Far East Journal of Mathematical Sciences (FJMS)
© 2026 Pushpa Publishing House, Prayagraj, India
https://pphmjopenaccess.com
https://doi.org/10.17654/0972087126028 **P-ISSN: 0972-0871**
Volume 143, Number 2, February 2026, Pages 489-502 **E-ISSN: 2584-1246**

HILBERT C^*-MODULE VALUED FUNCTIONS AND MULTIPLICATION OPERATORS

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Received: September 1, 2025; Accepted: October 1, 2025
2020 Mathematics Subject Classification: Primary 47B01; Secondary 46L08, 46L05.
Keywords and phrases: Hilbert C^*-module, compact operator, multiplication operator, multipliers.
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Communicated by K. K. Azad

How to cite this article: Mawoussi Todjro and Yaogan Mensah, Hilbert C^*-module valued functions and multiplication operators, Far East Journal of Mathematical Sciences (FJMS) 143(2) (2026), 489-502. https://doi.org/10.17654/0972087126028

Published Online: December 25, 2025

Abstract

Let M be a Hilbert module over a C^*-algebra A and let X be a locally compact space with a Radon measure. In this article, we introduce an A-valued product for M-valued functions on X which arises from the A-valued inner product of M. We associate a multiplication operator with the aforementioned product and investigate some of their major properties. Notably, we prove that every multiplication operator can be approximated by a net of elementary A-compact operators. Furthermore, assuming that X is a locally compact group, we provide a necessary and sufficient condition under which the multiplication operators are multipliers for integrable functions.

1. Introduction

The Hilbert C^*-module structure is a mathematical concept which generalizes in a natural way both the Hilbert space structure and the C^*-algebra structure. Hilbert C^*-modules were introduced by Kaplansky [5] who used them to establish that a derivation of every type I AW^*-algebra is inner. Paschke extended the notion to noncommutative C^*-algebras [9]. Subsequently, Rieffel formalized the concept of Hilbert C^*-module and used it as a toolkit to construct induced representations of C^*-algebras [11]. Some properties of Hilbert spaces are lost with Hilbert C^*-modules. The decomposition of a Hilbert C^*-module as an orthogonal summand of closed submodules, as well as the polar decomposition of an adjointable operator between Hilbert C^*-modules, may not hold. Hilbert modules play a crucial role in noncommutative geometry, in KK-theory and in Morita equivalence. We refer to [3, 7] to know more about Hilbert C^*-modules.

At the same time, the importance of Lebesgue spaces L^p is well known. They generalize the essential concepts of measures and integration. They permit to formalize the concept of convolution and transformations such as the Fourier transformation and the Laplace transformation and other transformations of the same kind [4, 10]. They are also the natural domain for several classes of linear and nonlinear operators, such as integral operators, compact operators, etc., [2]. They play an important role in the study of the geometry of Banach spaces. The L^p spaces unify fundamental concepts and provide invaluable tools for studying problems in pure and applied mathematics. In this article, we deal with Lebesgue type spaces constructed from the Bochner integral, the so-called Lebesgue-Bochner spaces.

In this paper, we introduce a product on the space of Hilbert C^*-module valued integrable functions on a locally compact space, generated from the A-valued inner product, since one cannot handle the point-wise product of these functions, because there is no such product on a Hilbert module. We also define a multiplication operator relative to the defined product and investigate some operator properties. Finally, we give a necessary and sufficient condition for which a multiplication operator is a multiplier. The paper is organized as follows. Section 1 addresses the definitions and the notations we will need in the sequel. Section 2 outlines our main findings.

2. Preliminaries and Notations

A Hilbert C^*-module over A is a right A-module M over a C^*-algebra A, equipped with an A-valued sesquilinear map $\langle \cdot, \cdot \rangle : M \times M \to A$ such that the following properties hold for all $x, y \in M$ and $a \in A$:

(i) A-linearity: $\langle x, ya \rangle = \langle x, y \rangle a$,

(ii) hermiticity: $\langle x, y \rangle = \langle y, x \rangle^*$,

(iii) positivity in A : $\langle x, x \rangle \geq 0$,

(iv) positive definiteness: $\langle x, x \rangle = 0 \Rightarrow x = 0$,

(v) completion: M is complete with respect to the norm $\| x \|_M = \| \langle x, x \rangle \|_A^{\frac{1}{2}}$, where $\| \cdot \|_A$ is the norm of A.

Remark 2.1. (1) The map $\langle \cdot, \cdot \rangle$ is called an *A-valued inner product*.

(2) If (v) lacks, then M is called a *pre-Hilbert A-module*.

Let us provide some examples.

