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ON THE PROPAGATION NUMBER OF SOME GRAPHS

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Abstract

Let $G = (V, E)$ be a graph and k be a positive integer. A set $S \subseteq V$ is a k-forcing set if its vertices are initially colored, while the remaining vertices are initially non-colored, and the graph is subjected to the following color change rule such that all vertices in G will eventually become colored. A colored vertex with at most k non-colored neighbors will cause each non-colored neighbor to become colored. The k-forcing number, denoted by $F_k(G)$, is the cardinality of a smallest k-forcing set. Let $v \in V$. The propagation number of G relative to v, denoted by $\rho_v(G)$, is the smallest k such that $\{v\}$ is a k-forcing set of G. This study also gives the propagation number of paths, cycles, complete graphs, the join of some classes of graphs, the corona of some classes of graphs, the Cartesian product of some classes of graphs, wheel-related graphs, cycle-related graphs, and uniform n-star split graphs.

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Introduction

A set of vertices of a graph is a *k*-forcing set if its vertices are initially colored, while the remaining vertices are initially non-colored, and the graph is subjected to the following color change rule until all the vertices will eventually become colored. A colored vertex with at most *k* non-colored neighbors will cause each non-colored neighbor to become colored. The *k*-forcing number of a graph is the cardinality of a smallest *k*-forcing set. Moreover, if *v* is a vertex of a graph *G*, then the propagation number of *G* relative to *v* is the smallest *k* such that $\{v\}$ is a *k*-forcing set of *G*.

The *k*-forcing concept is a generalization of the *zero forcing number* of a graph (the zero forcing number is the 1-forcing number). The concept was introduced by Barioli et al. [2] and independently, by Burgarth et al. [4]. These concepts have been studied in [1-25].

On the other hand, the concept of the propagation number is new. This study is its first investigation.

Let $k \in \mathbb{N}$ be a positive integer. A set $S \subseteq V$ is a *k-forcing set* if its vertices are initially colored, while the remaining vertices are initially non-colored, and the graph is subjected to the following color change rule such that all vertices in *G* will eventually become colored. A colored vertex with at most *k* non-colored neighbors will cause each of its non-colored neighbors to become colored. The *k-forcing number*, denoted by $F_k(G)$, is the cardinality of a smallest *k*-forcing set [23-25].

For example, consider graph G' in Figure 1. Then $S_1 = \{a\}$ is a 2-forcing set, while $S_2 = \{b\}$ is not. The 2-forcing number of G' is 1.

To see this, we note that *a* can 2-force *b* and *f*, *b* can 2-force *c* and *e*, and *c* can 2-force *d*. Hence, all the vertices of G' will eventually be colored. Thus, $S_1 = \{a\}$ is a 2-forcing set.

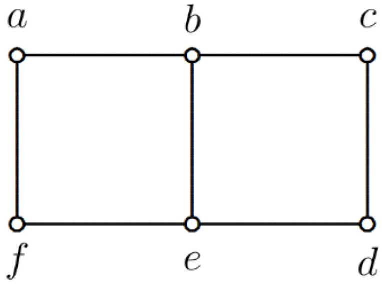


Figure 1. The graph G'.

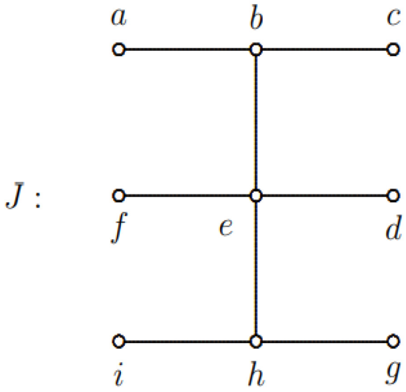
On the other hand, we observe that b cannot 2-force either a, e or c. Hence, color change cannot take effect. This shows that $S_2 = \{b\}$ is not a 2-forcing set.

Clearly, $S_1 = \{a\}$ is a minimum 2-forcing set. Thus, $F_2(G') = 1$.

Let $G = (V, E)$ be a connected graph and $v \in V$. The propagation number of G relative to v, denoted by $\rho_v(G)$, is the smallest k such that $\{v\}$ is a k-forcing set of G.

Consider again the graph G' in Figure 1. Note that $\{a\}$ is a 2-forcing set but not a 1-forcing set. Hence, $\rho_a(G') = 2$.

For example, consider graph J in Figure 1. Then $\rho_a(J) = 3$, and $\rho_e(J) = 4$.



To see this, we note that $\{a\}$ is neither a 1-forcing set, nor a 2-forcing set. However, it is a 3-forcing set. To see this, note that *a* 3-forces *b*; *b* 3-forces *c* and *e*; *e* 3-forces *d*, *f*, and *h*; and finally, *h* 3-forces *i*, and *g*. These sequences of actions color the entire vertex set. Hence, $\{a\}$ is a 3-forcing set. Since $\{a\}$ is neither a 1-forcing set, nor a 2-forcing set, it follows that 3 is the least integer *k* such that $\{a\}$ is a *k*-forcing set. Therefore, $\rho_a(J) = 3$.

