Far East Journal of Mathematical Sciences (FJMS)
© 2026 Pushpa Publishing House, Prayagraj, India
https://pphmjopenaccess.com
https://doi.org/10.17654/0972087126031 **P-ISSN: 0972-0871**
Volume 143, Number 2, February 2026, Pages 527-548 **E-ISSN: 2584-1246**

FIXED POINT RESULTS IN PARTIALLY ORDERED p-POMPEIU-HAUSDORFF METRIC SPACES

Arta Ekayanti[1,2,*], **Marjono**[1], **Mohamad Muslikh**[1] **and Sa'adatul Fitri**[1]

[1]Department of Mathematics
University of Brawijaya
Malang 65145, Indonesia
e-mail: arta_ekayanti@student.ub.ac.id
marjono@ub.ac.id
mslk@ub.ac.id
saadatulfitri@ub.ac.id

[2]Department of Mathematics Education
Muhammadiyah University of Ponorogo
Ponorogo 63471, Indonesia

Abstract

In this paper, we introduce a relation between sets that defines a partial order. Based on this relation, we construct a partially ordered

Received: October 31, 2025; Accepted: December 13, 2025
2020 Mathematics Subject Classification: 47H10, 26E25, 06A06.
Keywords and phrases: set-valued mapping, fixed point, partially ordered p-Pompeiu-Hausdorff metric spaces.

[*] Corresponding author

How to cite this article: Arta Ekayanti, Marjono, Mohamad Muslikh and Sa'adatul Fitri, Fixed point results in partially ordered p-Pompeiu-Hausdorff metric spaces, Far East Journal of Mathematical Sciences (FJMS) 143(2) (2026), 527-548.
https://doi.org/10.17654/0972087126031

p-Pompeiu-Hausdorff metric space. By using this partial order on $CB^p(X)$, we establish the existence of fixed points for set-valued mappings in partially ordered p-Pompeiu-Hausdorff metric spaces. Furthermore, the existence of common fixed points for such mappings is demonstrated through the use of a generalized contraction condition.

1. Introduction

Set-valued mappings constitute a significant area of study in mathematics, particularly within the frameworks of nonlinear analysis, optimization theory, game theory, economics, and fractal geometry [17]. Their theoretical and practical importance has been extensively investigated by numerous researchers, including Kuratowski, who addressed them in his seminal work topology [13]. Moreover, foundational contributions by Painlevé, Hausdorff, and Bouligand have underscored the relevance of set-valued mappings as mathematical constructs that frequently arise in various real-world contexts [4, 11, 19].

Set-valued mappings have gained significant attention among mathematicians working in game theory and economics since Kakutani introduced a fixed point theorem for set-valued mappings as a generalization of Brouwer's fixed point theorem [12]. Following this, Nadler extended Banach's fixed point theorem to the context of set-valued mappings [18]. Research on fixed points of set-valued mappings has continued to develop. Notably, in 2004, Feng and Liu established a fixed point theorem for set-valued mappings involving ordering relations between sets [9]. The order structure among these sets is described as follows.

Definition 1.1. Let X be a topology space and $\prec_X$ be a partial order endowed on X. For nonempty subsets A and B of X, the relations between A and B are denoted and defined as follows:

(1) $A \prec_1 B$, if for every $a \in A$, there exists $b \in B$ such that $a \prec_X b$.

(2) $A \prec_2 B$, if for every $b \in B$, there exists $a \in A$ such that $a \prec_X b$.

(3) $A \prec_3 B$, if $A \prec_1 B$ and $A \prec_2 B$.

Beg and Butt [2], also Gregorio and Macansantos [10] employed an order relation between sets to establish fixed point results for set-valued mappings in a partially ordered metric space. However, the order relation they used is a preorder, not a partial order, as it satisfies only reflexivity and transitivity, but not antisymmetry. The relation is defined by the partial order $\prec_X$ on X as follows [3, 5, 20].

Definition 1.2. A partial order $\prec_X$ is a binary relation on X which satisfies the following conditions:

(1) $x \prec_X x$ (reflexivity),

(2) if $x \prec_X y$ and $y \prec_X z$, then $x \prec_X z$ (transitivity),

(3) If $x \prec_X y$ and $y \prec_X x$, then $x = y$ (antisymmetry),

for all $x, y, z \in X$. A set with partial order $\prec_X$ is called *partially ordered sets*. When the relation $\prec_X$ satisfies only the reflexive and transitive properties, it is called a *preorder relation*.