Example 2.2. (i) Any C^*-algebra A is a Hilbert module over A under the A-valued inner product defined by

$$\langle a, b \rangle = a^* b, \quad a, b \in A.$$

(ii) The space $H_A = \left\{ (a_k) \subset A : \sum_k a_k^* a_k \text{ converges in } A \right\}$ is a Hilbert A-module over the C^*-algebra A. The A-valued inner product is defined by $\langle (a_k), (b_k) \rangle = \sum_k a_k^* b_k$. The space H_A is called the *standard Hilbert* C^*-*module*.

(iii) Let X be a compact space and let dx be a regular Borel measure on X. The space $C(X)$ of complex-valued functions on X is a Hilbert C^*-module over itself with the $C(X)$-valued inner product defined by

$$\langle u, v \rangle = \int_X \overline{u(x)} v(x) dx, \quad u, v \in C(X).$$

Let X be a locally compact space with a Radon measure μ. For all $1 \leq p < \infty$, we designate by $L^p(X, M)$ the space of M-valued Bochner p-th integrable (class of) functions on X. The map

$$\| \cdot \|_{L^p(X, M)} : L^p(X, M) \to \mathbb{R}_+, \quad f \mapsto \left(\int_X \| f(t) \|^p d\mu(t) \right)^{\frac{1}{p}}$$

is a norm on $L^p(X, M)$. The space $L^p(X, M)$ is a Banach space with respect to the norm $\|\cdot\|_{L^p(X,M)}$. Let us denote by $L^\infty(X, M)$ the space of measurable functions $f : X \to M$ such that there exists a positive constant λ such that $\| f(x)\|_M \leq \lambda$ for all $x \in X$ μ-almost everywhere (μ-a.e.). The space $L^\infty(X, M)$ is also a Banach space with respect to the norm $\|\cdot\|_{L^\infty(X,M)}$ defined by

$$\| f \|_{L^\infty(X,M)} = \inf\{\lambda \geq 0 : \| f(x)\|_M \leq \lambda \text{ for } \mu\text{-almost every } x\}.$$

If E and F are Banach spaces and $T : E \to F$ is a bounded linear operator, then the operator norm of T is defined by

$$\| T \| = \sup\{\| Tx \|_F : \| x \|_E \leq 1\}.$$

Furthermore, let $(M, \langle \cdot, \cdot \rangle_E)$ and $(N, \langle \cdot, \cdot \rangle_F)$ be two Hilbert A-modules. An operator $T : M \to N$ is said to be *adjointable* if there is a map $T^* : N \to M$ such that $\langle Tx, y\rangle_N = \langle x, T^*y\rangle_M$ for all $x \in M$ and $y \in N$. The map T^* is called the *adjoint* of T. Every adjointable operator is necessarily A-linear and bounded [6, page 8].

3. Main Results

Let M be a Hilbert module over the C^*-algebra A. Let f and g be M-valued functions on X. We set

$$(f \odot g)(x) = \langle f(x), g(x)\rangle, \quad x \in X. \tag{3.1}$$

The resulting function $f \odot g$ is an A-valued function on X.

Proposition 3.1. *Let* $1 \leq p < \infty$ *and let* q *be such that* $\frac{1}{p} + \frac{1}{q} = 1.$ *If* $f \in L^p(X, M)$ *and* $g \in L^q(X, M)$, *then* $f \odot g \in L^1(X, A)$ *and*

$$\| f \odot g \|_{L^1(X, A)} \leq \| f \|_{L^p(X, M)} \| g \|_{L^q(X, M)}.$$

Proof. Let $f \in L^p(X, M)$ and $g \in L^q(X, M)$. Then, we have

$$\int_X \| (f \odot g)(x) \|_A d\mu(x)$$

$$= \int_X \| \langle f(x), g(x) \rangle \|_A d\mu(x)$$

$$\leq \int_X \| f(x) \|_M \| g(x) \|_M d\mu(x)$$

$$\leq \left(\int_X \| f(x) \|_M^p d\mu(x) \right)^{\frac{1}{p}} \left(\int_X \| g(x) \|_M^q d\mu(x) \right)^{\frac{1}{q}} < \infty,$$

(using Hölder inequality).