Note that $\{e\}$ is neither a 1-forcing set, nor a 2-forcing set, nor a 3-forcing set. However, it is a 4-forcing set. To see this, we note that *e* 4-forces *b*, *d*, *f*, *h*; *b* 4-forces *a* and *c*; and finally, *h* 4-forces *i* and *g*. These sequences of actions color the entire vertex set. Hence, $\{e\}$ is a 4-forcing set. Since $\{e\}$ is neither a 1-forcing set, nor a 2-forcing set, nor a 3-forcing set, it follows that 4 is the least integer *k* such that $\{e\}$ is a *k*-forcing set. Therefore, $\rho_e(J) = 4$. □

1. Propagation Number

1.1. Propagation number of paths

Remark 1.1. Let $G = (V, E)$ be a graph and $v \in V$. If $k \leq \delta(v)$, then the color-change rule cannot start from *v*.

Theorem 1.1. *Let P_n be a path of order n. Then*

$$\rho_v(P_n) = \begin{cases} 1, & \text{if } \delta(v) = 1 \\ 2, & \text{otherwise.} \end{cases}$$

Proof. Let $P_n = (1, 2, ..., n)$ be a path of order *n*. Let $v \in V(P_n)$, and consider the following cases:

Case 1. $\delta(v) = 1$

Without loss of generality, we assume that $v = 1$. Then 1 can 1-force 2, 2 can 1-force 3, and so on, $n - 1$ can 1-force *n*. Hence, all the vertices of P_n

will eventually be colored. Thus, $\{1\}$ is a 1-forcing set. Thus, 1 is the smallest k such that $\{1\}$ is a 1-forcing set of P_n. Therefore, for this case, $\rho_v(P_n) = 1$.

Case 2. $\delta(v) = 2$

Without loss of generality, we assume that $v = 2$. Then, 2 can 2-force 1 and 3, 3 can 2-force 3, and so on, $n - 1$ can 2-force n. Hence, all the vertices of P_n will eventually be colored. Thus, $\{2\}$ is a 2-forcing set. By Remark 1.1, $\{2\}$ cannot be a 1-forcing set since it has two neighbors. Thus, 2 is the smallest k such that $\{2\}$ is a 2-forcing set of P_n. Therefore, for this case, $\rho_v(P_n) = 2$.

Remark 1.2. Let G be a graph. Then every k-forcing set in G is also a $k + 1$-forcing set.

Corollary 1.1. *Let G be a graph. Then* $F_k(G) \geq F_{k+1}(G)$ *for all* $k \in \mathbb{N}$.

Proof. Let $\mathcal{A}$ be the set of all k-forcing sets of G and $\mathcal{B}$ be the set of all $k + 1$-forcing sets of G. By Remark 1.2, every k-forcing set in G is also a $k +$ 1 forcing set. Hence, $\mathcal{A} \subseteq \mathcal{B}$. Thus, $\min\{|A| : A \in \mathcal{A}\} \geq \min\{|B| : B \in \mathcal{B}\}$, that is, $F_k(G) \geq F_{k+1}(G)$.

Remark 1.3. Let G be a connected graph. If S is a singleton k-forcing set of G, then $\delta(G) \leq k$.

1.2. Propagation number of cycles

Theorem 1.2. *Let* C_n *be a cycle of order n. Then* $\rho_v(C_n) = 1$ *for all* $v \in V(C_n)$.

Proof. Let $C_n = [1, 2, ..., n]$ be a cycle of order n, and $v \in V(C_n)$. Without loss of generality, we assume that $v = 1$. Then, 1 can 2-force 2 and n; 2 can 2-force 3, and at the same time n can 2-force $n - 1$; and so on. Hence, all the vertices of P_n will eventually be colored. Thus, $\{1\}$ is a

2-forcing set. By Remark 1.1, $\{1\}$ cannot be a 1-forcing set since it has two neighbors. Therefore, 2 is the smallest k such that $\{1\}$ is a 2-forcing set of C_n. Accordingly, $\rho_v(C_n) = 2$. □

1.3. Propagation number of complete graphs

Corollary 1.3. *Let* K_n *be a complete graph of order* n. *Then* $\rho_v(K_n) = n - 1$ *for all* $v \in V(K_n)$.

Proof. Let $K_n = (\{1, 2, \ldots, n\}, \{uv : u, v \in \{1, 2, \ldots, n\}\})$ be a complete graph of order n, and $v \in V(K_n)$. Without loss of generality, we assume that $v = 1$. Then, 1 can $n-1$-force vertices 1, 2, ..., n. Hence, all the vertices of K_n will eventually be colored. Thus, $\{1\}$ is an $n-1$-forcing set. By Remark 1.1, $\{1\}$ cannot be a k-forcing $(\text{with } k < n - 1)$ set since it has $n-1$ neighbors. Therefore, $n-1$ is the smallest k such that $\{1\}$ is an $n-1$-forcing set of K_n. Accordingly, $\rho_v(K_n) = n - 1$. □

1.4. Propagation number of the join of graphs

Theorem 1.4. *Let G and H be graphs of orders m and n, respectively. If* $v \in V(G)$, *then* $\rho_v(G + H) = \min\{m - \deg_G(v), n + \deg_G(v)\}$.

Proof. Let $v \in V(G)$, and consider the following cases:

Case 1. $m - \deg_G(v) > n + \deg_G(v)$

Let $k' = m - \deg_G(v)$. Then v can k-force the vertices of H. Let y be a vertex of H. Then y can k-force the vertices in $V(G)\backslash N_G(v)$. Hence, all the vertices of $G + H$ will eventually be colored. Thus, $\{v\}$ is a k'-forcing set. By Remark 1.1, $\{v\}$ cannot be a $k' - 1$-forcing set since y has k' neighbors. Therefore, k' is the smallest k such that $\{v\}$ is a k-forcing set of $G + H$.

