $0 < k < 1$.

(3.7) Fuzzy $\mathcal{F}$-contractive mapping [11]. For any $F \in \mathcal{F}([0, 1])$, T is said to be a *fuzzy $\mathcal{F}$-contractive mapping*, if there exists $k \in (0, 1)$ such that $kF(M(Tx, Ty, t)) \geq F(M(x, y, t))$ for all distinct $x, y \in X$ and $t > 0$.

4. Comparison of Fuzzy Contraction and Contractive Mappings

In the above contraction mappings, the respective authors assumed additional hypotheses on *T*, such as continuity or *t*-uniform continuity, α-admissibility, β-admissibility, and the completeness of *X*, in order to establish their fixed point results. At this point, we do not make any restriction on the statements, but at the end of this section, we impose additional restrictions as needed in order to establish the existence of a fixed point. We first establish a theorem of partial ordering for the above contraction principles. Note that a statement like $(a) \Rightarrow (b)$ means that any function that satisfies condition (a) also satisfies condition (b).

Theorem 4.1. *The following implications hold*:

(i) (3.1) $\Rightarrow$ (3.6)

(ii) (3.1) $\Leftrightarrow$ (3.2)

(iii) (3.1) $\Leftrightarrow$ (3.3)

(iv) (3.3) $\Leftrightarrow$ (3.2)

(v) (3.5) $\Rightarrow$ (3.2)

(vi) (3.7) $\Rightarrow$ (3.3)

(vii) (3.4) $\Rightarrow$ (3.1).

Proof. (i) The Picard iteration procedure, which underpins the proof of the fuzzy Banach contraction principle, also yields the fuzzy iterated contraction mapping by considering $y = Tx$ and $Ty = T^2x$. Therefore, fuzzy iterated contraction mapping is a direct consequence of the fuzzy Banach contraction principle. That is, for any $k \in (0, 1)$, $t \in \mathbb{R}^+$, set $kt = t_1$, and we have

$$M(Tx, Ty, kt) \geq M(x, y, t) \Rightarrow M(Tx, T^2x, t_1) \geq M\left(x, Tx, \frac{t_1}{k}\right).$$

Hence (3.1) $\Rightarrow$ (3.6).

(ii) Assume that, for any $k \in (0, 1)$, $M(Tx, Ty, kt) \geq M(x, y, t)$. Now for any $t \in \mathbb{R}^+$ and Lemma 2.5, we have

$$M(Tx, Ty, t) > M(Tx, Ty, kt) \geq M(x, y, t).$$

There exists $k_1 \in (0, 1)$ such that $k_1 M(Tx, Ty, t) = M(x, y, t)$. This implies that

$$\frac{1 - M(Tx, Ty, t)}{M(Tx, Ty, t)} \leq \frac{k_1(1 - M(x, y, t))}{M(x, y, t)}$$

$$\Rightarrow \frac{1}{M(Tx, Ty, t)} - 1 \leq k_1\left(\frac{1}{M(x, y, t)} - 1\right).$$

Hence (3.1) $\Rightarrow$ (3.2). Now, for converse, assume that (3.2) holds, and from Lemma 2.5, for any $k \in (0, 1)$, we have

$$\frac{1 - M(Tx, Ty, t)}{M(Tx, Ty, t)} \leq \frac{1 - M(Tx, Ty, kt)}{M(Tx, Ty, t)} < \frac{1 - M(x, y, t)}{M(Tx, Ty, t)}$$

$$\Rightarrow M(Tx, Ty, kt) \geq M(x, y, t).$$

This concludes that (3.1) $\Leftrightarrow$ (3.2).

(iii) Let $\psi : [0, 1] \to [0, 1]$ be a non-decreasing function such that $\psi(0) = 0$, $\psi(1) = 1$ and $\psi(r) > r$ for all $r \in (0, 1)$. First to prove (3.1) $\Rightarrow$ (3.3), assume that (3.1) holds and from Lemma 2.5, for every $t > 0$, there exists $k \in (0, 1)$ such that $M(Tx, Ty, t) > M(Tx, Ty, kt) \geq M(x, y, t)$. This implies that $\psi(M(Tx, Ty, t)) \geq \psi(M(Tx, Ty, kt)) \geq \psi(M(x, y, t))$. Now, there exist $k_1, k_2, k_3 \in (0, 1)$ such that

$$k_1\psi(M(Tx, Ty, t)) = M(Tx, Ty, t),$$

$$k_2\psi(M(Tx, Ty, t)) = \psi(M(Tx, Ty, kt))$$

and

$$k_3\psi(M(Tx, Ty, kt)) = \psi(M(x, y, t)).$$

Now, we get $\psi(M(x, y, t)) = \frac{k_2k_3}{k_1}M(Tx, Ty, t)$. It is clear that $k_2k_3 < k_1$ and $\frac{k_2k_3}{k_1} \in (0, 1)$. This implies that $M(Tx, Ty, t) \geq \psi(M(x, y, t))$ for all $x, y \in X$ and $t > 0$. Hence (3.1) $\Rightarrow$ (3.3). Now, to prove the converse (3.3) $\Rightarrow$ (3.1), let us assume that (3.3) holds. That is, $M(Tx, Ty, t) \geq \psi(M(x, y, t))$. We know that $\psi(M(x, y, t)) > M(x, y, t)$ and from Lemma 2.5, for every $t > 0$, there exists $k \in (0, 1)$ such that $M(Tx, Ty, t) > M(Tx, Ty, kt) \geq M(x, y, t)$. Now, there exist $k_4, k_5, k_6 \in (0, 1)$ such that $k_4M(Tx, Ty, t) = \psi(M(x, y, t))$, $k_5\psi(M(x, y, t)) = M(x, y, t)$, and $k_6M(Tx, Ty, t) = M(Tx, Ty, kt)$. Now, we get

$$\frac{k_4k_5}{k_6}M(Tx, Ty, kt) = M(x, y, t).$$

It is clear that $k_4k_5 < k_6$ and $\frac{k_4k_5}{k_6} \in (0, 1)$. This implies that $M(Tx, Ty, kt) \geq M(x, y, t)$ for all $x, y \in X$ and $t > 0$. Hence (3.3) $\Rightarrow$ (3.1), and hence (3.1) $\Leftrightarrow$ (3.3).

(iv) Assume that (3.3) holds. That is, $M(Tx, Ty, t) \geq \psi(M(x, y, t))$. Since ψ is non-decreasing and $\psi(p) > p$ for all $p \in (0, 1)$,

$$M(Tx, Ty, t) \geq \psi(M(x, y, t)) > M(x, y, t)$$

$$\Rightarrow M(Tx, Ty, t) > M(x, y, t)$$

$$\Rightarrow 1 - M(x, y, t) > 1 - M(Tx, Ty, t)$$

$$\Rightarrow [1 - M(x, y, t)]\frac{M(x, y, t)}{M(x, y, t)} > [1 - M(Tx, Ty, t)]\frac{M(Tx, Ty, t)}{M(Tx, Ty, t)}$$

$$\Rightarrow \left(\frac{1}{M(x, y, t)} - 1\right)\frac{M(x, y, t)}{M(Tx, Ty, t)} > \left(\frac{1}{M(Tx, Ty, t)} - 1\right).$$

By taking $k = \frac{M(x, y, t)}{M(Tx, Ty, t)} \in (0, 1)$, we get

$$\frac{1}{M(Tx, Ty, t)} - 1 \leq k\left(\frac{1}{M(x, y, t)} - 1\right).$$

Hence (3.3) $\Rightarrow$ (3.2). Now, to prove the reverse implication (3.2) $\Rightarrow$ (3.3), we assume that (3.2) holds. This implies that

$$M(Tx, Ty, t) \geq \frac{M(x, y, t)}{M(x, y, t) + k(1 - M(x, y, t))} = \psi(M(x, y, t)).$$

(For an Example of ψ_k, we refer [15]). It is clear that $M(Tx, Ty, t) \geq \psi(M(x, y, t))$. It concludes the implication (3.2) $\Rightarrow$ (3.3). Hence (3.2) $\Leftrightarrow$ (3.3).

