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MIXED SECOND-ORDER QUATERNIONIC DERIVATIVES: ADVANCES ON HYPERCOMPLEX

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Abstract

Over the past decades, the study of quaternions has advanced significantly, primarily in the field of mathematical analysis [1-5]. These advances have brought to light various generalizations of the Classical Theory of Complex Analysis, especially in the differentiation and integration of quaternionic functions. In the realm of quaternionic differentiation, new methods have been developed to handle the peculiarities of quaternionic functions, highlighting the Cauchy-Riemann Equations and a "closed" formulation for the Cauchy Integral [6, 7]. The main objective of this article is to demonstrate the equality of the mixed second quaternionic derivatives. This equality is fundamental for the theoretical development of quaternionic analysis and can provide new perspectives and applications related fields.

1. Introduction

Quaternionic analysis has offered new and valuable perspectives in various fields of science and engineering. Several fundamental theorems of complex analysis, such as the Cauchy theorem and the Cauchy integral formula, have analogous results in quaternionic analysis [8, 9]. However, due to the non-commutative nature of quaternions, there may be properties that significantly differ from the results obtained so far from the Complex Analysis. In this work, we present a new result: the equality of mixed second order quaternionic derivatives. This result demonstrates that, under certain conditions, the mixed second order quaternionic derivatives of quaternionic functions are equal. This discovery may not only consolidate the theory of quaternionic analysis but also open new possibilities for applications in physics, computing, and other fields.

2. Generalized Cauchy-Riemann Relations and Left and Right Quaternionic Derivatives

The following theorems present the generalized Cauchy-Riemann conditions and the right and left quaternionic derivatives. These results will be important for the purpose of this article.

Theorem 1. *For every pair of points a and b, and any path connecting them in a simply connected dimensional space, the integral* $\int f dq$ *is independent of given up way, only if there is a function* $F = F_1 + iF_2 + jF_3 + kF_4$, *with* $\int f dq = F(b) - F(a)$, *which satisfies the following relations*:

$$\frac{\partial F_1}{\partial q_1} = \frac{\partial F_2}{\partial q_2} = \frac{\partial F_3}{\partial q_3} = \frac{\partial F_4}{\partial q_4};$$

$$\frac{\partial F_2}{\partial q_1} = -\frac{\partial F_1}{\partial q_2} = \frac{\partial F_4}{\partial q_3} = -\frac{\partial F_3}{\partial q_4};$$

$$\frac{\partial F_3}{\partial q_1} = -\frac{\partial F_4}{\partial q_2} = -\frac{\partial F_1}{\partial q_3} = \frac{\partial F_2}{\partial q_4};$$

$$\frac{\partial F_4}{\partial q_1} = \frac{\partial F_3}{\partial q_2} = -\frac{\partial F_2}{\partial q_3} = -\frac{\partial F_1}{\partial q_4}. \tag{1}$$

Theorem 2. *For every pair of points a and b, and any path connecting them in a simply connected dimensional space, the integral* $\int dq f$ *is independent of given up way, only if there is a function* $G = G_1 + iG_2 + jG_3 + kG_4$, *with* $\int dq f = G(b) - G(a)$, *which satisfies the following relations*:

$$\frac{\partial G_1}{\partial q_1} = \frac{\partial G_2}{\partial q_2} = \frac{\partial G_3}{\partial q_3} = \frac{\partial G_4}{\partial q_4};$$

$$\frac{\partial G_2}{\partial q_1} = -\frac{\partial G_1}{\partial q_2} = -\frac{\partial G_4}{\partial q_3} = \frac{\partial G_3}{\partial q_4};$$

$$\frac{\partial G_3}{\partial q_1} = \frac{\partial G_4}{\partial q_2} = -\frac{\partial G_1}{\partial q_3} = -\frac{\partial G_2}{\partial q_4};$$

$$\frac{\partial G_4}{\partial q_1} = -\frac{\partial G_3}{\partial q_2} = \frac{\partial G_2}{\partial q_3} = \frac{\partial G_1}{\partial q_4}. \tag{2}$$

Theorem 3. *Given a function* $f(q)$ *over the ring of quaternions H, with differentiable coordinate functions that satisfy relations* (1), *and a function*

$h(q)$ *defined in terms of* $f(q)$ *by*:

$$h(q) = \frac{1}{4}\Bigg[\left(\frac{\partial f_1}{\partial q_1} + \frac{\partial f_2}{\partial q_2} + \frac{\partial f_3}{\partial q_3} + \frac{\partial f_4}{\partial q_4}\right) + i\left(\frac{\partial f_2}{\partial q_1} - \frac{\partial f_1}{\partial q_2} + \frac{\partial f_4}{\partial q_3} - \frac{\partial f_3}{\partial q_4}\right) + j\left(\frac{\partial f_3}{\partial q_1} - \frac{\partial f_4}{\partial q_2} - \frac{\partial f_1}{\partial q_3} + \frac{\partial f_2}{\partial q_4}\right) + k\left(\frac{\partial f_4}{\partial q_1} + \frac{\partial f_3}{\partial q_2} - \frac{\partial f_2}{\partial q_3} - \frac{\partial f_1}{\partial q_4}\right)\Bigg], \tag{3}$$

then $\int h(q)dq = f(q)$, *and therefore* $h(q)$ *can be formally treated as the left quaternionic derivative of* $f(q)$ *and denoted by* $h(q) = \frac{df_l(q)}{dq}$.