Definition 1.3. Let $(X, \prec_X)$ be a partially ordered set and $x, y \in X$. Then elements x and y are said to be *comparable* if either $x \prec_X y$ or $y \prec_X x$.

Meanwhile, research on fixed points of set-valued mappings in metric spaces continues to evolve, extending toward more generalized setting such as partial metric spaces as reflected in several results (e.g., [13, 17]). Motivated by these several results, the present study aims to investigate the existence of fixed points for set-valued mappings in spaces equipped with a partial metric, while also utilizing an order relation between sets that does satisfy the properties of a partial order. For this reason, the following section presents the definitions of partial metric spaces.

In 1994, Matthews proposed partial metric as a broader version of the standard metric, wherein an object's distance from itself need not always be zero [14, 15]. For $X \neq \emptyset$, a partial metric is a mapping $p : X \times X \to [0, \infty)$ which satisfies the following conditions:

(1) $p(x, y) = p(y, x)$,

(2) if $p(x, x) = p(x, y) = p(y, y)$, then $x = y$,

(3) $p(x, x) \leq p(x, y)$,

(4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$,

for all $x, y, z \in X$. Furthermore, a pair (X, p) is partial metric space.

Suppose that (X, p) is a partial metric space, and let $CB^p(X)$ denote the collection of all nonempty, closed, and bounded subsets of (X, p). For any $A, B \in CB^p(X)$, the mapping $H^p : CB^p(X) \times CB^p(X) \to [0, \infty)$ is defined by

$$H^p(A, B) = \max\{p(A, B), p(B, A)\}, \tag{1.1}$$

where $p(A, B) = \sup\{p(a, B) : a \in A\}$ and $p(a, B) = \inf\{p(a, b) : b \in B\}$.

The function H^p is called the *partial Pompeiu-Hausdorff metric* induced by p [1, 7, 16]. The pair $(CB^p(X), H^p)$ is referred to as a *p*-Pompeiu-Hausdorff metric space. It should be noted that every Hausdorff metric is a *p*-Pompeiu-Hausdorff metric; however, the converse is not necessarily true (we can see Example 2.6 and Remark 2.7 in [3]). For properties of this space, we refer to [1, 6, 7, 16] and references therein. Furthermore, the following properties hold for sequences in the *p*-Pompeiu-Hausdorff metric space [6-8, 16].

Definition 1.4. Suppose that $(CB^p(X), H^p)$ is a *p-Pompeiu-Hausdorff metric space*. Then a sequence (A_n) in $CB^p(X)$ converges to $A \in CB^p(X)$ if

$$\lim_{n\to\infty} H^p(A_n, A) = H^p(A, A). \tag{1.2}$$

Definition 1.5. Let $(CB^p(X), H^p)$ be a *p-Pompeiu-Hausdorff metric space*. Then a sequence (A_n) in $CB^p(X)$ properly converges to a set $A \in CB^p(X)$ if (A_n) converges to *A* and

$$\lim_{n\to\infty} H^p(A_n, A_n) = H^p(A, A). \tag{1.3}$$

Definition 1.6. Let $(CB^p(X), H^p)$ be a *p*-Pompeiu-Hausdorff metric space. Then a sequence (A_n) in $CB^p(X)$ is said to be a *Cauchy sequence* if

$$\lim_{n,m\to\infty} H^p(A_n, A_m) \tag{1.4}$$

exists and is finite.

Definition 1.7. A *p*-Pompeiu-Hausdorff metric space $(CB^p(X), H^p)$ is said to be *complete* if every Cauchy sequence properly converges in $CB^p(X)$.

Lemma 1.1. *Let* $A, B \in CB^p(X)$ *and suppose that* $H^p(A, B) < \varepsilon$ *for some* $\varepsilon > 0$. *Then, for every* $a \in A$, *there exists* $b \in B$ *such that*

$$p(a, b) < \varepsilon. \tag{1.5}$$

Lemma 1.2. *Let* (A_n) *be a sequence in* $CB^p(X)$ *such that*

$$\lim_{n\to\infty} H^p(A_n, A) = H^p(A, A) \tag{1.6}$$

for some $A \in CB^p(X)$. *If* $x_n \in A_n$ *for each* $n \in N$, *and*

$$\lim_{n\to\infty} p(x_n, x) = p(x, x), \tag{1.7}$$

then $x \in A$.