(v) For each $t > 0$, there exists $k \in (0, 1)$ such that $\phi(t) = kt$, where ϕ is a right continuous function defined as $\phi : [0, \infty) \to [0, \infty)$, with $\phi(t) < t$

for all $t > 0$. For instance, $\phi(t) = \frac{t}{2} = kt$ with $k = \frac{1}{2} \in (0, 1)$. Now, by assuming (3.5), from Definition 2.8(iii), we get

$$\frac{1}{M(Tx, Ty, t)} - 1 \leq \alpha(x, y, t)\left(\frac{1}{M(Tx, Ty, t)} - 1\right) \leq \phi\left(\frac{1}{M(x, y, t)} - 1\right)$$
$$= k\left(\frac{1}{M(x, y, t)} - 1\right).$$

Hence (3.5) $\Rightarrow$ (3.2).

(vi) Let us assume that (3.7) holds, that is, for any $k \in (0, 1)$, $kF(M(Tx, Ty, t)) \geq F(M(x, y, t))$ for all $x, y \in X$ and $t > 0$, where F is a strictly increasing function defined as $F : [0, 1] \to \mathbb{R}^+$ such that $F(r) < F(s)$ for all $r, s \in (0, 1)$ with $r < s$. Now, let $\psi : [0, 1] \to [0, 1]$ be a non-decreasing function defined as $\psi(0) = 0$, $\psi(1) = 1$ and $\psi(r) > r$ for all $r \in (0, 1)$. It is clear that $F(r) > \psi(r)$ for all $r \in (0, 1)$. Now, for every $r \in (0, 1)$, there exists $q \in (0, 1)$ such that $\psi(r) = qF(r)$. Now, we can find $k_1, k_2, k_3, k_4 \in (0, 1)$ such that $kk_1F(M(Tx, Ty, t)) = F(M(x, y, t))$, $k_2F(M(x, y, t)) = \psi(M(x, y, t))$, $k_3F(M(Tx, Ty, t)) = \psi(M(Tx, Ty, t))$, and $k_4\psi(M(Tx, Ty, t)) = M(Tx, Ty, t)$. Now, we get $\frac{kk_1k_2}{k_3k_4}M(Tx, Ty, t) = \psi(M(x, y, t))$, where $\frac{kk_1k_2}{k_3k_4} \in (0, 1)$ which implies that $M(Tx, Ty, t) \geq \psi(M(x, y, t))$. Hence (3.7) $\Rightarrow$ (3.3).

(vii) Let us start with (3.4), that is, for $k \in (0, 1)$, $\eta(M(Tx, Ty, t)) \leq k\eta(M(x, y, t))$ for all $x, y \in X$ and $t > 0$, where η is a strictly decreasing function defined as $\eta : (0, 1] \to [0, \infty)$, $\eta(r) > \eta(s)$ for all $r, s \in (0, 1)$ with $r < s$ and $\eta(1) = 0$. For every $x, y \in X$ and $t > 0$, there exists $\lambda \in (0, 1)$ such that $\eta(M(x, y, t)) \leq \lambda\eta(M(x, y, \lambda t))$. It is easy to verify by taking $\eta(r) = \frac{1}{\ln r}$ with the fuzzy metric space taken in the proof of implication (iii). Now, by Lemma 2.5, we get

$$\eta(M(Tx, Ty, t)) \leq \eta(M(Tx, Ty, kt)) \leq k\eta(M(x, y, kt))$$

$$\leq \eta(M(x, y, t)) \leq \eta(M(x, y, kt))$$

$$\Rightarrow \eta(M(Tx, Ty, kt)) \leq \eta(M(x, y, t))$$

$$\Rightarrow M(Tx, Ty, kt) \geq M(x, y, t).$$

Hence (3.4) $\Rightarrow$ (3.1). □

In the above Theorem 4.1, established equivalences between contraction and contractive mappings, expressed through "$\Leftrightarrow$", remain valid in any fuzzy metric space provided the corresponding necessary conditions and functions are satisfied. Nevertheless, when considering only one-sided implications "$\Rightarrow$", the converses do not generally hold. To substantiate this, it becomes essential to construct counterexamples that explicitly demonstrate the failure of the reverse implication. In this context, we now present the following examples:

Example 4.2. Let $X = \left[\frac{-1}{2}, \frac{1}{2}\right]$ be a metric space under the metric $M(x, y, t) = \frac{t}{t + |x - y|}$ for all $x, y \in X$, $t > 0$ and the continuous t-norm "$*$", defined as $a * b = ab$ for all $a, b \in [0, 1]$. Define $T : X \to X$ as $Tx = x^2$ for all $x \in X$. T satisfies fuzzy iterated contraction (3.6) but does not satisfy the fuzzy Banach contraction (3.1). Therefore, the implication (3.1) $\Rightarrow$ (3.6) holds but not (3.6) $\Rightarrow$ (3.1).

Example 4.3. Let $X = \mathbb{R}$ be a metric space under the metric $M(x, y, t) = \exp\left\{\frac{-|x - y|}{t}\right\}$ for all $x, y \in X$, $t > 0$ and the continuous t-norm "$*$"

defined as $a * b = ab$ for all $a, b \in [0, 1]$. Now, let $T : X \to X$ such that $Tx = \frac{x}{2}$ for all $x \in X$.

(i) Consider the functions $\alpha : X^2 \times \mathbb{R}^+ \to [0, \infty)$ as

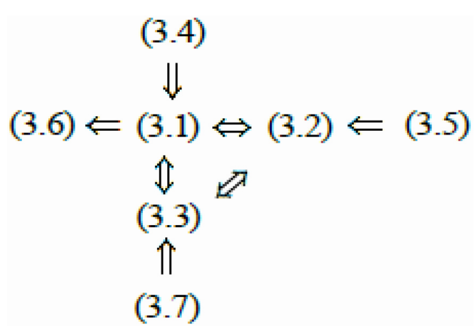
$$\alpha(x, y, t) = \begin{cases} x/y, & \text{if } x \geq y, \\ y/x, & \text{if } y \geq x \end{cases}$$

and $\phi : [0, \infty) \to [0, \infty)$ as $\phi(t) = \frac{t}{2}$. T satisfies the fuzzy contractive (3.2) but not satisfies the fuzzy α-ϕ-contractive (3.5). Hence (3.2) $\not\Rightarrow$ (3.5).

(ii) Consider the functions $F : [0, 1] \to [0, \infty)$ as $F(r) = \frac{-1}{\ln r}$ and $\psi : [0, 1] \to [0, 1]$ as $\psi(r) = \sqrt{r}$ for all $r \in (0, 1)$. T satisfies the fuzzy ψ-contractive (3.3), but for $0 < k \leq \frac{1}{2}$, the fuzzy $\mathcal{F}$-contractive (3.7) does not satisfy. Hence (3.3) $\not\Rightarrow$ (3.7).

(iii) Consider the function $\eta : (0, 1] \to [0, 1]$ as $\eta(r) = \ln\left(\frac{1}{r}\right)$ for all $r \in (0, 1]$. T satisfies the fuzzy Banach contraction (3.1), but it does not satisfy the fuzzy $\mathcal{H}$-contractive (3.4). Hence (3.1) $\not\Rightarrow$ (3.4).