Theorem 4. *Given a function* $f(q)$ *over the ring of quaternions H, with differentiable coordinate functions that satisfy relations* (2), *and a function* $g(q)$ *defined in terms of* $f(q)$ *by*:

$$g(q) = \frac{1}{4}\Bigg[\left(\frac{\partial f_1}{\partial q_1} + \frac{\partial f_2}{\partial q_2} + \frac{\partial f_3}{\partial q_3} + \frac{\partial f_4}{\partial q_4}\right) + i\left(\frac{\partial f_2}{\partial q_1} - \frac{\partial f_1}{\partial q_2} - \frac{\partial f_4}{\partial q_3} + \frac{\partial f_3}{\partial q_4}\right) + j\left(\frac{\partial f_3}{\partial q_1} + \frac{\partial f_4}{\partial q_2} - \frac{\partial f_1}{\partial q_3} - \frac{\partial f_2}{\partial q_4}\right) + k\left(\frac{\partial f_4}{\partial q_1} - \frac{\partial f_3}{\partial q_2} + \frac{\partial f_2}{\partial q_3} - \frac{\partial f_1}{\partial q_4}\right)\Bigg], \tag{4}$$

then $\int dqg(q) = f(q)$, *and therefore* $g(q)$ *can be formally treated as the right quaternionic derivative of* $f(q)$ *and denoted by* $g(q) = \frac{df_r(q)}{dq}$.

According to [10], if a complex function $f(z) = f_1(x, y) + if_2(x, y)$ is differentiable at a point $z = x + iy$, then at z the first-order partial derivatives of f_1 and f_2 must satisfy the Cauchy-Riemann equations

$$\frac{\partial f_1}{\partial x} = \frac{\partial f_2}{\partial y} \quad \text{and} \quad \frac{\partial f_1}{\partial y} = -\frac{\partial f_2}{\partial x}.$$

The relationships from Theorems 1 and 2 generalize the Cauchy-Riemann equations from the complex case to a four-dimensional domain.

Thus, when a function $f(q) = f_1(q) + if_2(q) + jf_3(q) + kf_4(q)$ satisfies the generalized Cauchy-Riemann conditions, the functions f_1, f_2, f_3 and f_4 satisfy a system of differential equations that ensures the smoothness of the function in a quaternionic sense.

Theorems 3 and 4 provide us with a formula to determine the right and left quaternionic derivatives of quaternionic functions that satisfy the generalized Cauchy-Riemann conditions. They will be important for investigating other properties related to higher-order quaternionic derivatives.

3. Mixed Second Quaternionic Derivatives

Starting from the left and right quaternionic derivatives given by the above theorems, we will define the mixed second derivatives left-right and right-left of a quaternionic function *f*, that is, $h_r(q) = \dfrac{d^2 f_{lr}(q)}{dq^2}$ and $g_l(q) = \dfrac{d^2 f_{rl}(q)}{dq^2}$, respectively.

Theorem 5. *If f is a quaternionic function that satisfies the generalized Cauchy-Riemann equations and h is its left quaternionic derivative, then the mixed second left-right derivative of f is given by*

$$h_r(q) = \frac{1}{16}\left(10\frac{\partial^2 f_1}{\partial q_1^2} - 2i\frac{\partial^2 f_2}{\partial q_1^2} + 2j\frac{\partial^2 f_3}{\partial q_1^2} + 2k\frac{\partial^2 f_4}{\partial q_1^2}\right). \tag{5}$$

Proof. From the coordinates of $h(q)$, we will determine $h_r(q)$ as follows:

$$\begin{aligned}
h_r(q) = \frac{1}{16}\Bigg[\Bigg(&\frac{\partial}{\partial q_1}\left(\frac{\partial f_1}{\partial q_1} + \frac{\partial f_2}{\partial q_2} + \frac{\partial f_3}{\partial q_3} + \frac{\partial f_4}{\partial q_4}\right)\\
&+ \frac{\partial}{\partial q_2}\left(\frac{\partial f_2}{\partial q_1} - \frac{\partial f_1}{\partial q_2} + \frac{\partial f_4}{\partial q_3} - \frac{\partial f_3}{\partial q_4}\right)\\
&+ \frac{\partial}{\partial q_3}\left(\frac{\partial f_3}{\partial q_1} - \frac{\partial f_4}{\partial q_2} - \frac{\partial f_1}{\partial q_3} + \frac{\partial f_2}{\partial q_4}\right)\\
&+ \frac{\partial}{\partial q_4}\left(\frac{\partial f_4}{\partial q_1} + \frac{\partial f_3}{\partial q_2} - \frac{\partial f_2}{\partial q_3} - \frac{\partial f_1}{\partial q_4}\right)\Bigg)\\
&+ i\Bigg(\frac{\partial}{\partial q_1}\left(\frac{\partial f_2}{\partial q_1} - \frac{\partial f_1}{\partial q_2} + \frac{\partial f_4}{\partial q_3} - \frac{\partial f_3}{\partial q_4}\right)\\
&- \frac{\partial}{\partial q_2}\left(\frac{\partial f_1}{\partial q_1} + \frac{\partial f_2}{\partial q_2} + \frac{\partial f_3}{\partial q_3} + \frac{\partial f_4}{\partial q_4}\right)\\
&- \frac{\partial}{\partial q_3}\left(\frac{\partial f_4}{\partial q_1} + \frac{\partial f_3}{\partial q_2} - \frac{\partial f_2}{\partial q_3} - \frac{\partial f_1}{\partial q_4}\right)\\
&+ \frac{\partial}{\partial q_4}\left(\frac{\partial f_3}{\partial q_1} - \frac{\partial f_4}{\partial q_2} - \frac{\partial f_1}{\partial q_3} + \frac{\partial f_2}{\partial q_4}\right)\Bigg)\\
&+ j\Bigg(\frac{\partial}{\partial q_1}\left(\frac{\partial f_3}{\partial q_1} - \frac{\partial f_4}{\partial q_2} - \frac{\partial f_1}{\partial q_3} + \frac{\partial f_2}{\partial q_4}\right)\\
&+ \frac{\partial}{\partial q_2}\left(\frac{\partial f_4}{\partial q_1} + \frac{\partial f_3}{\partial q_2} - \frac{\partial f_2}{\partial q_3} - \frac{\partial f_1}{\partial q_4}\right)\\
&- \frac{\partial}{\partial q_3}\left(\frac{\partial f_1}{\partial q_1} + \frac{\partial f_2}{\partial q_2} + \frac{\partial f_3}{\partial q_3} + \frac{\partial f_4}{\partial q_4}\right)\\
&- \frac{\partial}{\partial q_4}\left(\frac{\partial f_2}{\partial q_1} - \frac{\partial f_1}{\partial q_2} + \frac{\partial f_4}{\partial q_3} - \frac{\partial f_3}{\partial q_4}\right)\Bigg)
\end{aligned}$$

$$+ k\left(\frac{\partial}{\partial q_1}\left(\frac{\partial f_4}{\partial q_1} + \frac{\partial f_3}{\partial q_2} - \frac{\partial f_2}{\partial q_3} - \frac{\partial f_1}{\partial q_4}\right)\right.$$

$$- \frac{\partial}{\partial q_2}\left(\frac{\partial f_3}{\partial q_1} - \frac{\partial f_4}{\partial q_2} - \frac{\partial f_1}{\partial q_3} + \frac{\partial f_2}{\partial q_4}\right)$$

$$+ \frac{\partial}{\partial q_3}\left(\frac{\partial f_2}{\partial q_1} - \frac{\partial f_1}{\partial q_2} + \frac{\partial f_4}{\partial q_3} - \frac{\partial f_3}{\partial q_4}\right)$$

$$\left.\left.- \frac{\partial}{\partial q_4}\left(\frac{\partial f_1}{\partial q_1} + \frac{\partial f_2}{\partial q_2} + \frac{\partial f_3}{\partial q_3} + \frac{\partial f_4}{\partial q_4}\right)\right)\right].$$

In this way, we conclude that:

$$h_r(q) = \frac{1}{16}\left[\left(\frac{\partial^2 f_1}{\partial q_1^2} - \frac{\partial^2 f_1}{\partial q_2^2} - \frac{\partial^2 f_1}{\partial q_3^2} - \frac{\partial^2 f_1}{\partial q_4^2}\right.\right.$$

$$\left. + 2\frac{\partial^2 f_2}{\partial q_1 \partial q_2} + 2\frac{\partial^2 f_3}{\partial q_1 \partial q_3} + 2\frac{\partial^2 f_4}{\partial q_1 \partial q_4}\right)$$

$$+ i\left(\frac{\partial^2 f_2}{\partial q_1^2} - \frac{\partial^2 f_2}{\partial q_2^2} + \frac{\partial^2 f_2}{\partial q_3^2} + \frac{\partial^2 f_2}{\partial q_4^2}\right.$$

$$\left. - 2\frac{\partial^2 f_1}{\partial q_1 \partial q_2} - 2\frac{\partial^2 f_3}{\partial q_2 \partial q_3} - 2\frac{\partial^2 f_4}{\partial q_2 \partial q_4}\right)$$