2. Partial Order in *p*-Pompeiu-Hausdorff Metric Spaces

Based on Definition 1.1, a new approach to defining relations between sets is introduced as follows:

Definition 2.1. Suppose that $(X, p, \prec_X)$ is a partially ordered partial metric space and $CB^p(X)$ is a class of all nonempty, closed and bounded subsets of *X*. For every $A, B \in CB^p(X)$, the relation between sets *A* and *B*, denoted by $\prec_{CB^p(X)}$ and written as $A \prec_{CB^p(X)} B$, is defined as follows:

(1) For every $b \in B$, there exists $a \in A$ such that $a \prec_X b$.

(2) For every $a \in A$, there exists $b \in B$ such that $a = b$.

The relation $\prec_{CB^p(X)}$ defined in Definition 2.1 is a partial order relation. This can be seen in the following theorem:

Theorem 2.2. *The ordering relation* $\prec_{CB^p(X)}$, *as defined in Definition* 2.1, *is a partial order relation*.

Proof. It is known that a relation is said to be a *partial order* if it satisfies the properties of reflexivity, transitivity, and antisymmetry. We now show that the relation $\prec_{CB^p(X)}$, as defined in Definition 2.1, satisfies all three of these properties:

(1) Let $A \in CB^p(X)$. Then for every $a \in A$, clearly $a \in A$ (i.e., $a = a$, which implies $a \prec_X a$. This shows that $A \prec_{CB^p(X)} A$ for all $A \in CB^p(X)$. In other words, the relation $\prec_{CB^p(X)}$ is reflexive on $CB^p(X)$.

(2) Let $A, B, C \in CB^p(X)$ such that $A \prec_{CB^p(X)} B$ and $B \prec_{CB^p(X)} C$. Since $A \prec_{CB^p(X)} B$, for every $b \in B$, there exists $a \in A$ such that $a \prec_X b$, and for every $a \in A$, there exists $b \in B$ such that $a = b$. Similarly, since $B \prec_{CB^p(X)} C$, for every $c \in C$, there exists $b \in B$ such that $b \prec_X c$, and for every $b \in B$, there exists $c \in C$ such that $b = c$. Therefore, for every $c \in C$, there exists $a \in A$ such that $a \prec_X c$, and for every $a \in A$, there

exists $c \in C$ such that $a = c$; that is, $A \prec_{CB^p(X)} C$. Hence, the relation $\prec_{CB^p(X)}$ is transitive.

(3) Let $A, B \in CB^p(X)$ such that $A \prec_{CB^p(X)} B$ and $B \prec_{CB^p(X)} A$. Since $A \prec_{CB^p(X)} B$, for every $b \in B$, there exists $a \in A$ such that $a \prec_X b$, and for every $a \in A$, there exists $b \in B$ such that $a = b$. On the other hand, since $B \prec_{CB^p(X)} A$, for every $a \in A$, there exists $b \in B$ such that $b \prec_X a$, and for every $b \in B$, there exists $a \in A$ such that $b = a$. From these conditions, it follows that $A = B$. Hence, the relation $\prec_{CB^p(X)}$, as defined in Definition 2.1, satisfies the antisymmetry property.

Therefore, from points (1)-(3) above, it has been shown that the relation $\prec_{CB^p(X)}$ in Definition 2.1 is a partial order. This concludes the proof of Theorem 2.2. □

Since the relation $\prec_{CB^p(X)}$ defined in Definition 2.1 is a partial order, the pair $(CB^p(X), \prec_{CB^p(X)})$ forms a partially ordered collection of sets. Accordingly, we can define a partially ordered p-Pompeiu-Hausdorff partial metric space as follows.

Definition 2.3. Let $(X, p, \prec_X)$ be a partially ordered partial metric space. Then the triple $(CB^p(X), H^p, \prec_{CB^p(X)})$ is called a *partially ordered p-Pompeiu-Hausdorff partial metric space* if the pair $(CB^p(X), \prec_{CB^p(X)})$ forms a partially ordered set, and the pair $(CB^p(X), H^p)$ is a p-Pompeiu-Hausdorff partial metric space.