Remark 4.4. The implications of Theorem 4.1 are summarized in the diagram below:



Having derived certain implications, we now define a generalized contractive principle of the above contractions. Further results will be established in the complete fuzzy metric space. We begin with the following lemma:

Lemma 4.5. *Let* $(X, M, *)$ *be a complete fuzzy metric space. For some* $\lambda \geq 1$ *and* $0 < \theta \leq 1,$ *the functions* $\alpha, \beta : X^2 \times \mathbb{R}^+ \to \mathbb{R}^+$ *satisfy* $1 \leq \alpha(x, y, t) \leq \lambda$ *and* $0 < \beta(x, y, t) \leq \theta \leq 1$ *for all* $x, y \in X$ *and* $t > 0.$ *Further, for any sequence* $x_n \to x,$ $\alpha(x_n, x, t) \to 1$ *and* $\beta(x_n, x, t) \to 1.$ *Then for every* $t > 0,$ *there exist* $a, b, c \in [0, \infty)$ *with* $a + b + c \leq 1$ *and* $2b + c < 1$ *such that* $a\beta(x, y, t) + b\alpha(x, y, t) + c \leq 1$ *for all* $x, y \in X.$ *Moreover,* $\lim_{n\to\infty} (a\beta(x_n, x, t) + b\alpha(x_n, x, t) + c) = 1.$

Proof. Let $(X, M, *)$ be a complete fuzzy metric space and $\{x_n\}$ be any sequence in X that converges to $x \in X.$ That is, $x_n \to x$ as $n \to \infty.$ This implies that $\lim_{n\to\infty} \alpha(x_n, x, t) = 1$ and $\lim_{n\to\infty} \beta(x_n, x, t) = 1.$ Now, for $\varepsilon \in (0, 1)$ choose $b, c > 0$ with $2b + c < 1$ and $a + b + c \leq 1,$ set $b\lambda + c \leq \frac{\varepsilon}{2},$ $a = \min\left\{1 - b - c, \frac{1}{\theta}\left(1 - \frac{\varepsilon}{2}\right)\right\} > 0.$ Then for all $x, y \in X,$

$$a\beta(x, y, t) + b\alpha(x, y, t) + c \leq a\theta + b\lambda + c \leq \left(1 - \frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2} = 1.$$

Since $x_n \to x$ as $n \to \infty,$ $a\beta(x_n, x, t) + b\alpha(x_n, x, t) + c \to (a + b + c) \leq 1.$ Hence $\lim_{n\to\infty} (a\beta(x_n, x, t) + b\alpha(x_n, x, t) + c) = 1.$ □

Example 4.6. Let $(X, M, *)$ be a complete fuzzy metric space. Define $\beta(x, y, t) = M(x, y, t)$ and $\alpha(x, y, t) = \frac{1}{M(x, y, t)}$ for all $x, y \in X$ and $t > 0.$ Then $a\beta(x_n, x, t) + b\alpha(x_n, x, t) + c \to (a + b + c) \leq 1$ as $n \to \infty.$

Definition 4.7. Let $(X, M, *)$ be a complete fuzzy metric space. An α and β admissible mapping $T : X \to X$ is said to be *generalized $\mathcal{H}$-$\mathcal{F}$-contractive*, if for every $\eta \in \mathcal{H}((0, 1])$ and $F \in \mathcal{F}([0, 1])$, there exist $a, b, c \in [0, \infty)$ with $2b + c < 1$ and $a + b + c \leq 1$ such that

$$M(Tx, Ty, t) \geq a\beta(x, y, t)F(\min\{M(x, y, t), M(x, Ty, t), M(y, Ty, t)\})$$
$$+ b\alpha(x, y, t)\eta(\max\{M(x, y, t), M(x, Ty, t), M(y, Ty, t)\})$$
$$+ c\min\{M(x, y, t), M(x, Tx, t), M(y, Ty, t)\} \tag{4.1}$$

for all $x, y \in X$ and $t > 0$.

To substantiate this definition, we present the following proposition, which encapsulates a fundamental implication and provides a rigorous justification.

Proposition 4.8. (2.1) $\Rightarrow$ (4.1).

Proof. Let $(X, M, *)$ be a complete fuzzy metric space and $T : X \to X$ be the Banach contraction mapping. That is, for $k \in (0, 1)$, $M(Tx, Ty, kt) \geq M(x, y, t)$ for all $x, y \in X$ and $t > 0$. From Lemma 2.5, for any $k \in (0, 1)$, we have $M(Tx, Ty, t) \geq M(Tx, Ty, kt) \geq M(x, y, t)$. From Definition 2.4 (GV-4), for any $t_1, s_1 > 0$, we get

$$M(Tx, Ty, t_1) * M(y, Ty, s_1) \geq M(x, y, t_1) * M(y, Ty, s_1)$$
$$\Rightarrow M(y, Tx, t) \geq M(x, Ty, t)$$

and

$$M(Tx, Ty, t_1) * M(y, Tx, s_1) \geq M(x, y, t_1) * M(y, Tx, s_1)$$
$$\Rightarrow M(y, Ty, t) \geq M(x, Tx, t).$$

From equation (4.1), we have

$$RHS = a\beta(x, y, t)F(\min\{M(x, y, t), M(x, Ty, t), M(y, Ty, t)\})$$

$$+ b\alpha(x, y, t)\eta(\max\{M(x, y, t), M(x, Ty, t), M(y, Ty, t)\})$$

$$+ c\min\{M(x, y, t), M(x, Tx, t), M(y, Ty, t)\}$$

$$RHS \leq a\beta(x, y, t)F(\min\{M(Tx, Ty, t), M(y, Tx, t), M(y, Ty, t)\})$$

$$+ b\alpha(x, y, t)\eta(\max\{M(Tx, Ty, t), M(y, Tx, t), M(y, Ty, t)\})$$

$$+ c\min\{M(Tx, Ty, t), M(y, Ty, t), M(y, Ty, t)\}$$

$$RHS \leq a\beta(x, y, t)F(\min\{M(Tx, Ty, t), M(y, Tx, t), M(y, Ty, t)\})$$

$$+ b\alpha(x, y, t)\eta(\max\{M(Tx, Ty, t), M(y, Tx, t), M(y, Ty, t)\})$$

$$+ c\min\{M(Tx, Ty, t), M(y, Ty, t)\}. \tag{4.2}$$

Case (i). If $\min\{M(Tx, Ty, t), M(y, Ty, t)\} = M(Tx, Ty, t)$, that is, $M(Tx, Ty, t) \leq M(y, Ty, t)$, and

$$\max\{M(Tx, Ty, t), M(y, Tx, t), M(y, Ty, t)\} = M(y, Ty, t),$$

then $\min\{M(Tx, Ty, t), M(y, Tx, t), M(y, Ty, t)\} = M(Tx, Ty, t)$, and from equation (4.2), we get

$$RHS \leq a\beta(x, y, t)F(M(Tx, Ty, t)) + b\alpha(x, y, t)\eta(M(y, Ty, t))$$

$$+ cM(Tx, Ty, t).$$

Since $F \in \mathcal{F}([0, 1])$ is a strictly increasing function defined as $F(q) > q$ and $\eta \in \mathcal{H}((0, 1])$ is a strictly decreasing function defined as $\eta(q) < q$ for all $q \in (0, 1)$, there exist $p_1, p_2, p_3 \in (0, 1)$ such that $p_1 F(q) = q$, $\eta(q) = p_2 q$ and $M(Tx, Ty, t) = p_3 M(y, Ty, t)$. Thus

$$RHS \leq \frac{a}{p_1^1}\beta(x, y, t)M(Tx, Ty, t) + bp_2^1\alpha(x, y, t)M(y, Ty, t)$$

$$+ cM(Tx, Ty, t)$$

$$\leq \frac{a}{p_1^1}\beta(x, y, t)M(Tx, Ty, t) + \frac{bp_2^1}{p_3^1}\alpha(x, y, t)M(Tx, Ty, t)$$

$$+ cM(Tx, Ty, t)$$

$$\leq \left(\frac{a}{p_1^1}\beta(x, y, t) + \frac{bp_2^1}{p_3^1}\alpha(x, y, t) + c\right)M(Tx, Ty, t),$$

where $0 < p_1^1 = \dfrac{M(Tx, Ty, t)}{F(M(Tx, Ty, t))}$, $p_2^1 = \dfrac{\eta(M(y, Ty, t))}{M(y, Ty, t)}$ and $p_3^1 = \dfrac{M(Tx, Ty, t)}{M(y, Ty, t)} < 1.$

