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TORSION FREE LCA GROUPS, GROUPS WITH UNIQUE ROOTS AND A QUESTION OF A. G. MYASNIKOV: (I) CT GROUPS

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Abstract

Let us say a group G is *LCA* provided every abelian subgroup of G is locally cyclic. A. G. Myasnikov posed the question of whether or not every torsion free LCA group must be commutative transitive in the sense that the relation of commutativity is transitive on the nonidentity

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elements. Pavel Shumyatsky observed that a counterexample to Myasnikov's question is contained in a classical result of Adyan. This paper is meant to be expository and light reading. None the less, we do prove one theorem, namely, a torsion free LCA group G is commutative transitive if and only if roots in G (when they exist) are unique.

1. Preliminaries

Let E be the isomorphism class of the one element group. E is the *trivial variety* of groups. All other varieties of groups are *nontrivial*. If V is any nontrivial variety of groups, then for each cardinal $r \geq 1$, there is a group $F_r(V)$, unique up to isomorphism, free in V on r generators.

Convention: The trivial group $\{1\}$ is free in V on the empty set and has rank 0. Recall that a group G is *Hopfian* provided it cannot be isomorphic to any proper quotient group, equivalently, provided every epiendomorphism is an automorphism. In her classic book, Hanna Neumann [7] shows that if r is a finite cardinal, then a relatively free group $F_r(V)$ is Hopfian is and only if every set of r elements that generates the group generates it freely. ([7] 41.33). She also shows that a finite rank free polynilpotent group (of any fixed type) is Hopfian. ([7] Theorem 41.52).

Putting these together we see that if G is any free polynilpotent group (of any fixed type) having rank r, then every set of r generators of G are free generators. Killing a fly with a hammer, the application of the above in this paper is that in every group free abelian of rank 2 every 2 element generating set must be linearly independent over $\mathbb{Z}$.

A group G is a *U-group* provided roots, when they exist, are unique. That is, for each integer $n \geq 2$, G satisfies the sentence

$$\forall x,\ y\ ((x^n = y^n) \to (x = y)).$$

It follows that the class of U-groups is closed under taking subgroups. There is a converse to the assertion that any universally axiomatizable class of models is closed under taking submodels. For example, it is shown in Hodges [5], that if T is a theory in a first order language with equality and $T_{\forall}$ is the set of all universal sentences of the language in question which are logical consequences of T, then the model class of $T_{\forall}$ consists of all submodels of models of T. Thus, any first order axiomatizable class of structures closed under substructures must have at least one set of universal axioms.

En route to showing that every torsion free nilpotent group admits a Mal'cev completion, Baumslag shows in [2] that every torsion free nilpotent group is a U-group. Although this is of no great import we cannot here resist pointing out that an independent proof of that result follows from Stallings Property-S. Indeed, weaker variants of Property-S will suffice. (See [3]).

We observe that every U-group is torsion free but not conversely. Indeed, in a U-group,

$$x^n = 1 \Rightarrow x^n = 1^n \Rightarrow x = 1.$$

Let $N_2 = \langle a_1, a_2; a_1^2 a_2^2 = 1 \rangle$ be the nonorientable surface group of genus 2. Since the relator is not a proper power in the ambient free group, N_2 is torsion free. From $a_1^2 = (a_2^{-1})^2$ we cannot conclude $a_1 = a_2^{-1}$ as $a_1 a_2$ is not the identity in N_2. Thus N_2 is not a U-group. One other observation before we go on to consider LCA-groups and commutative transitive groups. Suppose the group G is not a U-group. We claim the least integer n such that there exist $x \neq y$ in G with $x^n = y^n$ must be prime. To see that let p be a prime divisor of this least integer n. Suppose $x \neq y$ but $x^n = y^n$. By minimality, $x^{n/p} \neq y^{n/p}$. Letting $u = x^{n/p}$ and $w = y^{n/p}$, we have $u \neq w$ but $u^p = w^p$. By minimality $n \leq p$. But $p | n$ so $n = p$ is the only possibility and the claim is established.

Fixing notation, if G is a group and $g \in G$, we let $C_G(g)$ be the centralizer of g in G and $Z(G) = \bigcap_{g \in G} C_G(g)$ be the center of G.