$$+ j\left(\frac{\partial^2 f_3}{\partial q_1^2} + \frac{\partial^2 f_3}{\partial q_2^2} - \frac{\partial f_3}{\partial q_3^2} + \frac{\partial^2 f_3}{\partial q_4^2}\right.$$

$$\left. - 2\frac{\partial^2 f_1}{\partial q_1 \partial q_3} - 2\frac{\partial^2 f_2}{\partial q_2 \partial q_3} - 2\frac{\partial^2 f_4}{\partial q_3 \partial q_4}\right)$$

$$+ k\left(\frac{\partial^2 f_4}{\partial q_1^2} + \frac{\partial^2 f_4}{\partial q_2^2} + \frac{\partial^2 f_4}{\partial q_3^2} - \frac{\partial^2 f_4}{\partial q_4^2}\right.$$

$$\left.\left. - 2\frac{\partial^2 f_1}{\partial q_1 \partial q_4} - 2\frac{\partial^2 f_2}{\partial q_2 \partial q_4} - 2\frac{\partial^2 f_3}{\partial q_3 \partial q_4}\right)\right]. \qquad (6)$$

Partially differentiating all terms of the equalities in (2) resulting from the generalized Cauchy-Riemann conditions with respect to q_1, we have:

$$\frac{\partial^2 f_1}{\partial q_1^2} = \frac{\partial^2 f_2}{\partial q_1 \partial q_2} = \frac{\partial^2 f_3}{\partial q_1 \partial q_3} = \frac{\partial^2 f_4}{\partial q_1 \partial q_4}$$

$$\frac{\partial^2 f_2}{\partial q_1^2} = -\frac{\partial^2 f_1}{\partial q_1 \partial q_2} = -\frac{\partial^2 f_4}{\partial q_1 \partial q_3} = \frac{\partial^2 f_3}{\partial q_1 \partial q_4}$$

$$\frac{\partial^2 f_3}{\partial q_1^2} = \frac{\partial^2 f_4}{\partial q_1 \partial q_2} = -\frac{\partial^2 f_1}{\partial q_1 \partial q_3} = -\frac{\partial^2 f_2}{\partial q_1 \partial q_4}$$

$$\frac{\partial^2 f_4}{\partial q_1^2} = -\frac{\partial^2 f_3}{\partial q_1 \partial q_2} = \frac{\partial^2 f_2}{\partial q_1 \partial q_3} = -\frac{\partial^2 f_1}{\partial q_1 \partial q_4}. \tag{7}$$

Partially differentiating all terms of the equalities in (2) resulting from the generalized Cauchy-Riemann conditions with respect to q_2, we have:

$$\frac{\partial^2 f_1}{\partial q_1 \partial q_2} = \frac{\partial^2 f_2}{\partial q_2^2} = \frac{\partial^2 f_3}{\partial q_3 \partial q_2} = \frac{\partial^2 f_4}{\partial q_4 \partial q_2}$$

$$\frac{\partial^2 f_2}{\partial q_1 \partial q_2} = -\frac{\partial^2 f_1}{\partial q_2^2} = -\frac{\partial^2 f_4}{\partial q_3 \partial q_2} = \frac{\partial^2 f_3}{\partial q_4 \partial q_2}$$

$$\frac{\partial^2 f_3}{\partial q_1 \partial q_2} = \frac{\partial^2 f_4}{\partial q_2^2} = -\frac{\partial^2 f_1}{\partial q_3 \partial q_2} = -\frac{\partial^2 f_2}{\partial q_4 \partial q_2}$$

$$\frac{\partial^2 f_4}{\partial q_1 \partial q_2} = -\frac{\partial^2 f_3}{\partial q_2^2} = \frac{\partial^2 f_2}{\partial q_3 \partial q_2} = -\frac{\partial^2 f_1}{\partial q_4 \partial q_2}. \tag{8}$$

Partially differentiating all terms of the equalities in (2) resulting from the generalized Cauchy-Riemann conditions with respect to q_3, we have:

$$\frac{\partial^2 f_1}{\partial q_1 \partial q_3} = \frac{\partial^2 f_2}{\partial q_2 \partial q_3} = \frac{\partial^2 f_3}{\partial q_3^2} = \frac{\partial^2 f_4}{\partial q_4 \partial q_3}$$

$$\frac{\partial^2 f_2}{\partial q_1 \partial q_3} = -\frac{\partial^2 f_1}{\partial q_2 \partial q_3} = -\frac{\partial^2 f_4}{\partial q_3^2} = \frac{\partial^2 f_3}{\partial q_4 \partial q_3}$$

$$\frac{\partial^2 f_3}{\partial q_1 \partial q_3} = \frac{\partial^2 f_4}{\partial q_2 \partial q_3} = -\frac{\partial^2 f_1}{\partial q_3^2} = -\frac{\partial^2 f_2}{\partial q_4 \partial q_3}$$

$$\frac{\partial^2 f_4}{\partial q_1 \partial q_3} = -\frac{\partial^2 f_3}{\partial q_2 \partial q_3} = \frac{\partial^2 f_2}{\partial q_3^2} = -\frac{\partial^2 f_1}{\partial q_4 \partial q_3}. \tag{9}$$