Thus $RHS \leq (a'\beta(x, y, t) + b'\alpha(x, y, t) + c')M(Tx, Ty, t),$

where $a' = \dfrac{a}{p_1^1}$, $b' = \dfrac{bp_2^1}{p_3^1}$ and $c' = c.$

The procedure in Case (i) is adopted in following cases:

Case (ii). If $\min\{M(Tx, Ty, t), M(y, Ty, t)\} = M(Tx, Ty, t)$, that is, $M(Tx, Ty, t) \leq M(y, Ty, t)$, and

$$\max\{M(Tx, Ty, t), M(y, Tx, t), M(y, Ty, t)\} = M(y, Tx, t),$$

then $\min\{M(Tx, Ty, t), M(y, Tx, t), M(y, Ty, t)\} = M(Tx, Ty, t).$

Case (iii). If $\min\{M(Tx, Ty, t), M(y, Ty, t)\} = M(Tx, Ty, t)$, that is, $M(Tx, Ty, t) \leq M(y, Ty, t)$, and

$$\max\{M(Tx, Ty, t), M(y, Tx, t), M(y, Ty, t)\} = M(y, Ty, t),$$

then $\min\{M(Tx, Ty, t), M(y, Tx, t), M(y, Ty, t)\} = M(y, Tx, t).$

Case (iv). If $\min\{M(Tx, Ty, t), M(y, Ty, t)\} = M(y, Ty, t)$, that is, $M(Tx, Ty, t) \geq M(y, Ty, t)$, and

$$\max\{M(Tx, Ty, t), M(y, Tx, t), M(y, Ty, t)\} = M(y, Tx, t),$$

then $\min\{M(Tx, Ty, t), M(y, Tx, t), M(y, Ty, t)\} = M(y, Ty, t)$.

Case (v). If $\min\{M(Tx, Ty, t), M(y, Ty, t)\} = M(y, Ty, t)$, that is, $M(Tx, Ty, t) \geq M(y, Ty, t)$, and

$$\max\{M(Tx, Ty, t), M(y, Tx, t), M(y, Ty, t)\} = M(Tx, Ty, t),$$

then $\min\{M(Tx, Ty, t), M(y, Tx, t), M(y, Ty, t)\} = M(y, Ty, t)$.

Case (vi). If $\min\{M(Tx, Ty, t), M(y, Ty, t)\} = M(y, Ty, t)$, that is, $M(Tx, Ty, t) \geq M(y, Ty, t)$, and

$$\max\{M(Tx, Ty, t), M(y, Tx, t), M(y, Ty, t)\} = M(Tx, Ty, t),$$

then $\min\{M(Tx, Ty, t), M(y, Tx, t), M(y, Ty, t)\} = M(y, Tx, t)$.

Equation (4.2) can be expressed as

$$RHS \leq (\tilde{a}\beta(x, y, t) + \tilde{b}\alpha(x, y, t) + \tilde{c})M(Tx, Ty, t).$$

By Lemma 4.5, corresponding to each case, there exist some $\tilde{a}, \tilde{b}, \tilde{c} \in [0, \infty)$ with $2\tilde{b} + \tilde{c} < 1$ and $\tilde{a} + \tilde{b} + \tilde{c} \leq 1$ such that $\tilde{a}\beta(x, y, t) + \tilde{b}\alpha(x, y, t) + \tilde{c} \leq 1$. This implies that $RHS \leq M(Tx, Ty, t)$. Hence

$$\begin{aligned} M(Tx, Ty, t) \geq\ & a\beta(x, y, t)F(\min\{M(x, y, t), M(x, Ty, t), M(y, Ty, t)\}) \\ & + b\alpha(x, y, t)\eta(\max\{M(x, y, t), M(x, Ty, t), M(y, Ty, t)\}) \\ & + c\min\{M(x, y, t), M(x, Tx, t), M(y, Ty, t)\}. \end{aligned}$$

This concludes the proof. □

Lemma 4.9. *Let* $(X, M, *)$ *be a complete fuzzy metric space. For every* a_k, b_k, c_k *in* $[0, \infty)$ *with* $2b_k + c_k < 1$ *and* $a_k + b_k + c_k \leq 1$, *if* $\{a_k\beta(x_{n_k}, y_{n_k}, t) + b_k\alpha(x_{n_k}, y_{n_k}, t) + c_k\}_{k=0}^{\infty}$ *is an s-increasing sequence, then for every* $\varepsilon > 0$ *and* $t > 0$, *there exists* $N \in \mathbb{N}$ *such that*

$$\prod_{k=0}^{m}(a_k\beta(x_{n_k}, y_{n_k}, t) + b_k\alpha(x_{n_k}, y_{n_k}, t) + c_k) > 1 - \varepsilon \text{ for all } m > N.$$

Moreover,

$$\lim_{m, n \to \infty}\prod_{k=0}^{m}(a_k\beta(x_{n_k}, y_{n_k}, t) + b_k\alpha(x_{n_k}, y_{n_k}, t) + c_k) = 1 \text{ for all } m, n > N.$$

Proof. For every $x, y \in X$ and $t > 0$, there exist $a_k, b_k, c_k \in [0, \infty)$ with $a_k + b_k + c_k \leq 1$ and $2b_k + c_k < 1$ such that

$$a_k\beta(x_{n_k}, y_{n_k}, t) + b_k\alpha(x_{n_k}, y_{n_k}, t) + c_k = q_k \in (0, 1).$$

Now, for $m \in \mathbb{N}$, let $s_m = \frac{1}{q_k^m}$ be an s-increasing sequence. That is, there exists $N \in \mathbb{N}$ such that $s_m + 1 \leq s_{m+1}$ for all $m > N$. This implies that $\frac{1}{q_k^m} + 1 \leq \frac{1}{q_k^{m+1}} \Rightarrow q_k + q_k^{m+1} < 1 \Rightarrow \lim_{m\to\infty} q_k^m = 1$. Hence for every $\varepsilon > 0$ and $t > 0$, there exists $N \in \mathbb{N}$ such that

$$\prod_{k=0}^{m}(a_k\beta(x_{n_k}, y_{n_k}, t) + b_k\alpha(x_{n_k}, y_{n_k}, t) + c_k) > 1 - \varepsilon$$

for all $m > N$. Further,

$$\lim_{m, n \to \infty}\prod_{k=0}^{m}(a_k\beta(x_{n_k}, y_{n_k}, t) + b_k\alpha(x_{n_k}, y_{n_k}, t) + c_k) = 1$$

for all $m, n > N$. □

Theorem 4.10. *Let* $(X, M, *)$ *be a complete fuzzy metric space. If an* α *and* β *admissible self-mapping* $T : X \to X$ *is generalized* $\mathcal{H}$-$\mathcal{F}$-*contractive, then T has a unique fixed point.*