Definition 1.1. A group G is *commutative transitive*, briefly CT, provided it satisfies the universal sentence

$$\forall x, y, z \quad ((y \neq 1) \wedge (xy = yx) \wedge (yz = zy)) \rightarrow (xz = zx).$$

Thus, the class of CT groups is closed under taking subgroups.

Proposition 1 (Harrison [4])**.** *Let G be a group. Then the following three statements are equivalent in pairs.*

(1) *G is CT.*

(2) *For all* $g \in G \backslash \{1\}$, $C_G(g)$ *is abelian.*

(3) *If* M_1 *and* M_2 *are maximal abelian subgroups in* G, *then* $M_1 \cap M_2 = \{1\}$ *unless* $M_1 = M_2$.

Clearly a nonabelian CT group must be centerless. In particular a nilpotent group is CT if and only if it is abelian.

The next result may be found in Kurosh [6].

Proposition 2. *Let G be a torsion free CT group. Then G is a U-group.*

Proof. Suppose $x^n = y^n$ with $x \neq y$ in G. So $n \geq 2$. Since G is torsion free, we cannot have $x^n = 1 = y^n$ so $x^n = z = y^n$ with $z \neq 1$.

Now x commutes with z which commutes with y so, by CT, x and y commute. Then $x^n = y^n \Rightarrow (xy^{-1})^n = 1 \Rightarrow xy^{-1} = 1 \Rightarrow x = y$ since G is torsion free. □

Definition 1.2. Let G and H be groups. *H discriminates G* provided to every finite subset $S \subseteq G \backslash \{1\}$ there is a homomorphism $\varphi : G \rightarrow H$ such that $\varphi(g) \neq 1$ for all $g \in S$.

It is well-known and easy to prove that if $H \leq G$, then a sufficient condition for G and H to satisfy the same universal sentences is that H discriminates G. Indeed

Proposition 3. *If $H \leq G$ and H discriminates G, then H and G are universally equivalent.*

Proof. Clearly any universal sentence of G holds in H. Also two groups are universally equivalent if and only if they are existentially equivalent. Let σ be an existential sentence true in G. We need to prove σ holds in H. Put the matrix of σ in disjunctive normal form. So we have σ is equivalent to the following disjunction

$$\bigvee_j \exists \bar{x} \left(\left(\bigwedge_i u_{i,j}(\bar{x}) = 1 \right) \wedge \left(\bigwedge_k w_{k,j}(\bar{x}) \neq 1 \right) \right)$$

where $\bar{x} = (x_1, \ldots, x_n)$ is a tuple of variables and u_{ij}, w_{kj} are group words of G in the variables of $\bar{x}$. This holds in G if and only if some disjunct does. Say the disjunct σ_0:

$$\exists \bar{x} \left(\left(\bigwedge_i u_i(\bar{x}) = 1 \right) \wedge \left(\bigwedge_k w_k(\bar{x}) \neq 1 \right) \right)$$

holds in G (where we suppress notationally the dependence on j). Let $\bar{x} = \bar{y}$ verify σ_0 in G. Then, since H discriminates G, there is a homomorphism $\varphi : G \to H$ such that $w_k(\varphi(\bar{y})) = \varphi(w_k(\bar{y})) \neq 1$ in H for all k and, of course, $u_i(\varphi(\bar{y})) = \varphi(u_i(\bar{y})) = 1$ for all i. Hence, σ_0, and therefore also σ, holds in H. □

Definition 1.3. A group G is *LCA* provided every abelian subgroup of G is locally cyclic.

Let G be LCA and $H \leq G$. Then any finitely generated noncyclic abelian subgroup of H would also be a finitely generated noncyclic abelian subgroup of G - a contradiction. It follows that the LCA property is inherited by subgroups. In contradistinction to the U and CT properties, LCA is not first order axiomatizable. Indeed, $\mathbb{Z}$ discriminates $\mathbb{Z} \times \mathbb{Z}$ so $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$ are universally equivalent by Proposition 3. But $\mathbb{Z}$ is LCA while $\mathbb{Z} \times \mathbb{Z}$ is not. Since LCA is closed under taking subgroups, it cannot be first order. For if it were axiomatizable, as previously mentioned, it would have a set of universal axioms and so be closed under universal equivalence.

Suppose G is a torsion free group. Observe that the following three statements are equivalent in pairs.