Case 2. $m - \deg_G(v) \leq n + \deg_G(v)$

Let $k' = n - 1.$ Then *v* can *k*-force the vertices of *H*. Let *y* be a vertex of *H*. Then *y* can *k*-force the vertices in $V(G)\backslash N_G(v).$ Hence, all the vertices of $G + H$ will eventually be colored. Thus, $\{v\}$ is a k'-forcing set. By Remark 1.1, $\{v\}$ cannot be a $k' - 1$-forcing set since *v* has k' neighbors. Therefore k' is the smallest *k* such that $\{v\}$ is a *k*-forcing set of $G + H.$

Accordingly, $\rho_v(G + H) = \min\{m - \deg_G(v), n + \deg_G(v)\}.$

Corollary 1.4.1. *Let* $K_{m,n}$ *be a complete bipartite graph* $\overline{K_m} + \overline{K_n}.$ *If* $v \in V(\overline{K_m}),$ *then* $\rho_v(K_{m,n})\min\{m - 1, n\}.$

Corollary 1.4.2. *Let* $K_{1,n}$ *be a star of order* $n + 1.$ *Then*

$$\rho_v(K_{1,n}) = \begin{cases} n, & \textit{if } \delta(v) = n \\ n - 1, & \textit{otherwise.} \end{cases}$$

1.5. Propagation number of the corona of graphs

Theorem 1.5. *Let G and H be graphs of orders m and n, respectively. Then* $\rho_v(G \circ H) = \rho_v(G) + n$ *for all* $v \in G \circ H.$

Proof. Let $v \in V(G \circ H),$ and consider the following cases:

Case 1. $v \in V(G)$

Let $k' = \rho_v(G) + n.$ Then initially *v* can k'-force its neighbors. Now, every time the color-change rule colors an *i*th vertex in *G*, it successively can k'-force all the vertices of the *i*th copy of *H*. Hence, all the vertices of $G \circ H$ will eventually be colored. Thus, $\{v\}$ is a k'-forcing set. By Remark 1.1, $\{v\}$ cannot be a $k' - 1$-forcing set, since *v* has k' neighbors. Therefore, k' is the smallest *k* such that $\{v\}$ is a *k*-forcing set of $G \circ H.$

Case 2. $v \in V(H)$

Let $k' = \rho_v(G) + n.$ Then initially *v* can k'-force its neighbors. Now, every time the color-change rule colors an ith vertex in G, it successively can k'-force all the vertices of the ith copy of H. Hence, all the vertices of $G \circ H$ will eventually be colored. Thus, $\{v\}$ is a k'-forcing set. By Remark 1.1, $\{v\}$ cannot be a $k' - 1$-forcing set since v has k' neighbors. Therefore, k' is the smallest k such that $\{v\}$ is a k-forcing set of $G \circ H$.

Accordingly, $\rho_v(G \circ H) = \rho_v(G) + n$. ☐

1.6. Propagation number of the Cartesian product of some classes of graphs

Remark 1.6. Let $G = (V, E)$ be a graph with maximum degree Δ. Then $\rho_v(G) \leq \Delta$ for all $v \in V$. Equality holds when $\deg(v) = \Delta$.

Theorem 1.6.1. *Let P_m and P_n be paths of orders m and n, respectively. Then*

$$\rho_v(P_m \times P_n) = \begin{cases} 2, & \textit{if } \deg(v) = 2 \\ 3, & \textit{if } \deg(v) = 3 \\ 4, & \textit{if } \deg(v) = 4. \end{cases}$$

Proof. Let $P_m = (1, 2, ..., m)$ and $P_n = (1, 2, ..., n)$ be paths of orders m and n, respectively. Without loss of generality, suppose that $m \leq n$. Let $S = \{(1, 1), (2, 1), ..., (m, 1)\}$. Then for all $i = 1, 2, ..., m$, $(i, 1)$ can 1-force $(i, 2)$, $(i, 2)$ can 1-force $(i, 3)$, ..., $(i, n-1)$ can 1-force (i, n). Hence, S is a 1-forcing set. Thus, $F_1(P_m \times P_n) \leq \min\{m, n\}$. Let $S = \{(1, 1)\}$. It can easily be shown that S is a 2-forcing set. Hence, $F_2(P_m \times P_n) = 1$. By Corollary 1.1, $F_k(P_m \times P_n) = 1$ for all positive integers $k \geq 2$. Hence, by Remark 1.6 the theorem follows. ☐

Theorem 1.6.2. *Let C_m and C_n be cycles of orders m and n, respectively. Then* $\rho_v(C_m \times C_n) = 4$.