Partially differentiating all terms of the equalities in (2) resulting from the generalized Cauchy-Riemann conditions with respect to q_4, we have:

$$\frac{\partial^2 f_1}{\partial q_1 \partial q_4} = \frac{\partial^2 f_2}{\partial q_2 \partial q_4} = \frac{\partial^2 f_3}{\partial q_3 \partial q_4} = \frac{\partial^2 f_4}{\partial q_4^2}$$

$$\frac{\partial^2 f_2}{\partial q_1 \partial q_4} = -\frac{\partial^2 f_1}{\partial q_2 \partial q_4} = -\frac{\partial^2 f_4}{\partial q_3 \partial q_4} = \frac{\partial^2 f_3}{\partial q_4^2}$$

$$\frac{\partial^2 f_3}{\partial q_1 \partial q_4} = \frac{\partial^2 f_4}{\partial q_2 \partial q_4} = -\frac{\partial^2 f_1}{\partial q_3 \partial q_4} = -\frac{\partial^2 f_2}{\partial q_4^2}$$

$$\frac{\partial^2 f_4}{\partial q_1 \partial q_4} = -\frac{\partial^2 f_3}{\partial q_2 \partial q_4} = \frac{\partial^2 f_2}{\partial q_3 \partial q_4} = -\frac{\partial^2 f_1}{\partial q_4^2}. \tag{10}$$

The sequences of equalities given in (7), (8), (9), and (10) result in

$$\frac{\partial^2 f_1}{\partial q_1^2} = -\frac{\partial^2 f_1}{\partial q_2^2} = -\frac{\partial^2 f_1}{\partial q_3^2} = -\frac{\partial^2 f_1}{\partial q_4^2}$$

$$\frac{\partial^2 f_2}{\partial q_1^2} = -\frac{\partial^2 f_2}{\partial q_2^2} = -\frac{\partial^2 f_2}{\partial q_3^2} = -\frac{\partial^2 f_2}{\partial q_4^2}$$

$$\frac{\partial^2 f_3}{\partial q_1^2} = -\frac{\partial^2 f_3}{\partial q_2^2} = -\frac{\partial^2 f_3}{\partial q_3^2} = -\frac{\partial^2 f_3}{\partial q_4^2}$$

$$\frac{\partial^2 f_4}{\partial q_1^2} = -\frac{\partial^2 f_4}{\partial q_2^2} = -\frac{\partial^2 f_4}{\partial q_3^2} = -\frac{\partial^2 f_4}{\partial q_4^2}. \tag{11}$$

By using the equalities (7), (8), (9), (10), and (11) in (6), we simplify the structure of h_r and obtain

$$h_r(q) = \frac{1}{16}\left(10\frac{\partial^2 f_1}{\partial q_1^2} - 2i\frac{\partial^2 f_2}{\partial q_1^2} + 2j\frac{\partial^2 f_3}{\partial q_1^2} + 2k\frac{\partial^2 f_4}{\partial q_1^2}\right). \tag{12}$$

Theorem 6. *If f is a quaternionic function that satisfies the generalized Cauchy-Riemann equations and g is its right quaternionic derivative, then the mixed second right-left derivative of f is given by*

$$g_l(q) = \frac{1}{16}\left(10\frac{\partial^2 f_1}{\partial q_1^2} - 2i\frac{\partial^2 f_2}{\partial q_1^2} + 2j\frac{\partial^2 f_3}{\partial q_1^2} + 2k\frac{\partial^2 f_4}{\partial q_1^2}\right). \tag{13}$$

Proof. From the coordinates of $g(q)$, we will determine $g_l(q)$ as follows:

$$g_l(q) = \frac{1}{16}\Bigg[\Bigg(\frac{\partial}{\partial q_1}\left(\frac{\partial f_1}{\partial q_1} + \frac{\partial f_2}{\partial q_2} + \frac{\partial f_3}{\partial q_3} + \frac{\partial f_4}{\partial q_4}\right)$$

$$+ \frac{\partial}{\partial q_2}\left(\frac{\partial f_2}{\partial q_1} - \frac{\partial f_1}{\partial q_2} - \frac{\partial f_4}{\partial q_3} + \frac{\partial f_3}{\partial q_4}\right)$$

$$+ \frac{\partial}{\partial q_3}\left(\frac{\partial f_3}{\partial q_1} + \frac{\partial f_4}{\partial q_2} - \frac{\partial f_1}{\partial q_3} - \frac{\partial f_2}{\partial q_4}\right)$$

$$+ \frac{\partial}{\partial q_4}\left(\frac{\partial f_4}{\partial q_1} - \frac{\partial f_3}{\partial q_2} + \frac{\partial f_2}{\partial q_3} - \frac{\partial f_1}{\partial q_4}\right)\Bigg)$$