(1) G is LCA

(2) G contains no copy of $\mathbb{Z} \times \mathbb{Z}$

(3) Every pair of commuting elements in G is linearly dependent over $\mathbb{Z}$.

2. Myasnikov's Question

Alexei Myasnikov asked us if a torsion free LCA group must be CT. We began our investigation by proving the following:

Theorem 2.1. *Let G be a torsion free LCA group. Then a sufficient condition for G to be CT is that it be a U-group.*

In view of Proposition 2, we have the immediate:

Corollary 1. *Let G be a torsion free LCA group. Then G is CT if and only if it is a U-group.*

Remark 2.2. In general being a U-group is not sufficient for CT. Indeed, any nonabelian torsion free nilpotent group is a counterexample.

Proof of Theorem 2.1. Let G be a LCA group which is a U-group. (In particular, G must be torsion free). Let $y \neq 1$ in G and let x and z each commute with y.

Since G is LCA, each of $\langle x, y\rangle$ and $\langle y, z\rangle$ is cyclic. Say $\langle x, y\rangle = \langle u\rangle$ and $\langle y, z\rangle = \langle w\rangle$. Then there are integers m, n, p, q such that $x = u^m$, $y = u^n$, $y = w^p$, $z = w^q$; moreover, we may choose u and w such that n and p are positive.

Now

$$\begin{aligned} w^p &= y \\ &= u^{-1}yu \\ &= u^{-1}w^p u \\ &= (u^{-1}wu)^p. \end{aligned}$$

Then $w = u^{-1}wu$ since G is a U-group. Hence, u and w commute; so, $\langle u, w\rangle$ is cyclic. Say $\langle u, w\rangle = \langle s\rangle$. Then there are integers k and l such that $u = s^k$ and $w = s^l$ and so $x = u^m = s^{km}$ and $z = w^q = s^{lq}$ commute. □

At this point we knew that, if a counterexample existed, it would have to be a torsion free LCA group G which is not a U-group. Assume that G is such a counterexample. Then the minimum integer p such that there are $x \neq y$ with $x^p = y^p$ is prime. Say $a_1^p = a_2^p$ with $a_1 \neq a_2$, where p is minimum. Since the LCA property is inherited by subgroups, the 2-generator subgroup $G_0 = \langle a_1, a_2\rangle$ would also be a counterexample. Note that $a_1^p = z = a_2^p$ is a nontrivial central element in G_0.

Next we observe that the central quotient $G_0/\langle z\rangle$ is a torsion group. This is so, since if $g \in G_0$ is arbitrary, then the abelian subgroup $\langle g, z\rangle$ is cyclic. Say $\langle g, z\rangle = \langle h\rangle$. Then $g = h^m$ and $z = h^n$, where we may choose h such that n is positive. Then $g^n = h^{nm} = z^m \equiv 1 (\mathrm{mod}\langle z\rangle)$.

So our search for a counterexample is narrowed considerably by the above observations. None the less at this point we were hopelessly stuck but had the good sense to consult MATH PUB FORUM. To our surprise and delight we got an immediate response from Pavel Shumyatsky who pointed out that the counterexample of our dreams was already in the literature!

Theorem 2.3 (Adyan [1]). *There is a torsion free group G with cyclic center* $\langle z\rangle$ *such that* $G/\langle z\rangle$ *has large prime exponent*.

Corollary 2. *Adyan's group G is a torsion free LCA group which is not CT*.

Proof. Since G is nonabelian with nontrivial center, it cannot be CT. It remains to show that G is LCA. Suppose x and y commute.

Let p be the exponent of $G/\langle z\rangle$. Then $x^p \equiv 1 \equiv y^p (\mathrm{mod}\langle z\rangle)$. Say $x^p = z^m$ and $y^p = z^n$. Then $x^{np} = z^{mn} = y^{mp}$. If $m = 0$, then $x^{np} = 1$ so $x = 1$ as G is torsion free. In that event $\langle x, y\rangle = \langle 1, y\rangle = \langle y\rangle$ is cyclic. So we may assume $m \neq 0$. Then x and y are linearly dependent over $\mathbb{Z}$ as $x^{np}y^{-mp} = 1$ with $mp \neq 0$. Hence, $\langle x, y\rangle$ is cyclic in that event also. $\square$

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