Proof. Note that $C_m \times C_n$ is 4-regular. Let $vw \in E$ and $S = \{v, w\}$. Let $u \in G \backslash S$. Since G is connected, there exists a shortest path $(v = v_1, v_2, \ldots, v_n = u)$ in G connecting v and u. Note that, v_1 can $(\Delta - 1)$-force v_2, v_2 can $(\Delta - 1)$-force $v_3, \ldots, v_{n-1}$ can $(\Delta - 1)$-force $v_n = u$. Hence, u will eventually be colored. Since u is arbitrary, this implies all the vertices of G will eventually be colored. Therefore, S is a $(\Delta - 1)$-forcing set. Thus, $F_{\Delta-1}(G) \leq 2$. Suppose $F_{\Delta-1}(G) = 1$. Let S be a minimum $(\Delta - 1)$-forcing set, say $S = \{v\}$. Let $u \in N(S) \backslash S$. Since $\deg_G(v) = \Delta$, v cannot $(\Delta - 1)$-force u, that is, the color-change rule cannot take effect. This is a contradiction. Therefore, $F_{\Delta-1}(G) \geq 2$. Accordingly, $F_{\Delta-1}(G) = 2$. Hence, $F_3(G) = 2$. Thus, by Remark 1.6, $\rho_v(C_m \times C_n) = 4$. □

1.7. Propagation number of wheel-related graphs

Theorem 1.7.1. *Let* SF_n *be a sunflower graph of order* $2n + 1$. *If* $\deg(v) = 2$, *then* $\rho_v(SF_n) = 3$.

Proof. Let SF_n be the sunflower graph of order $2n + 1$ obtained from $W_n = (\{v\}, \varnothing) + [v_1, v_2, \ldots, v_n]$ by adding vertices u_i joined by edges to vertices v_i and $v_{i+1 (\operatorname{mod} n)}$ for $i = 1, 2, \ldots, n$. Let $S = \{u_1\}$ and consider the following cases.

Case 1. n is even

If n is even, then u_1 can 3-force v_1 and v_2; v_1 can 3-force u_n and v_n, and v_2 can 3-force u_2, v_3, and v; v_n can 3-force u_{n-1} and v_{n-1}, and v_3 can 3-force u_3 and v_4; and so on. Until eventually, $v_{(n+4)/2}$ can 3-force $u_{(n+2)/2}$, and $v_{n/2}$ can 3-force $u_{n/2}$ and $v_{(n+2)/2}$.

Case 2. n is odd

If n is odd, then u_1 can 3-force v_1 and v_2; v_1 can 3-force u_n and v_n, and v_2 can 3-force u_2, v_3, and v; v_n can 3-force u_{n-1} and v_{n-1}, and v_3

can 3-force u_3 and v_4; and so on. Until eventually, $v_{(n+3)/2}$ can 3-force $u_{(n+1)/2}$, and $v_{\lfloor n/2 \rfloor}$ can 3-force $u_{\lfloor n/2 \rfloor}$ and $v_{(n+1)/2}$.

In any case, all the vertices of SF_n will eventually be colored. Hence, S is a 3-forcing set. Thus, $F_2(SF_n) = 1$. By Corollary 1.1, $F_k(SF_n) = 1$ for all positive integers $k \geq 3$. In any case, all the vertices of SF_n will eventually be colored. Hence, S is a 2-forcing set. Thus, $F_2(SF_n) \leq 2$. By Remark 1.3, a 2-forcing set of SF_n cannot be singleton. Therefore, $F_2(SF_n) = 2$. Stating that $F_2(SF_n) = 2$, $\rho_v(SF_n)$ cannot be 2. Hence, $\rho_v(SF_n) = 3$. □

Theorem 1.7.2. *Let* SF_n *be a sunflower graph of order* $2n + 1$. *Then*

$$\rho_v(SF_n) = \begin{cases} 5, & \text{if } \deg(v) = 5 \\ n, & \text{if } \deg(v) = n. \end{cases}$$

Proof. Let SF_n be the sunflower graph of order $2n + 1$ obtained from $W_n = (\{v\}, \varnothing) + [v_1, v_2, \ldots, v_n]$ by adding vertices u_i joined by edges to vertices v_i and $v_{i+1(\bmod n)}$ for $i = 1, 2, \ldots, n$. Let $S = \{u_1\}$ and consider the following cases.

Case 1. *n* is even

If *n* is even, then u_1 can 3-force v_1 and v_2; v_1 can 3-force u_n and v_n, and v_2 can 3-force u_2, v_3, and v; v_n can 3-force u_{n-1} and v_{n-1}, and v_3 can 3-force u_3 and v_4; and so on. Until eventually, $v_{(n+4)/2}$ can 3-force $u_{(n+2)/2}$, and $v_{n/2}$ can 3-force $u_{n/2}$ and $v_{(n+2)/2}$.

Case 2. *n* is odd

If *n* is odd, then u_1 can 3-force v_1 and v_2; v_1 can 3-force u_n and v_n, and v_2 can 3-force u_2, v_3, and v; v_n can 3-force u_{n-1} and v_{n-1}, and v_3 can 3-force u_3 and v_4; and so on. Until eventually, $v_{(n+3)/2}$ can 3-force $u_{(n+1)/2}$, and $v_{\lfloor n/2 \rfloor}$ can 3-force $u_{\lfloor n/2 \rfloor}$ and $v_{(n+1)/2}$.

In any case, all the vertices of SF_n will eventually be colored. Hence, S is a 3-forcing set. Thus, $F_2(SF_n) = 1.$ By Corollary 1.1, $F_k(SF_n) = 1$ for all positive integers $k \geq 3.$ Hence, by Remark 1.6, the theorem follows. □

Theorem 1.7.3. *Let* LC_n *be a lotus-inside-circle graph of order* $2n + 1.$ *If* $\deg(v) = 3,$ *then* $\rho_v(LC_n) = 3.$

Proof. Let $C_n = [v_1, v_2, ..., v_n]$ be a cycle of order n, and $K_{1,n} = (\{u\}, \varnothing) + (\{u_1, u_2, ..., u_n\}, \varnothing)$ be a star of order $n + 1.$ Let LC_n be the lotus inside circle graph of order $2n + 1$ obtained from C_n and $K_{1,n}$ by joining each vertex u_i to vertices v_i and $v_{i+1(\mathrm{mod}\, n)}$ for $i = 1, 2, ..., n.$ Let $S = \{u_1\}$ and consider the following cases.