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0, t) \geq 1$ and $\beta(x_0, Tx_0, t) \leq 1$, and $\{x_n\}$ be a sequence defined by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}_0$. If there exists $n \in \mathbb{N}_0$ such that $x_n = x_{n+1} = Tx_n$, then x_n is a fixed point of T, and the proof is completed. Now assume that $x_n \neq x_{n+1} = Tx_n$ for all $n \in \mathbb{N}_0$. Since $\alpha(x_0, Tx_0, t) = \alpha(x_0, x_1, t) \geq 1$ and $\beta(x_0, Tx_0, t) = \beta(x_0, x_1, t) \leq 1$ for all $t > 0$, by continuing this process, we get

$$\alpha(x_n, Tx_n, t) = \alpha(x_n, x_{n+1}, t) \geq 1 \text{ and}$$

$$\beta(x_n, Tx_n, t) = \beta(x_n, x_{n+1}, t) \leq 1, \quad \forall t > 0, \forall n \in \mathbb{N}_0.$$

This implies that T is α and β admissible. Since $(X, M, *)$ is a complete fuzzy metric space, $x_n \to x$ as $n \to \infty$ for some $x \in X$. That is, for every $\delta \in (0, 1)$ and $t > 0$, there exists $N \in \mathbb{N}_0$ such that $M(x_n, x, t) > 1 - \delta$. This implies that

$$\lim_{n\to\infty} M(x_n, x, t) = 1, \quad \lim_{n\to\infty} \alpha(x_n, x, t) = 1 \quad \text{and} \quad \lim_{n\to\infty} \beta(x_n, x, t) = 1.$$

Since T is a generalized $\mathcal{H}$-$\mathcal{F}$-contractive mapping, there exist $a, b, c \in [0, \infty)$ with $2b + c < 1$ and $a + b + c \leq 1$, satisfying equation (4.1). Now, by taking $x = x_n$ and $y = x_{n+1}$, we get

$$\begin{aligned} &M(Tx_n, Tx_{n+1}, t) \\ &\geq a\beta(x_n, x_{n+1}, t)F(\min\{M(x_n, x_{n+1}, t), \\ &\quad M(x_n, Tx_{n+1}, t), M(x_{x+1}, Tx_{n+1}, t)\}) \\ &\quad + b\alpha(x_n, x_{n+1}, t)\eta(\max\{M(x_n, x_{n+1}, t), \end{aligned}$$

$$M(x_n, Tx_{n+1}, t), M(x_{x+1}, Tx_{n+1}, t)\})$$

$$+ c\min\{M(x_n, x_{n+1}, t), M(x_n, Tx_n, t), M(x_{n+1}, Tx_{n+1}, t)\}.$$

By the procedure of Proposition 4.8, we get

$$\begin{aligned} M(Tx_n, Tx_{n+1}, t) \geq\ & a\beta(x_n, x_{n+1}, t)F(M(x_n, Tx_{n+1}, t)) \\ & + b\alpha(x_n, x_{n+1}, t)\eta(M(x_{x+1}, Tx_{n+1}, t)) \\ & + cM(x_n, x_{n+1}, t), \end{aligned}$$

where $F \in \mathcal{F}([0, 1])$ is a strictly increasing function defined as $F(q) > q$ and $\eta \in \mathcal{H}((0, 1])$ is a strictly decreasing function defined as $\eta(q) < q$ for all $q \in (0, 1)$. This implies that

$$\eta(M(x_n, x_{n+1}, t)) < M(x_n, x_{n+1}, t) < F(M(x_n, x_{n+1}, t))$$

for all $n \in \mathbb{N}_0$. There exist $r_1, s_1 \in (0, 1)$ such that $\eta(M(x_n, x_{n+1}, t)) = r_1M(x_n, x_{n+1}, t)$ and $M(x_n, x_{n+1}, t) = s_1F(M(x_n, x_{n+1}, t))$. Thus

$$\begin{aligned} M(Tx_n, Tx_{n+1}, t) \geq\ & \frac{a}{s_1}\beta(x_n, x_{n+1}, t)M(x_n, Tx_{n+1}, t) \\ & + br_1\alpha(x_n, x_{n+1}, t)M(x_{x+1}, Tx_{n+1}, t) \\ & + cM(x_n, x_{n+1}, t). \end{aligned}$$

Similarly, we can find some $r_2, s_2 \in (0, 1)$ such that $M(x_n, Tx_{n+1}, t) = r_2M(x_n, x_{n+1}, t)$ and $M(x_n, x_{n+1}, t) = s_2M(x_{n+1}, Tx_{n+1}, t)$. Thus

$$\begin{aligned} M(Tx_n, Tx_{n+1}, t) \geq\ & \frac{ar_2}{s_1}\beta(x_n, x_{n+1}, t)M(x_n, x_{n+1}, t) \\ & + \frac{br_1}{s_2}\alpha(x_n, x_{n+1}, t)M(x_n, x_{n+1}, t) \\ & + cM(x_n, x_{n+1}, t). \end{aligned}$$

We chose positive real numbers a, b, c such that $a' = \frac{ar_2}{s_1}$, $b' = \frac{br_1}{s_2}$, $c' = c$ satisfy $a' + b' + c' \leq 1$ and $2b' + c' < 1$. Thus

$$M(Tx_n, Tx_{n+1}, t) \geq (a'\beta(x_n, x_{n+1}, t) + b'\alpha(x_n, x_{n+1}, t) + c')M(x_n, x_{n+1}, t). \tag{4.3}$$

Now, from equations (4.1) and (4.3), for $k \in \mathbb{N}_0$, we can find $a_k, b_k, c_k \in [0, \infty)$ with $a_k, b_k, c_k \leq 1$ and $2b_k + c_k < 1$, such that

$$M(Tx_{n+k}, Tx_{n+k+1}, t) \geq (a_k\beta(x_{n+k}, x_{n+k+1}, t)$$
$$+ b_k\alpha(x_{n+k}, x_{n+k+1}, t) + c_k)M(x_{n+k}, x_{n+k+1}, t)$$
$$\Rightarrow M(x_{n+k+1}, x_{n+k+2}, t) \geq (a_k\beta(x_{n+k}, x_{n+k+1}, t)$$
$$+ b_k\alpha(x_{n+k}, x_{n+k+1}, t) + c_k)M(x_{n+k}, x_{n+k+1}, t).$$

Similarly, for $(k-1) \in \mathbb{N}$, we can find $a_{k-1}, b_{k-1}, c_{k-1} \in [0, \infty)$ with $a_{k-1} + b_{k-1} + c_{k-1} \leq 1$ and $2b_{k-1} + c_{k-1} < 1$, such that

$$M(x_{n+k}, x_{n+k+1}, t) \geq (a_{k-1}\beta(x_{n+k-1}, x_{n+k}, t)$$
$$+ b_{k-1}\alpha(x_{n+k-1}, x_{n+k}, t) + c_{k-1}) \cdot M(x_{n+k-1}, x_{n+k}, t)$$
$$\Rightarrow M(x_{n+k+1}, x_{n+k+2}, t) \geq (a_k\beta(x_{n+k}, x_{n+k+1}, t)$$
$$+ b_k\alpha(x_{n+k}, x_{n+k+1}, t) + c_k)$$
$$\cdot (a_{k-1}\beta(x_{n+k-1}, x_{n+k}, t) + b_{k-1}\alpha(x_{n+k-1}, x_{n+k}, t) + c_{k-1})$$
$$\cdot M(x_{n+k-1}, x_{n+k}, t).$$