Hence, $f \odot g \in L^1(X, A)$ and

$$\| f \odot g \|_{L^1(X, A)} \leq \| f \|_{L^p(X, M)} \| g \|_{L^q(X, M)}. \qquad \square$$

Proposition 3.2. *Let $1 \leq p < \infty$ and let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence in $L^p(X, M)$ which converges to f in $L^p(X, M)$. Then, for all $g \in L^q(X, M)$, the sequence $(f_n \odot g)_{n \in \mathbb{N}}$ converges to $f \odot g$ in $L^1(X, M)$.*

Proof. We have

$$\| f_n \odot g - f \odot g \|_{L^1(X, A)} = \int_X \| \langle f_n(x), g(x) \rangle - \langle f(x), g(x) \rangle \|_A d\mu(x)$$

$$= \int_X \| \langle f_n(x) - f(x), g(x) \rangle \|_A d\mu(x)$$

$$\leq \| f_n - f \|_{L^p(X, M)} \| g \|_{L^q(X, M)}.$$

The result is obtained since $\| f_n - f \|_{L^p(X, M)}$ tends to 0 as n goes to infinity. $\square$

Proposition 3.3. *Let* $(f_n, g_n)_{n\in\mathbb{N}}$ *be a sequence in* $L^p(X, M)\times L^q(X, M)$ *such that* $(f_n)_{n\in\mathbb{N}}$ *converges to f in* $L^p(X, M)$ *and* $(g_n)_{n\in\mathbb{N}}$ *converges to g in* $L^q(X, M)$. *Then, the sequence* $(f_n \odot g_n)_{n\in\mathbb{N}}$ *converges to* $f \odot g$ *in* $L^1(X, A)$.

Proof. We have

$$\begin{aligned}
&\| f_n \odot g_n - f \odot g \|_{L^1(X, A)}\\
&= \| f_n \odot g_n - f_n \odot g + f_n \odot g - f \odot g \|_{L^1(X, A)}\\
&\le \| f_n \odot g_n - f_n \odot g \|_{L^1(X, A)} + \| f_n \odot g - f \odot g \|_{L^1(X, A)}\\
&\le \| f_n \|_{L^p(X, M)} \| g_n - g \|_{L^q(X, M)} + \| g \|_{L^q(X, M)} \| f_n - f \|_{L^p(X, M)}\\
&\le \| f_n - f \|_{L^p(X, M)} \| g_n - g \|_{L^q(X, M)} + \| f \|_{L^p(X, M)} \| g_n - g \|_{L^q(X, M)}\\
&\quad + \| g \|_{L^q(X, M)} \| f_n - f \|_{L^p(X, M)}.
\end{aligned}$$

Therefore, $\| f_n \odot g_n - f \odot g \|_{L^1(X, A)}$ tends to 0 as n goes to ∞. □

Let φ and f be M-valued functions on X. Consider the A-linear operator T_φ defined by the formal expression

$$T_\varphi f = \varphi \odot f.$$

Proposition 3.4. *Let* $\varphi \in L^\infty(X, M)$. *Then, the operator* $T_\varphi : L^1(X, M) \to L^1(X, A)$ *is bounded with* $\| T_\varphi \| \le \| \varphi \|_{L^\infty(X, M)}$.

Proof. Let $\varphi \in L^\infty(X, M)$ and $f \in L^1(X, M)$. Then, we have

$$\int_X \| (\varphi \odot f)(x) \|_A d\mu(x) = \int_X \| \langle \varphi(x), f(x) \rangle \|_A d\mu(x)$$

$$\leq \int_X \| \varphi(x) \|_M \| f(x) \|_M \, d\mu(x)$$

$$\leq \| \varphi \|_{L^\infty(X, M)} \| f \|_{L^1(X, M)} < \infty.$$

Thus, T_φ is bounded with $\| T_\varphi \| \leq \| \varphi \|_{L^\infty(X, M)}$. □

Proposition 3.5. *Let* $\varphi \in L^2(X, M)$ *with* $\varphi \neq 0$. *Then, the operator* $T_\varphi : L^2(X, M) \to L^1(X, A)$ *is bounded and* $\| T_\varphi \| = \| \varphi \|_{L^2(X, M)}$.

Proof. Let $\varphi, f \in L^2(X, M)$. We have

$$\| T_\varphi f \|_{L^1(X, A)} = \int_X \| (\varphi \odot f)(x) \| d\mu(x) \leq \| \varphi \|_{L^2(X, M)} \| f \|_{L^2(X, M)}.$$

Hence, $\| T_\varphi \| \leq \| \varphi \|_{L^2(X, M)}$. Moreover, if we set $f = \dfrac{\varphi}{\| \varphi \|_{L^2(X, M)}} \in L^2(X, M)$, then we have

$$\| T_\varphi f \|_{L^1(X, A)} = \int_X \| T_\varphi f(x) \|_A \, d\mu(x)$$

$$= \int_X \| \langle \varphi(x), f(x) \rangle \|_A \, d\mu(x)$$

$$= \frac{1}{\| \varphi \|_{L^2(X, M)}} \int_X \| \langle \varphi(x), \varphi(x) \rangle \|_A \, d\mu(x)$$

$$= \frac{1}{\| \varphi \|_{L^2(X, M)}} \int_X \| \varphi(x) \|_M^2 \, d\mu(x)$$

$$= \| \varphi \|_{L^2(X, M)}.$$

Hence, $\| \varphi \|_{L^2(X, M)} = \| T_\varphi f \|_{L^1(X, A)} \leq \| T_\varphi \|$. Finally, $\| T_\varphi \| = \| \varphi \|_{L^2(X, M)}$.