Case 1. n is even

If n is even, then u_1 can 3-force v, v_1 and v_2; v_1 can 3-force u_n and v_n, and v_2 can 3-force u_2 and v_3; v_n can 3-force u_{n-1} and v_{n-1}, and v_3 can 3-force u_3 and v_4; and so on. Until eventually, $v_{(n+4)/2}$ can 3-force $u_{(n+2)/2}$, and $v_{n/2}$ can 3-force $u_{n/2}$ and $v_{(n+2)/2}$.

Case 2. n is odd

If n is odd, then u_1 can 3-force v, v_1 and v_2; v_1 can 3-force u_n and v_n, and v_2 can 3-force u_2 and v_3; v_n can 3-force u_{n-1} and v_{n-1}, and v_3 can 3-force u_3 and v_4; and so on. Until eventually, $v_{(n+3)/2}$ can 3-force $u_{(n+1)/2}$, and $v_{\lfloor n/2 \rfloor}$ can 3-force $u_{\lfloor n/2 \rfloor}$ and $v_{(n+1)/2}$.

In any case, all the vertices of LC_n will eventually be colored. Hence, S is a 3-forcing set. Thus, $F_3(LC_n) = 1.$ By Corollary 1.1, $F_k(LC_n) = 1$ for all positive integers $k \geq 3.$

In any case, all the vertices of LC_n will eventually be colored. Hence, S

is a 2-forcing set. Thus, $F_2(LC_n) \leq 2.$ By Remark 1.3, a 2-forcing set of SF_n cannot be singleton. Therefore, $F_2(LC_n) = 2.$ Stating that $F_2(LC_n) = 2,$ $\rho_v(LC_n)$ cannot be 2. Hence, $\rho_v(LC_n) = 3.$ □

Theorem 1.7.4. *Let* LC_n *be a lotus-inside-circle graph of order* $2n+1.$ *If* $\deg(v) = 3,$ *then* $\rho_v(LC_n) = 3.$

$$\rho_v(LC_n) = \begin{cases} 4, & \text{if } \deg(v) = 4 \\ n, & \text{if } \deg(v) = n. \end{cases}$$

Proof. By Theorem 1.7.3, $F_k(SF_n) = 1$ for all $k \geq 3.$ Hence, by Remark 1.6 the theorem follows. □

Theorem 1.7.5. *Let* H_n *be a helm graph of order* $2n+1.$ *If* $\deg(v) = 2,$ *then* $\rho_v(H_n) = 3.$

Proof. Let H_n be the helm graph of order $2n+1$ obtained from the wheel $W_n = (\{v\}, \varnothing) + [v_1, v_2, ..., v_n]$ by attaching pendant edges v_iu_i for $i = 1, 2, ..., n.$ Let $S = \{u_1\}$ and consider the following cases.

Case 1. *n* is even

If *n* is even, then u_1 can 3-force $v_1;$ v_1 can 3-force $v,$ v_n and $v_2;$ v_2 can 3-force u_2 and $v_3,$ and v_n can 3-force u_n and $v_{n-1};$ v_3 can 3-force u_3 and $v_4,$ and v_{n-1} can 3-force u_{n-1} and $v_{n-2};$ and so on. Until eventually, $v_{(n+2)/2}$ can 3-force $u_{(n+2)/2}.$

Case 2. *n* is odd

If *n* is even, then u_1 can 3-force $v_1;$ v_1 can 3-force $v,$ v_n and $v_2;$ v_2 can 3-force u_2 and $v_3,$ and v_n can 3-force u_n and $v_{n-1};$ v_3 can 3-force u_3 and $v_4,$ and v_{n-1} can 3-force u_{n-1} and $v_{n-2};$ and so on. Until eventually, $v_{\lceil n/2 \rceil}$ can 3-force $u_{\lceil n/2 \rceil}.$

In any case, all the vertices of H_n will eventually be colored. Hence, S is a 3-forcing set. Thus, $F_3(H_n) = 1.$ By Corollary 1.1, $F_k(H_n) = 1$ for all positive integers $k \geq 3.$

Also, let H_n be the helm graph of order $2n + 1$ obtained from the wheel $W_n = (\{v\}, \varnothing) + [v_1, v_2, ..., v_n]$ by attaching pendant edges v_iu_i for $i = 1, 2, ..., n.$ Let $S = \{v, u_1\}$ and consider the following cases.

Case 1. n is even

If n is even, then u_1 can 2-force v_1; v_1 can 2-force v_n and v_2; v_2 can 2-force u_2 and v_3, and v_n can 2-force u_n and v_{n-1}; v_3 can 2-force u_3 and v_4, and v_{n-1} can 2-force u_{n-1} and v_{n-2}; and so on. Until eventually, $v_{(n+2)/2}$ can 2-force $u_{(n+2)/2}$.