By continuing this procedure, we get

$$M(x_{n+k+1}, x_{n+k+2}, t) \geq (a_k\beta(x_{n+k}, x_{n+k+1}, t)$$
$$+ b_k\alpha(x_{n+k}, x_{n+k+1}, t) + c_k)$$
$$\cdot (a_{k-1}\beta(x_{n+k-1}, x_{n+k}, t) + b_{k-1}\alpha(x_{n+k-1}, x_{n+k}, t) + c_{k-1})$$

$$\cdot\,(a_{k-2}\beta(x_{n+k-2},\, x_{n+k-1},\, t) + b_{k-2}\alpha(x_{n+k-2},\, x_{n+k-1},\, t) + c_{k-2})$$

$$\vdots$$

$$(a_0\beta(x_n,\, x_{n+1},\, t) + b_0\alpha(x_n,\, x_{n+1},\, t) + c_0)M(x_n,\, x_{n+1},\, t)$$

$$\Rightarrow M(x_{n+k+1},\, x_{n+k+2},\, t) \geq \prod_{i=0}^{k}(a_i\beta(x_{n+i},\, x_{n+i+1},\, t)$$

$$+\, b_i\alpha(x_{n+i},\, x_{n+i+1},\, t) + c_i)M(x_n,\, x_{n+1},\, t).$$

Hence by Lemma 4.9, for every $\varepsilon > 0$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\prod_{i=0}^{k}(a_i\beta(x_{n+i},\, x_{n+i+1},\, t) + b_i\alpha(x_{n+i},\, x_{n+i+1},\, t) + c_i)M(x_n,\, x_{n+1},\, t) > 1 - \varepsilon$$

for all $k > n_0$.

Moreover,

$$\lim_{n,\, k\to\infty}\prod_{i=0}^{k}(a_i\beta(x_{n+i},\, x_{n+i+1},\, t) + b_i\alpha(x_{n+i},\, x_{n+i+1},\, t) + c_i)M(x_n,\, x_{n+1},\, t) = 1$$

$$\Rightarrow \lim_{k,\, n\to\infty} M(x_{n+k+1},\, x_{n+k+2},\, t) = 1.$$

This implies that $\{x_n\}$ is a Cauchy sequence in X, and $\{x_n\}$ converges to some $x \in X$.

Now, we see that $x \in X$ is a fixed point of T. From equation (4.3), for every $t > 0$, there exist $a, b, c \in [0, \infty)$ with $2b + c < 1$ and $a + b + c \leq 1$ such that

$$M(Tx,\, Tx_n,\, t) \geq (a\beta(x,\, x_n,\, t) + b\alpha(x,\, x_n,\, t) + c)M(x,\, x_n,\, t)$$

$$\Rightarrow \lim_{n\to\infty} M(Tx,\, Tx_n,\, t) \geq \lim_{n\to\infty}(a\beta(x,\, x_n,\, t) + b\alpha(x,\, x_n,\, t) + c)M(x,\, x_n,\, t)$$

$$\Rightarrow \lim_{n\to\infty} M(Tx, Tx_{n+1}, t) \geq (a\beta(x, x, t) + b\alpha(x, x, t) + c)M(x, x, t)$$

$$\Rightarrow M(Tx, x, t) \geq (a + b + c)$$

$$\Rightarrow M(Tx, x, t) = 1.$$

Hence $Tx = x$.

Finally, to prove uniqueness, we assume that there is another fixed point y in X. That is $Ty = y$. Now, from equation (4.3), for $t > 0$, we get

$$M(x, y, t) = M(Tx, Ty, t) \geq (a\beta(x, y, t) + b\alpha(x, y, t) + c)M(x, y, t).$$

Since $a\beta(x, y, t) + b\alpha(x, y, t) + c \leq 1$, for all $x, y \in X$ and $t > 0$, $M(x, y, t) = M(x, y, t) \Leftrightarrow x = y$. This concludes the proof. □

Now, we strengthen the above Theorem 4.10 with the following examples.

Example 4.11. Let $X = [0, 1]$ be a complete fuzzy metric space equipped with the fuzzy metric $M(x, y, t) = \exp\left\{\frac{-|x - y|}{t}\right\}$ for all $x, y \in X$, $t > 0$ and "$*$" be the continuous t-norm defined for any $a, b \in [0, 1]$ as $a * b = ab$. Now, for every $t > 0$, $\alpha(x, y, t) = 1 + \frac{|x - y|}{t}$ and $\beta(x, y, t) = 1 - \frac{|x - y|}{t}$ for all $x, y \in X$. Now, let $\eta \in \mathcal{H}((0, 1])$ and $F \in \mathcal{F}([0, 1])$ with $\eta(q) = 1 - \sqrt{q}$ and $F(q) = \sqrt{q}$, respectively. Now, consider the continuous self-mapping $T : X \to X$ defined as $Tx = 1 - \exp\{-x\}$ for all $x \in X$. From the graph in Figure 1(a), it is clear that $\alpha(x, y, t) \geq 1$ and $\beta(x, y, t) \leq 1$, that is, $\alpha(x, y, t)$ is bounded below by 1 and $\beta(x, y, t)$ is bounded above by 1. Also, it is noticed that as the number of iterations increases, $\alpha(x, y, t)$ and $\beta(x, y, t)$ converge to 1.

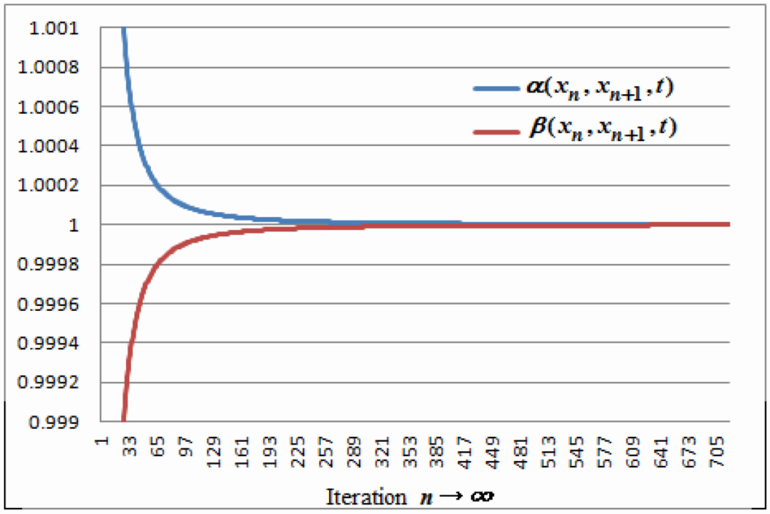


Figure 1(a). α and β functions.