$$+ i\Bigg(\frac{\partial}{\partial q_1}\left(\frac{\partial f_2}{\partial q_1} - \frac{\partial f_1}{\partial q_2} - \frac{\partial f_4}{\partial q_3} + \frac{\partial f_3}{\partial q_4}\right)$$

$$- \frac{\partial}{\partial q_2}\left(\frac{\partial f_1}{\partial q_1} + \frac{\partial f_2}{\partial q_2} + \frac{\partial f_3}{\partial q_3} + \frac{\partial f_4}{\partial q_4}\right)$$

$$+ \frac{\partial}{\partial q_3}\left(\frac{\partial f_4}{\partial q_1} - \frac{\partial f_3}{\partial q_2} + \frac{\partial f_2}{\partial q_3} - \frac{\partial f_1}{\partial q_4}\right)$$

$$
\begin{aligned}
&- \frac{\partial}{\partial q_4}\left(\frac{\partial f_3}{\partial q_1} + \frac{\partial f_4}{\partial q_2} - \frac{\partial f_1}{\partial q_3} - \frac{\partial f_2}{\partial q_4}\right)\bigg)\\
&+ j\bigg(\frac{\partial}{\partial q_1}\left(\frac{\partial f_3}{\partial q_1} + \frac{\partial f_4}{\partial q_2} - \frac{\partial f_1}{\partial q_3} - \frac{\partial f_2}{\partial q_4}\right)\\
&- \frac{\partial}{\partial q_2}\left(\frac{\partial f_4}{\partial q_1} - \frac{\partial f_3}{\partial q_2} + \frac{\partial f_2}{\partial q_3} - \frac{\partial f_1}{\partial q_4}\right)\\
&- \frac{\partial}{\partial q_3}\left(\frac{\partial f_1}{\partial q_1} + \frac{\partial f_2}{\partial q_2} + \frac{\partial f_3}{\partial q_3} + \frac{\partial f_4}{\partial q_4}\right)\\
&+ \frac{\partial}{\partial q_4}\left(\frac{\partial f_2}{\partial q_1} - \frac{\partial f_1}{\partial q_2} - \frac{\partial f_4}{\partial q_3} + \frac{\partial f_3}{\partial q_4}\right)\bigg)\\
&+ k\bigg(\frac{\partial}{\partial q_1}\left(\frac{\partial f_4}{\partial q_1} - \frac{\partial f_3}{\partial q_2} + \frac{\partial f_2}{\partial q_3} - \frac{\partial f_1}{\partial q_4}\right)\\
&+ \frac{\partial}{\partial q_2}\left(\frac{\partial f_3}{\partial q_1} + \frac{\partial f_4}{\partial q_2} - \frac{\partial f_1}{\partial q_3} - \frac{\partial f_2}{\partial q_4}\right)\\
&- \frac{\partial}{\partial q_3}\left(\frac{\partial f_2}{\partial q_1} - \frac{\partial f_1}{\partial q_2} - \frac{\partial f_4}{\partial q_3} + \frac{\partial f_3}{\partial q_4}\right)\\
&- \frac{\partial}{\partial q_4}\left(\frac{\partial f_1}{\partial q_1} + \frac{\partial f_2}{\partial q_2} + \frac{\partial f_3}{\partial q_3} + \frac{\partial f_4}{\partial q_4}\right)\bigg)\bigg].
\end{aligned}
$$

In this way, we conclude that:

$$
\begin{aligned}
g_l(q) = \frac{1}{16}\bigg[&\bigg(\frac{\partial^2 f_1}{\partial q_1^2} - \frac{\partial^2 f_1}{\partial q_2^2} - \frac{\partial^2 f_1}{\partial q_3^2} - \frac{\partial^2 f_1}{\partial q_4^2}\\
&+ 2\frac{\partial^2 f_2}{\partial q_1 \partial q_2} + 2\frac{\partial^2 f_3}{\partial q_1 \partial q_3} + 2\frac{\partial^2 f_4}{\partial q_1 \partial q_4}\bigg)\\
&+ i\bigg(\frac{\partial^2 f_2}{\partial q_1^2} - \frac{\partial^2 f_2}{\partial q_2^2} + \frac{\partial^2 f_2}{\partial q_3^2} + \frac{\partial^2 f_2}{\partial q_4^2}\\
&- 2\frac{\partial^2 f_1}{\partial q_1 \partial q_2} - 2\frac{\partial^2 f_3}{\partial q_2 \partial q_3} - 2\frac{\partial^2 f_4}{\partial q_2 \partial q_4}\bigg)
\end{aligned}
$$

$$+ j\left(\frac{\partial^2 f_3}{\partial q_1^2} + \frac{\partial^2 f_3}{\partial q_2^2} - \frac{\partial f_3}{\partial q_3^2} + \frac{\partial^2 f_3}{\partial q_4^2}\right.$$