Case 2. n is odd

If n is even, then u_1 can 2-force v_1; v_1 can 2-force v_n and v_2; v_2 can 2-force u_2 and v_3, and v_n can 2-force u_n and v_{n-1}; v_3 can 2-force u_3 and v_4, and v_{n-1} can 2-force u_{n-1} and v_{n-2}; and so on. Until eventually, $v_{\lceil n/2 \rceil}$ can 2-force $u_{\lceil n/2 \rceil}$.

In any case, all the vertices of H_n will eventually be colored. Hence, S is a 2-forcing set. Thus, $F_2(H_n) \leq 2.$ By Remark 1.3, a 2-forcing set of H_n cannot be singleton. Therefore, $F_2(H_n) = 2.$ Stating that $F_2(H_n) = 2,$ $\rho_v(H_n)$ cannot be 2. Hence, $\rho_v(H_n) = 3.$ □

Theorem 1.7.6. *Let* H_n *be a helm graph of order* $2n + 1.$ *Then*

$$\rho_v(H_n) = \begin{cases} 4, & \text{if } \deg(v) = 4 \\ n, & \text{if } \deg(v) = n. \end{cases}$$

Proof. By Theorem 1.7.5, $F_k(H_n) = 1$ for all $k \geq 3.$ Hence, by Remark 1.6, the theorem follows. □

Theorem 1.7.7. *Let* G_n *be a gear graph of order* $2n+1$. *If* $\deg(v) = 2$, *then* $\rho_v(G_n) = 2$.

Proof. Let G_n be the gear graph obtained from $W_n = (\{v\}, \varnothing) + [v_1, v_2, ..., v_n]$ by adding vertices u_i in between adjacent vertices v_i and $v_{i+1(\text{mod}\, n)}$ for $i = 1, 2, ..., n$. Let $S = \{v, u_1\}$ and consider the following cases.

Case 1. *n* is even

If *n* is even, then u_1 can 2-force v_1 and v_2; v_1 can 2-force u_n and *v*, and v_2 can 2-force u_2; u_n can 2-force v_n, and v_3 can 2-force u_3; and so on. Until eventually, $v_{(n+2)/2}$ can 2-force $u_{(n+2)/2}$.

Case 2. *n* is odd

If *n* is odd, then u_1 can 2-force v_1 and v_2; v_1 can 2-force u_n and v_n, and v_2 can 2-force u_2 and v_3; v_n can 2-force u_{n-1} and v_{n-1}, and v_3 can 2-force u_3 and v_4; and so on. Until eventually, $v_{\lceil n/2 \rceil}$ can 2-force $u_{\lceil n/2 \rceil}$.

In any case, all the vertices of G_n will eventually be colored. Hence, *S* is a 2-forcing set. Thus, $F_2(G_n) = 1$. By Corollary 1.1, $F_k(G_n) = 1$ for all positive integers $k \geq 2$.

Also, let G_n be the gear graph obtained from $W_n = (\{v\}, \varnothing) + [v_1, v_2, ..., v_n]$ by adding vertices u_i in between adjacent vertices v_i and $v_{i+1(\text{mod}\, n)}$ for $i = 1, 2, ..., n$. Let $S = \{v, v_1, u_1\}$ and consider the following cases.

Case 1. *n* is even

If *n* is even, then u_1 can 1-force v_2, and v_1 can 1-force u_n; v_2 can 1-force u_2, and u_n can 1-force v_n; u_2 can 1-force v_3, and v_n can 1-force u_{n-1}; and so on. Until eventually, $v_{(n+2)/2}$ can 1-force $u_{(n+2)/2}$.

Case 2. n is odd

If n is odd, then u_1 can 1-force v_2, and v_1 can 1-force u_n; v_2 can 1-force u_2, and u_n can 1-force v_n; u_2 can 1-force v_3, and v_n can 1-force u_{n-1}; and so on. Until eventually, $v_{\lceil n/2 \rceil}$ can 1-force $u_{\lceil n/2 \rceil}$.

In any case, all the vertices of G_n will eventually be colored. Hence, S is a 1-forcing set. Thus, $F_1(G_n) \leq 3$. Note that a 1-forcing set of G_n cannot have less than 3 elements. Therefore, $F_1(G_n) = 3$. Stating that $F_1(G_n) = 3$, $\rho_v(G_n)$ cannot be 1. Hence, $\rho_v(G_n) = 2$. □

Theorem 1.7.8. *Let* G_n *be a gear graph of order* $2n + 1$. *Then*

$$\rho_v(G_n) = \begin{cases} 3, & \text{if } \deg(v) = 3 \\ n, & \text{if } \deg(v) = n. \end{cases}$$

Proof. By Theorem 1.7.7, $F_k(G_n) = 1$ for all $k \geq 2$. Hence, by Remark 1.6, the theorem follows. □

1.8. Propagation number of cycle-related graphs

Theorem 1.8.1. *Let* S_n *be a sun graph of order* $2n$. *If* $\deg(v) = 2$, *then* $\rho_v(S_n) = 1$.

Proof. Let S_n be the sun graph of order $2n$ obtained from $C_n = [v_1, v_1, \ldots, v_n]$ by adding vertices u_i joined by edges to vertices v_i and $v_{i+1(\bmod n)}$ for $i = 1, 2, \ldots, n$. Let $S = \{u_1\}$ and consider the following cases.