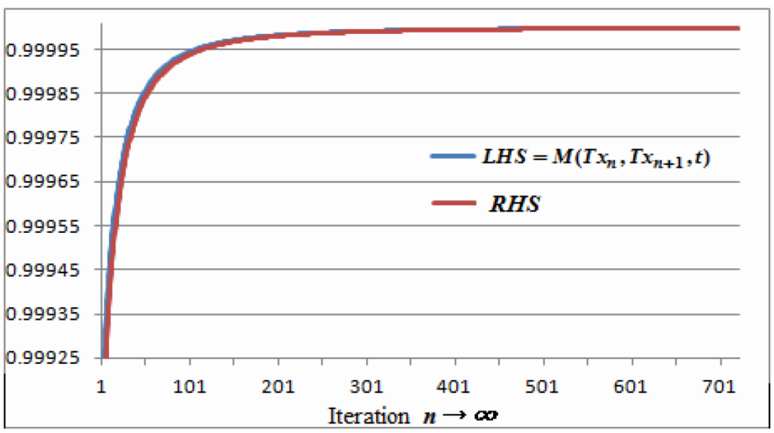


Figure 1(b). Existence of $\mathcal{H}$-$\mathcal{F}$-contractive condition, where

$$RHS = a\beta(x, y, t)F(\min\{M(x, y, t), M(x, Ty, t), M(y, Ty, t)\})$$
$$+ b\alpha(x, y, t)\eta(\max\{M(x, y, t), M(x, Ty, t), M(y, Ty, t)\})$$
$$+ c\min\{M(x, y, t), M(x, Tx, t), M(y, Ty, t)\}.$$

Also, it is clear that T is a α and β admissible, and for every $t > 0$, $x, y \in X$ with $x \neq y$, we have

$$M(Tx, Ty, t) > M(Tx, Ty, kt) > M(x, y, t).$$

From the graph given in Figure 1(b), it is clear that for every $t > 0$ and for all $x, y \in X$, T is generalized $\mathcal{H}$-$\mathcal{F}$-contractive. Hence from Theorem 4.10, T has a unique fixed point $x (= 0)$ in X. To verify the above calculations, the reader is referred to the computational data for $t = 2$, $a = 0.35$, $b = 0.25$, and $c = 0.4$ as presented in Table 1.

Table 1. Computation of fixed point for $t = 2$, $a = 0.35$, $b = 0.25$, and $c = 0.4$, where

$$\begin{aligned} RHS = {} & a\beta(x, y, t)F(\min\{M(x, y, t), M(x, Ty, t), M(y, Ty, t)\}) \\ & + b\alpha(x, y, t)\eta(\max\{M(x, y, t), M(x, Ty, t), M(y, Ty, t)\}) \\ & + c\min\{M(x, y, t), M(x, Tx, t), M(y, Ty, t)\} \end{aligned}$$

n	x_n	$Tx_n = x_{n+1}$	$\alpha(x_n, x_{n+1}, t)$	$\beta(x_n, x_{n+1}, t)$	$M(x_n, x_{n+1}, t)$	$LHS = M(Tx_n, Tx_{n+1}, t)$	RHS
1	0.5	0.39346934	1.05326533	0.94673467	0.948128412	0.966483868	0.700897286
2	0.39346934	0.325287996	1.034090672	0.965909328	0.966483868	0.976477425	0.718074632
3	0.325287996	0.277680702	1.023803647	0.976196353	0.976477425	0.982544462	0.727501333
…	…	…	…	…	…	…	…
99	0.1019399224	0.01921227	1.000093477	0.999906523	0.999906527	0.999908315	0.749908956
100	0.01921227	0.01902889	1.00009169	0.99990831	0.999908315	0.999910051	0.749910696
101	0.01902889	0.018848984	1.000089953	0.999910047	0.999910051	0.994722224	0.749912387
…	…	…	…	…	…	…	…
3362	0.00059395539	0.00059377903	1.00000008818	0.99999991182	0.99999991182	0.99999991187	0.74999991403
3363	0.00059377903	0.00059360278	1.00000008813	0.99999991187	0.99999991187	0.99999991193	0.74999991408
3364	0.00059360278	0.00059342663	1.00000008807	0.99999991193	0.99999991193	0.99999991198	0.74999991413
…	…	…	…	…	…	…	…
$n \to \infty$	$x_n \to 0$	$Tx_n \to 0$	$\alpha(x_n, x_{n+1}, t) \to 1$	$\beta(x_n, x_{n+1}, t) \to 1$	$M(x_n, x_{n+1}, t) \to 1$	$M(Tx_n, Tx_{n+1}, t) \to 1$	$RHS \to 1$

Example 4.12. Let $X = [0, \infty)$ be a complete fuzzy metric space under the fuzzy metric $M(x, y, t) = \exp\left\{\frac{-|x - y|}{t}\right\}$ for all $x, y \in X$ and $t > 0$ with continuous t-norm $r * s = \min\{r, s\}$ for all $r, s \in [0, 1]$. Now, define a self-mapping $T : X \to X$ as $Tx = \begin{cases} \frac{x^2}{4}, & \text{if } x \in [0, 1], \\ 4x, & \text{if } x \in (0, \infty). \end{cases}$

Now, let F be a strictly increasing function on $[0, 1]$ defined as $F(p) = \frac{-1}{\ln p}$ for $0 < p \leq 1$, and let η be a strictly decreasing function on $[0, 1]$ defined as $\eta(p) = \exp\left\{\frac{-1}{p}\right\}$ for $0 < p \leq 1$.

Also, let $\beta : X^2 \times \mathbb{R} \to \mathbb{R}$ be defined as

$$\beta(x, y, t) = \begin{cases} \frac{1}{2}, & \text{for } x, y \in [0, 1], \\ 1, & \text{for } x, y \in (1, \infty) \end{cases}$$

and let $\alpha : X^2 \times \mathbb{R} \to \mathbb{R}$ be defined as

$$\alpha(x, y, t) = \begin{cases} 1, & \text{for } x, y \in [0, 1], \\ 2, & \text{for } x, y \in (1, \infty). \end{cases}$$

Case (i). For $x, y \in [0, 1]$, we have

$$\begin{aligned} & a\beta(x, y, t)F(\min\{M(x, y, t), M(x, Ty, t), M(y, Ty, t)\}) \\ & + b\alpha(x, y, t)\eta(\max\{M(x, y, t), M(x, Ty, t), M(y, Ty, t)\}) \\ & + c\min\{M(x, y, t), M(x, Tx, t), M(y, Ty, t)\} \\ \leq\ & a\beta(x, y, t)F(M(x, Ty, t)) + b\alpha(x, y, t)\eta(M(y, Ty, t)) \\ & + cM(x, Ty, t) \\ \leq\ & \frac{a}{2}F(M(x, Ty, t)) + b\eta(M(Tx, Ty, t)) + cM(x, Ty, t). \end{aligned}$$

For $0 \leq a \leq \frac{1}{4}$, $0 \leq b \leq \frac{1}{2}$ and $0 \leq c < 1$, we get

$$\begin{aligned} & a\beta(x, y, t)F(\min\{M(x, y, t), M(x, Ty, t), M(y, Ty, t)\}) \\ & + b\alpha(x, y, t)\eta(\max\{M(x, y, t), M(x, Ty, t), M(y, Ty, t)\}) \end{aligned}$$

$$+ c\min\{M(x, y, t), M(x, Tx, t), M(y, Ty, t)\}$$

$$\leq M(Tx, Ty, t).$$

Hence T is generalized $\mathcal{H}$-$\mathcal{F}$-contractive, and hence there exists a unique fixed point $x(= 0) \in X$.

Case (ii). For $x, y \in (1, \infty)$, we have

$$a\beta(x, y, t)F(\min\{M(x, y, t), M(x, Ty, t), M(y, Ty, t)\})$$

$$+ b\alpha(x, y, t)\eta(\max\{M(x, y, t), M(x, Ty, t), M(y, Ty, t)\})$$

$$+ c\min\{M(x, y, t), M(x, Tx, t), M(y, Ty, t)\}$$

$$\leq a\beta(x, y, t)F(M(x, Ty, t)) + b\alpha(x, y, t)\eta(M(x, y, t)) + cM(x, Ty, t)$$

$$\leq \frac{a}{2}F(M(x, Ty, t)) + b\eta(M(x, y, t)) + cM(x, Ty, t).$$

For any positive real numbers a, b, c with $a + b + c \leq 1$ and $2b + c < 1$, we get

$$a\beta(x, y, t)F(\min\{M(x, y, t), M(x, Ty, t), M(y, Ty, t)\})$$

$$+ b\alpha(x, y, t)\eta(\max\{M(x, y, t), M(x, Ty, t), M(y, Ty, t)\})$$

$$+ c\min\{M(x, y, t), M(x, Tx, t), M(y, Ty, t)\}$$

$$\geq M(Tx, Ty, t).$$

Hence T is not generalized $\mathcal{H}$-$\mathcal{F}$-contractive in $(1, \infty)$, and there is no fixed point in the interval $(1, \infty)$.