□

Consider the set $\mathcal{M} = \{T_\varphi : \varphi \in L^2(X, M)\}$ endowed with the (strong) operator norm. Then, from Proposition 3.5, we deduce the following corollary.

Corollary 3.6. *The mapping* $\varphi \mapsto T_\varphi$ *is an isometry from* $L^2(X, M)$ *onto the space* $\mathcal{M}$.

The Hilbert C^*-module structure of $L^2(X, M)$ with respect to the following A-valued inner product was studied in [1, 12]:

$$\langle f, g\rangle_{L^2(X, M)} = \int_X \langle f(x), g(x)\rangle d\mu(x). \tag{3.2}$$

Notably, it was proved that the space $L^2(X, M)$ is a Hilbert C^*-module over A in [1].

Proposition 3.7. *Let* $\varphi \in L^\infty(X, M)$. *Then, the operator* $T_\varphi : L^2(X, M) \to L^2(X, A)$ *is adjointable and its adjoint operator* $T_\varphi^* : L^2(X, A) \to L^2(X, M)$ *is defined by* $T_\varphi^* g(x) = \varphi(x) g(x)$.

Proof. Let $f \in L^2(X, M)$ and $g \in L^2(X, A)$. Then, we have

$$\begin{aligned}
\langle T_\varphi f, g\rangle_{L^2(X, A)} &= \langle \varphi \odot f, g\rangle_{L^2(X, A)} \\
&= \int_X [(\varphi \odot f)(x)]^* g(x) d\mu(x) \\
&= \int_X \langle \varphi(x), f(x)\rangle^* g(x) d\mu(x) \\
&= \int_X \langle f(x), \varphi(x)\rangle g(x) d\mu(x) \\
&= \int_X \langle f(x), \varphi(x) g(x)\rangle d\mu(x).
\end{aligned}$$

Set $T_\varphi^* : L^2(X, A) \to L^2(X, M)$, $g \mapsto T_\varphi^* g$, where $T_\varphi^* g(x) = \varphi(x) g(x)$, $x \in X$. The map T_φ^* is well defined. Indeed, for $g \in L^2(X, A)$,

$$\int_X \| T_\varphi^* g(x) \|_M^2 \, d\mu(x)$$

$$= \int_X \| \varphi(x) g(x) \|_M^2 \, d\mu(x) \le \int_X \| \varphi(x) \|_M^2 \| g(x) \|_A^2 \, d\mu(x)$$

$$\le \| \varphi \|_{L^\infty(X, M)}^2 \int_X \| g(x) \|_A^2 \, d\mu(x) < \infty.$$

Hence, $T_\varphi^* g \in L^2(X, M)$ and $\langle T_\varphi f, g \rangle_{L^2(X, A)} = \langle f, T_\varphi^* g \rangle_{L^2(X, M)}$. On the other hand,

$$\begin{aligned}
\| T_\varphi^* g \|_{L^2(X, M)} &= \| \langle T_\varphi^* g, T_\varphi^* g \rangle_{L^2(X, M)} \|_A \\
&\le \int_X \| \langle T_\varphi^* g(x), T_\varphi^* g(x) \rangle \|_A \, d\mu(x) \\
&= \int_X \| T_\varphi^* g(x) \|_M^2 \, d\mu(x) \\
&= \int_X \| \varphi(x) g(x) \|_M^2 \, d\mu(x) \\
&\le \int_X \| \varphi(x) \|_M^2 \| g(x) \|_A^2 \, d\mu(x) \\
&\le \| \varphi \|_{L^\infty(X, M)}^2 \| g \|_{L^2(X, A)}^2.
\end{aligned}$$

Hence, $\| T_\varphi^* g \|_{L^2(X, M)} \le \| \varphi \|_{L^\infty(X, M)} \| g \|_{L^2(X, A)}$. Therefore, T_φ^* is bounded A-linear operator with $\| T_\varphi^* \| \le \| \varphi \|_{L^\infty(X, M)}$. Consequently, T_φ is adjointable. □

In what follows, we recall the definition of an elementary A-compact operator. Then, we prove that every multiplication operator can be approached by a net of elementary A-compact operators.