Case 1. n is even

If n is even, then u_1 can 2-force v_1 and v_2; v_1 can 2-force u_n and v_n, and v_2 can 2-force u_2 and v_3; v_n can 2-force u_{n-1} and v_{n-1}, and v_3 can 2-force u_3 and v_4; and so on. Until eventually, $v_{(n+4)/2}$ can 2-force $u_{(n+2)/2}$, and $v_{n/2}$ can 2-force $u_{n/2}$ and $v_{(n+2)/2}$.

Case 2. n is odd

If n is odd, then u_1 can 2-force v_1 and v_2; v_1 can 2-force u_n and v_n, and v_2 can 3-force u_2 and v_3; v_n can 2-force u_{n-1} and v_{n-1}, and v_3 can 2-force u_3 and v_4; and so on. Until eventually, $v_{(n+3)/2}$ can 2-force $u_{(n+1)/2}$, and $v_{\lfloor n/2 \rfloor}$ can 2-force $u_{\lfloor n/2 \rfloor}$ and $v_{(n+1)/2}$.

In any case, all the vertices of S_n will eventually be colored. Hence, S is a 2-forcing set. Thus, $F_2(S_n) = 1$. By Corollary 1.1, $F_k(S_n) = 1$ for all positive integers $k \geq 2$.

Also, let S_n be the sun graph of order $2n$ obtained from $C_n = [v_1, v_1, ..., v_n]$ by adding vertices u_i joined by edges to vertices v_i and $v_{i+1(\text{mod}\, n)}$ for $i = 1, 2, ..., n$. Let $S = \{v_1, v_1, ..., v_n\}$. Then for each $i = 1, 2, ..., n$, v_i can 1-force u_i. Hence, all the vertices of S_n will eventually be colored. Hence, S is a 1-forcing set. Note that a 1-forcing set of S_n cannot have less than n elements. Therefore, $F_1(S_n) = n$. Stating that $F_1(S_n) = n$, $\rho_v(S_n)$ cannot be 1. Hence, $\rho_v(S_n) = 2$.

Theorem 1.8.2. *Let* S_n *be a sun graph of order* $2n$. *If* $\deg(v) = 4$, *then* $\rho_v(S_n) = 4$.

Proof. Let L_n be the sunlet graph of order $2n$ obtained from $C_n = [v_1, v_1, ..., v_n]$ by attaching pendant edges $v_i u_i$ for $i = 1, 2, ..., n$. Let $S = \{u_1\}$ and consider the following cases.

Case 1. n is even

If n is even, then u_1 can 2-force v_1; v_1 can 2-force v_n and v_2; v_2 can 2-force u_2 and v_3, and v_n can 2-force u_n and u_{n-1}; v_3 can 2-force u_3 and v_4, and v_{n-1} can 2-force u_{n-1} and v_{n-2}; and so on. Until eventually, $v_{(n+2)/2}$ can 2-force $u_{(n+2)/2}$.

Case 2. n is odd

If n is even, then u_1 can 2-force v_1; v_1 can 2-force v_n and v_2; v_2 can 2-force u_2 and v_3, and v_n can 2-force u_n and v_{n-1}; v_3 can 2-force u_3 and v_4, and v_{n-1} can 2-force u_{n-1} and v_{n-2}; and so on. Until eventually, $v_{\lfloor n/2 \rfloor}$ can 2-force $u_{\lfloor n/2 \rfloor}$.

In any case, all the vertices of L_n will eventually be colored. Hence, S is a 2-forcing set. Thus, $F_2(L_n) = 1$. By Corollary 1.1, $F_k(L_n) = 1$ for all positive integers $k \geq 2$. Hence, by Remark 1.6, the theorem follows. □

Theorem 1.8.3. *Let* L_n *be a sunlet graph of order* $2n$. *If* $\deg(v) = 1$, *then* $\rho_v(L_n) = 2$.

Proof. By Theorem 1.8.2, $F_k(L_n) = 1$ for all $k \geq 2$. Let L_n be the sunlet graph of order $2n$ obtained from $C_n = [v_1, v_1, ..., v_n]$ by attaching pendant edges $v_i u_i$ for $i = 1, 2, ..., n$. Let $S = \{v_i : i \equiv 1 \pmod 4$ or $i \equiv 2 \pmod 4\}$. Without loss of generality, assume that n is even. Then for each i with $i \equiv 1 \pmod 4$ or $i \equiv 2 \pmod 4$, u_i can 1-force v_i; and, v_i can 1-force the vertex in $N(v_i) \backslash S$. Thus, all the vertices of L_n will eventually be colored. Hence, S is a 1-forcing set. Thus, $F_1(L_n) \leq \lceil n/2 \rceil$. Note that a 1-forcing set of L_n cannot have less than $\lceil n/2 \rceil$ elements. Therefore, $F_1(L_n) = \lceil n/2 \rceil$, stating that $F_1(L_n) = \lceil n/2 \rceil$, $\rho_v(L_n)$ cannot be 1. Hence, $\rho_v(L_n) = 2$. □

Theorem 1.8.4. *Let* L_n *be a sunlet graph of order* $2n$. *If* $\deg(v) = 3$, *then* $\rho_v(L_n) = 1$.