$$\left. - 2\frac{\partial^2 f_1}{\partial q_1 \partial q_3} - 2\frac{\partial^2 f_2}{\partial q_2 \partial q_3} - 2\frac{\partial^2 f_4}{\partial q_3 \partial q_4}\right)$$

$$+ k\left(\frac{\partial^2 f_4}{\partial q_1^2} + \frac{\partial^2 f_4}{\partial q_2^2} + \frac{\partial^2 f_4}{\partial q_3^2} - \frac{\partial^2 f_4}{\partial q_4^2}\right.$$

$$\left.\left. - 2\frac{\partial^2 f_1}{\partial q_1 \partial q_4} - 2\frac{\partial^2 f_2}{\partial q_2 \partial q_4} - 2\frac{\partial^2 f_3}{\partial q_3 \partial q_4}\right)\right]. \tag{14}$$

Partially differentiating all terms of the equalities in (1) resulting from the generalized Cauchy-Riemann conditions with respect to q_1, we have:

$$\frac{\partial^2 f_1}{\partial q_1^2} = \frac{\partial^2 f_2}{\partial q_1 \partial q_2} = \frac{\partial^2 f_3}{\partial q_1 \partial q_3} = \frac{\partial^2 f_4}{\partial q_1 \partial q_4}$$

$$\frac{\partial^2 f_2}{\partial q_1^2} = -\frac{\partial^2 f_1}{\partial q_1 \partial q_2} = \frac{\partial^2 f_4}{\partial q_1 \partial q_3} = -\frac{\partial^2 f_3}{\partial q_1 \partial q_4}$$

$$\frac{\partial^2 f_3}{\partial q_1^2} = -\frac{\partial^2 f_4}{\partial q_1 \partial q_2} = -\frac{\partial^2 f_1}{\partial q_1 \partial q_3} = \frac{\partial^2 f_2}{\partial q_1 \partial q_4}$$

$$\frac{\partial^2 f_4}{\partial q_1^2} = \frac{\partial^2 f_3}{\partial q_1 \partial q_2} = -\frac{\partial^2 f_2}{\partial q_1 \partial q_3} = -\frac{\partial^2 f_1}{\partial q_1 \partial q_4}. \tag{15}$$

Partially differentiating all terms of the equalities in (1) resulting from the generalized Cauchy-Riemann conditions with respect to q_2, we have:

$$\frac{\partial^2 f_1}{\partial q_1 \partial q_2} = \frac{\partial^2 f_2}{\partial q_2^2} = \frac{\partial^2 f_3}{\partial q_3 \partial q_2} = \frac{\partial^2 f_4}{\partial q_4 \partial q_2}$$

$$\frac{\partial^2 f_2}{\partial q_1 \partial q_2} = -\frac{\partial^2 f_1}{\partial q_2^2} = \frac{\partial^2 f_4}{\partial q_3 \partial q_2} = -\frac{\partial^2 f_3}{\partial q_4 \partial q_2}$$

$$\frac{\partial^2 f_3}{\partial q_1 \partial q_2} = -\frac{\partial^2 f_4}{\partial q_2^2} = -\frac{\partial^2 f_1}{\partial q_3 \partial q_2} = \frac{\partial^2 f_2}{\partial q_4 \partial q_2}$$

$$\frac{\partial^2 f_4}{\partial q_1 \partial q_2} = \frac{\partial^2 f_3}{\partial q_2^2} = -\frac{\partial^2 f_2}{\partial q_3 \partial q_2} = -\frac{\partial^2 f_1}{\partial q_4 \partial q_2}. \tag{16}$$

Partially differentiating all terms of the equalities in (1) resulting from the generalized Cauchy-Riemann conditions with respect to q_3, we have:

$$\frac{\partial^2 f_1}{\partial q_1 \partial q_3} = \frac{\partial^2 f_2}{\partial q_2 \partial q_3} = \frac{\partial^2 f_3}{\partial q_3^2} = \frac{\partial^2 f_4}{\partial q_4 \partial q_3}$$

$$\frac{\partial^2 f_2}{\partial q_1 \partial q_3} = -\frac{\partial^2 f_1}{\partial q_2 \partial q_3} = \frac{\partial^2 f_4}{\partial q_3^2} = -\frac{\partial^2 f_3}{\partial q_4 \partial q_3}$$

$$\frac{\partial^2 f_3}{\partial q_1 \partial q_3} = -\frac{\partial^2 f_4}{\partial q_2 \partial q_3} = -\frac{\partial^2 f_1}{\partial q_3^2} = \frac{\partial^2 f_2}{\partial q_4 \partial q_3}$$

$$\frac{\partial^2 f_4}{\partial q_1 \partial q_3} = \frac{\partial^2 f_3}{\partial q_2 \partial q_3} = -\frac{\partial^2 f_2}{\partial q_3^2} = -\frac{\partial^2 f_1}{\partial q_4 \partial q_3}. \tag{17}$$