Acknowledgements

The authors sincerely thank the referee for the thorough reading of the manuscript and for the valuable comments and detailed suggestions, which have significantly contributed to improving the quality of the paper.

References

[1] S. C. Arora and V. Sharma, Fixed point theorems for fuzzy mappings, Fuzzy Sets and Systems 110 (2000), 127-130. https://doi.org/10.1016/S0165-0114(97)00366-7.

[2] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations integrals, Fundamenta Mathematicae 3(1) (1922), 133-181. http://eudml.org/doc/213289.

[3] I. Beg and A. Azam, Fixed points of asymptotically regular multivalued mappings, J. Austral. Math. Soc. 53 (1992), 313-326. https://doi.org/10.1017/S1446788700036491.

[4] R. K. Bose and D. Sahani, Fuzzy mappings and fixed point theorems, Fuzzy Sets and Systems 21 (1987), 53-58. https://doi.org/10.1016/0165-0114(87)90152-7.

[5] Y. J. Cho, S. Sedghi and N. Shobe, Generalized fixed point theorems for compatible mappings with some types in fuzzy metric spaces, Chaos Solitons Fractals 39 (2009), 2233-2244. https://doi.org/10.1016/j.chaos.2007.06.108.

[6] A. George and P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets and Systems 64 (1994), 395-399. https://doi.org/10.1016/0165-0114(94)90162-7.

[7] D. Gopal and C. Vetro, Some new fixed point theorems in fuzzy metric spaces, Iranian Journal of Fuzzy Systems 11(3) (2014), 95-107. https://doi.org/10.22111/ijfs.2014.1572.

[8] M. Grabiec, Fixed points in fuzzy metric spaces, Fuzzy Sets and Systems 27 (1988-1989), 385-389. https://doi.org/10.1016/0165-0114(88)90064-4.

[9] V. Gregori and A. Sapena, On fixed point theorems in fuzzy metric spaces, Fuzzy Sets and Systems 125 (2002), 245-252. https://doi.org/10.1016/S0165-0114(00)00088-9.

[10] S. Heilpern, Fuzzy mappings and fixed point theorem, J. Math. Anal. Appl. 83 (1981), 566-569. https://doi.org/10.1016/0022-247X(81)90141-4.

[11] H. Huang, B. Carić, T. Došenović, D. Rakić and M. Brdar, Fixed-point theorems in fuzzy metric spaces via fuzzy F-contraction, Mathematics 9(6) (2021), 641. https://doi.org/10.3390/math9060641.

[12] Koon S. Wong, Zabidin Salleh and Che M. I. Che Taib, Fixed-point results for fuzzy generalized β-F-contraction mappings in fuzzy metric spaces and their applications, Fixed Point Theory and Algorithms for Sciences and Engineering 8 (2023), 2-22. https://doi.org/10.1186/s13663-023-00746-x.

[13] I. Kramosil and J. Michálek, Fuzzy metric and statistical metric spaces, Kybernetika 11 (1975), 326-334. https://www.kybernetika.cz/content/1975/5/336/paper.pdf.

[14] B. S. Lee and S. J. Cho, A fixed point theorems for contractive type fuzzy mappings, Fuzzy Sets and Systems 61 (1994), 309-312. https://doi.org/10.1016/0165-0114(94)90173-2.

[15] D. Mihet, Fuzzy ψ-contractive mappings in non-Archimedean fuzzy metric spaces, Fuzzy Sets and Systems 159 (2008), 739-744. https://doi.org/10.1016/j.fss.2007.07.006.

[16] J. Y. Park and J. U. Jeong, Fixed point theorems for fuzzy mappings, Fuzzy Sets and Systems 87 (1997), 111-116. https://doi.org/10.1016/S0165-0114(96)00013-9.

[17] R. K. Saini and V. Gupta, Common coincidence points of *R*-weakly commuting fuzzy maps, Thai Journal of Mathematics, Mathematical Assoc. of Thailand 6(1) (2008), 109-115. https://thaijmath2.in.cmu.ac.th/index.php/thaijmath/article/view/116.

[18] R. K. Saini and V. Gupta, Fuzzy version of some fixed points theorems on expansion type maps in fuzzy metric space, Thai Journal of Mathematics, Mathematical Assoc. of Thailand 5(2) (2007), 245-252. https://thaijmath2.in.cmu.ac.th/index.php/thaijmath/article/view/90.

[19] B. Schweizer and A. Sklar, Statistical metric spaces, Pacific J. Math. 10 (1960), 313-334. http://dx.doi.org/10.2140/pjm.1960.10.313.

[20] Seong Hoon Cho, On common fixed point theorems in fuzzy metric spaces, Int. Math. Forum 1(10) (2006), 471-479. http://dx.doi.org/10.12988/imf.2006.06038.

[21] S. Sharma, Common fixed point theorems in fuzzy metric spaces, Fuzzy Sets and Systems 127(3) (2002), 345-352. https://doi.org/10.1016/S0165-0114(01)00112-9.

[22] G. Song, Comments on a common fixed point theorem in fuzzy metric spaces, Fuzzy Sets and Systems 135(3) (2003), 409-413. https://doi.org/10.1016/S0165-0114(02)00131-8.

[23] Taixiang Sun, Caihong Han, Guangwang Su, Bin Qin and Lue Li, The periodic points of ε-contractive maps in fuzzy metric spaces, Appl. Gen. Topol. 22(2) (2021), 311-319. https://doi.org/10.4995/agt.2021.14449.

[24] Tatjana Zikic, On fixed point theorems of Gregori and Sapena, Fuzzy Sets and Systems 144 (2004), 421-429. https://doi.org/10.1016/S0165-0114(03)00179-9.

[25] R. Vasuki, A common fixed point theorem in fuzzy metric spaces, Fuzzy Sets and Systems 97 (1998), 395-397. https://doi.org/10.1016/S0165-0114(96)00342-9.

[26] D. Wardowski, Fuzzy contractive mappings and fixed points in fuzzy metric spaces, Fuzzy Sets and Systems 222 (2013), 108-114. https://doi.org/10.1016/j.fss.2013.01.012.

[27] L. Xia and Y. H. Tang, Some fixed point theorems for fuzzy iterated contraction maps in fuzzy metric spaces, Journal of Applied Mathematics and Physics 6 (2018), 224-227. https://doi.org/10.4236/jamp.2018.61021.

[28] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965), 338-353. https://doi.org/10.1016/S0019-9958(65)90241-X.

[29] Akbar Azam, Saqib Hussain and Muhammad Arshad, Common fixed points of Kannan type fuzzy mappings on closed balls, Appl. Math. Inf. Sci. Lett. 1(2) (2013), 7-10. https://www.naturalspublishing.com/Article.asp?ArtcID=3193.

[30] Akbar Azam, Saqib Hussain and Muhammad Arshad, Common fixed points of Chatterjea type fuzzy mappings on closed balls, Neural Comput. and Appl. 21 (2012), 313-317. http://dx.doi.org/10.1007/s00521-012-0907-4.

[31] Ljubomir Ciric, Some new results for Banach contractions and Edelstein contractive mappings on fuzzy metric spaces, Chaos Solitons Fractals 42 (2009), 146-154. http://dx.doi.org/10.1016/j.chaos.2008.11.010.