Proof. By Theorem 1.8.3, $F_k(L_n) = 1$ for all $k \geq 2$. Hence, by Remark 1.6, the theorem follows. □

1.9. Propagation number of uniform *n*-star split graphs

Theorem 1.9.1. *Let* $K_{1,n} = (\{x\}, \varnothing) + (\{u_1, u_2, ..., u_n\}, \varnothing)$ *be a star of order* $n + 1$, *and* SS_n^r *be the uniform n-star split graph obtained by adding to* $K_{1,n}$ *stars* $K_{1,r} = (\{u_i\}, \varnothing) + (\{v_1, v_1, ..., v_r\}, \varnothing)$ *for each* $i = 1, 2, ..., n$. *Then*

$$\rho_v(SS_n^r) = \begin{cases} \max\{n, r\}, & \text{if } v = x \\ \max\{n-1, r+1\}, & \text{if } v = u_i \text{ for some } i = 1, 2, ..., n \\ \max\{n-1, r\}, & \text{if } v = v_i \text{ for some } i = 1, 2, ..., r. \end{cases}$$

Proof. Let $v \in V(SS_n^r)$, and consider the following cases:

Case 1. $v = x$

Subcase 1. $r \leq n$

If $r \leq n$, then clearly $\{x\}$ is a k-forcing set precisely when $n \leq k$. Thus, $\rho_v(SS_n^r) = n$.

Subcase 2. $n < r$

If $n < r$, then clearly $\{x\}$ is a k-forcing set precisely when $r \leq k$. Thus, $\rho_v(SS_n^r) = r$.

Therefore, for this case, $\rho_v(SS_n^r) = \max\{n, r\}$.

Case 2. $v = u_i$ for some $i = 1, 2, ..., n$

Subcase 1. $r + 1 \leq n - 1$

If $r \leq n$, then clearly $\{u_i\}$ is a k-forcing set precisely when $n - 1 \leq k$. Thus, $\rho_v(SS_n^r) = n - 1$.

Subcase 2. $n - 1 < r + 1$

If $n < r$, then clearly $\{x\}$ is a k-forcing set precisely when $r + 1 \leq k$. Thus, $\rho_v(SS_n^r) = r + 1$.

Therefore, for this case, $\rho_v(SS_n^r) = \max\{n - 1, r + 1\}$.

Case 3. $v = v_j$ for some $j = 1, 2, \ldots, r$

Subcase 1. $r \leq n - 1$

If $r \leq n$, then clearly $\{v_j\}$ is a k-forcing set precisely when $n - 1 \leq k$. Thus, $\rho_v(SS_n^r) = n - 1$.

Subcase 2. $n - 1 < r$

If $n < r$, then clearly $\{x\}$ is a k-forcing set precisely when $r \leq k$. Thus, $\rho_v(SS_n^r) = r$.

Therefore, for this case, $\rho_v(SS_n^r) = \max\{n - 1, r\}$. □

Theorem 1.9.2. *Let* $K_{1,n} = (\{x\}, \varnothing) + (\{u_1, u_2, \ldots, u_n\}, \varnothing)$ *be a star of order* $n + 1$, *and* $SS(n, r)$ *be the graph obtained by adding to* $K_{1,n}$ *edges* $u_i w_i$ *and stars* $K_{1,r} = (\{w_i\}, \varnothing) + (\{v_1, v_2, \ldots, v_r\}, \varnothing)$ *for each* $i = 1, 2, \ldots, n$. *Then*

$$\rho_v(SS(n, r)) = \begin{cases} \max\{n, r\}, & \textit{if } v = x \\ \max\{n - 1, r + 1\}, & \textit{if } v = u_i \textit{ for some } i = 1, 2, \ldots, n \\ \max\{n - 1, r\}, & \textit{if } v = v_j \textit{ for some } j = 1, 2, \ldots, r. \end{cases}$$

Proof. Let $v \in V(SS(n, r))$, and consider the following cases:

Case 1. $v = x$

Subcase 1. $r \leq n$

If $r \leq n$, then clearly $\{x\}$ is a k-forcing set precisely when $n \leq k$. Thus, $\rho_v(SS(n, r)) = n$.

Subcase 2. $n < r$

If $n < r$, then clearly $\{x\}$ is a k-forcing set precisely when $r \leq k$. Thus, $\rho_v(SS(n, r)) = r$.

Therefore, for this case, $\rho_v(SS(n, r)) = \max\{n, r\}$.

Case 2. $v = u_i$ for some $i = 1, 2, \ldots, n$

Subcase 1. $r + 1 \leq n - 1$

If $r \leq n$, then clearly $\{u_i\}$ is a k-forcing set precisely when $n - 1 \leq k$. Thus, $\rho_v(SS(n, r)) = n - 1$.

Subcase 2. $n - 1 < r + 1$

If $n < r$, then clearly $\{x\}$ is a k-forcing set precisely when $r + 1 \leq k$. Thus, $\rho_v(SS(n, r)) = r + 1$.

Therefore, for this case, $\rho_v(SS(n, r)) = \max\{n - 1, r + 1\}$.

Case 3. $v = v_j$ for some $j = 1, 2, \ldots, r$

Subcase 1. $r \leq n - 1$

If $r \leq n$, then clearly $\{v_j\}$ is a k-forcing set precisely when $n - 1 \leq k$. Thus, $\rho_v(SS(n, r)) = n - 1$.

Subcase 2. $n - 1 < r$

If $n < r$, then clearly $\{x\}$ is a k-forcing set precisely when $r \leq k$. Thus, $\rho_v(SS(n, r)) = r$. Therefore, for this case, $\rho_v(SS(n, r)) = \max\{n - 1, r\}$. $\square$

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