Definition 3.8 [13]**.** Let M and N be two Hilbert modules over a C^*-algebra A. An elementary A-compact operator $\theta_{x,y} : M \to N$ with $x \in N$, $y \in M$ is defined by $\theta_{x,y}(z) = x\langle y, z\rangle$.

Proposition 3.9. *Let X be a locally compact space with a regular Borel measure μ such that $\mu(X) < \infty$. Then, for $\varphi \in L^2(X, M)$, the operator $T_\varphi : L^2(X, M) \to L^2(X, M)$ is the limit of a net of elementary A-compact operators.*

Proof. We know that every C^*-algebra admits an approximate unit [8]. Then, let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for A. For each $\lambda \in \Lambda$, let us consider the constant function $h_\lambda : M \to M$ defined by $h_\lambda(x) = u_\lambda$. Since μ is a finite measure, h_λ belongs to $L^2(X, M)$. Now, define $\theta_{h_\lambda,\varphi}(\cdot) = h_\lambda\langle\varphi, \cdot\rangle$. For $f \in L^2(X, M)$ and $x \in X$, we have

$$\begin{aligned}\theta_{h_\lambda,\varphi}f(x) &= h_\lambda(x)\langle\varphi(x), f(x)\rangle \\ &= u_\lambda\langle\varphi(x), f(x)\rangle.\end{aligned}$$

Thus, we obtain

$$\lim_\lambda \theta_{h_\lambda,\varphi}f(x) = \lim_\lambda u_\lambda\langle\varphi(x), f(x)\rangle = \langle\varphi(x), f(x)\rangle = T_\varphi f(x).$$

Therefore, $\lim_\lambda \theta_{h_\lambda,\varphi} = T_\varphi.$ □

From now on, let X be a locally compact group. We may take μ to be a left Haar measure on X. For $s \in X$. Consider the translation operator τ_s defined on $L^1(X, M)$ by $(\tau_s f)(x) = f(s^{-1}x)$, $x \in X$. We consider the following notion of multiplier.

Definition 3.10. The operator $T : L^1(X, M) \to L^1(X, A)$ is called a *multiplier* for $(L^1(X, M), L^1(X, A))$ if T commutes with the translation operators τ_s, for all $s \in X$; that is $\forall s \in X, T(\tau_s f) = \tau_s(Tf)$.

We obtain the following characterization of multiplication operators which are multipliers for $(L^1(X, M), L^1(X, A))$.

Proposition 3.11. *The operator* T_φ *is a multiplier for* $(L^1(X, M),$ $L^1(X, A))$ *if and only if* φ *is a left invariant function*.

Proof. Let us assume that T_φ is a multiplier for $(L^1(X, M), L^1(X, A))$. Let $s \in X$. For $f \in L^1(X, M)$ and $x \in X$, we have $\tau_s T_\varphi f(x) = T_\varphi \tau_s f(x)$. Therefore, $\langle \varphi(s^{-1}x), f(s^{-1}x)\rangle = \langle \varphi(x), f(s^{-1}x)\rangle$. The latter implies $\langle \varphi(s^{-1}x) - \varphi(x), f(s^{-1}x)\rangle = 0$ for all $f \in L^1(X, M)$. Since f is an arbitrary element of $L^1(X, M)$, $\forall x \in X$, $\varphi(s^{-1}x) - \varphi(x) = 0$. Thus, φ is left invariant.

Reciprocally, assume that φ is left invariant. Let $x, s \in X$. Then, we have

$$\begin{aligned} T_\varphi(\tau_s f)(x) &= \langle \varphi(x), \tau_s f(x)\rangle \\ &= \langle \varphi(x), f(s^{-1}x)\rangle \\ &= \langle \tau_s\varphi(x), \tau_s f(x)\rangle \quad \text{(left invariance of } \varphi) \\ &= \tau_s(\langle \varphi(\cdot), f(\cdot)\rangle)(x) \\ &= \tau_s(T_\varphi f)(x). \end{aligned}$$

Hence, $T_\varphi(\tau_s f) = \tau_s(T_\varphi f)$. □

4. Conclusion

In this article, we defined a product for Hilbert C^*-module valued maps. We then considered the corresponding multiplication operators in the framework of Hilbert C^*-modules. We studied their boundedness among some Lebesgue-Bochner spaces. Moreover, we established that the multiplication operators are limit of a net of elementary A-compact operators. Finally, we obtained a characterization of multiplication operators which are multipliers.

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