Partially differentiating all terms of the equalities in (1) resulting from the generalized Cauchy-Riemann conditions with respect to q_4, we have:

$$\frac{\partial^2 f_1}{\partial q_1 \partial q_4} = \frac{\partial^2 f_2}{\partial q_2 \partial q_4} = \frac{\partial^2 f_3}{\partial q_3 \partial q_4} = \frac{\partial^2 f_4}{\partial q_4^2}$$

$$\frac{\partial^2 f_2}{\partial q_1 \partial q_4} = -\frac{\partial^2 f_1}{\partial q_2 \partial q_4} = \frac{\partial^2 f_4}{\partial q_3 \partial q_4} = -\frac{\partial^2 f_3}{\partial q_4^2}$$

$$\frac{\partial^2 f_3}{\partial q_1 \partial q_4} = -\frac{\partial^2 f_4}{\partial q_2 \partial q_4} = -\frac{\partial^2 f_1}{\partial q_3 \partial q_4} = \frac{\partial^2 f_2}{\partial q_4^2}$$

$$\frac{\partial^2 f_4}{\partial q_1 \partial q_4} = \frac{\partial^2 f_3}{\partial q_2 \partial q_4} = -\frac{\partial^2 f_2}{\partial q_3 \partial q_4} = -\frac{\partial^2 f_1}{\partial q_4^2}. \tag{18}$$

The sequences of equalities given in (15), (16), (17), and (18) result in

$$\frac{\partial^2 f_1}{\partial q_1^2} = -\frac{\partial^2 f_1}{\partial q_2^2} = -\frac{\partial^2 f_1}{\partial q_3^2} = -\frac{\partial^2 f_1}{\partial q_4^2}$$

$$\frac{\partial^2 f_2}{\partial q_1^2} = -\frac{\partial^2 f_2}{\partial q_2^2} = -\frac{\partial^2 f_2}{\partial q_3^2} = -\frac{\partial^2 f_2}{\partial q_4^2}$$

$$\frac{\partial^2 f_3}{\partial q_1^2} = -\frac{\partial^2 f_3}{\partial q_2^2} = -\frac{\partial^2 f_3}{\partial q_3^2} = -\frac{\partial^2 f_3}{\partial q_4^2}$$

$$\frac{\partial^2 f_4}{\partial q_1^2} = -\frac{\partial^2 f_4}{\partial q_2^2} = -\frac{\partial^2 f_4}{\partial q_3^2} = -\frac{\partial^2 f_4}{\partial q_4^2}. \tag{19}$$

By using the equalities (15), (16), (17), (18), and (19) in (14), we simplify the structure of g_l and obtain

$$g_l(q) = \frac{1}{16}\left(10\frac{\partial^2 f_1}{\partial q_1^2} - 2i\frac{\partial^2 f_2}{\partial q_1^2} + 2j\frac{\partial^2 f_3}{\partial q_1^2} + 2k\frac{\partial^2 f_4}{\partial q_1^2}\right). \tag{20}$$

Theorem 7. *If f is a quaternionic function that satisfies the generalized Cauchy-Riemann equations, then its mixed second quaternionic derivatives are equal.*

Proof. In fact, from (5) and (13), we conclude that h_r and g_l are equal.

The Clairaut-Schwarz Theorem, which applies to the equality of mixed partial derivatives, states that, under certain conditions of continuity and existence, the mixed derivatives of a function of several variables are equal. Thus, when considering the mixed quaternionic derivatives $h_r(q)$ and $g_l(q)$, we can expect that, under the appropriate conditions of smoothness and continuity, we have $h_r(q) = g_l(q)$. This relations not only reinforces the consistency of quaternionic analysis but also ensures that the differential properties of quaternionic functions align with the known results of multivariable analysis, enhancing our understanding of quaternionic functions and their applications.

4. Concluding Remarks

The derivatives of quaternionic functions are generally not commutative, which makes this type of function peculiar and the generalizations made from results known from Classical Complex Analysis cannot be applied widely to Quaternionic Functions.

It was found that the higher-order derivatives of quaternionic functions can be commutative as long as they satisfy conditions already stipulated for these types of functions. These conditions are known as Generalized Riemann-Cauchy Conditions, which are two sets of conditions, one on the right and the other on the left, which relate the partial derivatives of the components of the functions in question.

The result determined in this work makes it possible to establish conditions for the mixed second-order derivatives to be equal, which has important impacts on the analysis of Quaternionic Functions. The first relevant impact is that conditions for the commutativity of the second-order derivatives on the right and left have already been established, and now, with the results established for mixed derivatives, it completes the study of the derivatives of this type of function and demonstrates that their derivatives will not necessarily be non-commutative. Secondly, it is possible to draw new conclusions about derivatives of order greater than 2, since the results for order 2 have already been established.

Finally, the study presented here could generate new results about the derivation and integration of this type of function, making it possible to find out more about the derivatives of this type of function.

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