Example 2.4. Let $X = [0, \infty)$ be equipped with the partial metric

$$p(x, y) = \max\{x, y\}$$

and the partial order $x \prec_X y$ be defined as $x \leq y$. Then we can show that $(X, p, \prec_X)$ is a partially ordered partial metric space. Let

$$A = [0, 2] \quad \text{and} \quad B = [0, 3].$$

Then both A and B are nonempty, bounded and closed in X. For subsets $A, B \subseteq X$, we have $p(A, B) = 2$ and $p(B, A) = 3$. Hence,

$$H^p(A, B) = \max\{2, 3\} = 3.$$

Let $CB^p(X) = \{A \subseteq X : A$ is nonempty, closed, and bounded in $X\}$. Then for $A, B \in CB^p(X)$, the relation $A \subseteq B$ fulfills the properties of Definition 2.1 for $\prec_X$ which is the usual order $\leq$ on X. Hence, $A \subseteq B$, it is a partial order on $CB^p(X)$. Hence the triple $(CB^p(X), H^p, \prec_{CB^p(X)})$ is a partially ordered p-Pompeiu-Hausdorff partial metric space.

The completeness property in the partial ordered p-Pompeiu-Hausdorff metric space differs from the completeness property in the classical Pompeiu-Hausdorff metric space. The main distinction lies in the type of convergence of Cauchy sequences used: the classical Pompeiu-Hausdorff metric space involves ordinary convergence of Cauchy sequences, whereas the p-Pompeiu-Hausdorff metric space involves properly convergence of Cauchy sequences. Furthermore, the completeness property in the partial ordered p-Pompeiu-Hausdorff metric space can be derived as stated in Definition 2.5. The formal statement is given below.

Definition 2.5. A triple $(CB^p(X), H^p, \prec_{CB^p(X)})$ is called a *complete partial ordered p-Pompeiu-Hausdorff metric space* if $(CB^p(X), \prec_{CB^p(X)})$ is a partially ordered set and $(CB^p(X), H^p)$ is a complete p-Pompeiu-Hausdorff metric space.

3. Fixed Point of a Set-valued Mapping in Partially Ordered *p*-Pompeiu-Hausdorff Metric Spaces

Theorem 3.1. *Let* $(CB^p(X), H^p, \prec_{CB^p(X)})$ *be a partially ordered p-Pompeiu-Hausdorff metric space, where X is a complete partially ordered partial metric space. Consider a mapping* $F : X \to CB^p(X)$ *that satisfies the following conditions*:

(1) *There exists* $x_0 \in X$ *such that* $x_1 \in F(x_0)$ *and* $x_0 \prec_X x_1$.

(2) *For any* $x, y \in X$ *with* $y \prec_X x$, *it holds that* $F(y) \prec_{CB^p(X)} F(x)$.

(3) *There exists* $\alpha \in (0, 1)$ *such that*

$$H^p(F(x), F(y)) \leq \alpha p(x, y),$$

for all $x, y \in X$ *with* $y \prec_X x$.

If for any sequence (x_n) *in X with* $x_n \to x$ *and consecutive terms comparable such that* $x \prec_X x_n$ *for all n*, *then there exists* $x \in X$ *such that* $x \in F(x)$.

Proof. Let $x_0 \in X$. Then by condition (1), there exists $x_1 \in F(x_0)$ such that $x_1 \prec_X x_0$, and hence $p(x_0, x_1) < 1$. Since $x_1 \prec_X x_0$, by condition (3), we obtain

$$H^p(F(x_0), F(x_1)) \leq \alpha p(x_0, x_1) < \alpha \cdot 1 = \alpha.$$

Furthermore, since $x_1 \prec_X x_0$, by condition (2), it follows that $F(x_1) \prec_{CB^p(X)} F(x_0)$. Because $x_1 \in F(x_0)$, by the definition of the order in $CB^p(X)$, there exists $x_2 \in F(x_1)$ such that $x_2 \prec_X x_1$. Then, by Lemma 1.1, we obtain $p(x_1, x_2) < \alpha$, since $H^p(F(x_0), F(x_1)) < \alpha$. Similarly, because $x_2 \prec_X x_1$, by condition (3), we obtain

$$H^p(F(x_1), F(x_2)) \leq \alpha p(x_1, x_2) < \alpha \cdot \alpha = \alpha^2.$$

Since $x_2 \prec_X x_1$, condition (2) implies $F(x_2) \prec_{CB^p(X)} F(x_1)$. From $x_2 \in F(x_1)$, the order definition on $CB^p(X)$ ensures that there exists $x_3 \in F(x_2)$ with $x_3 \prec_X x_2$. By Lemma 1.1,

$$p(x_2, x_3) < \alpha^2.$$

Then condition (3) gives

$$H^p(F(x_2), F(x_3)) \leq \alpha p(x_2, x_3) < \alpha \cdot \alpha^2 = \alpha^3.$$

Continuing this process, for each $n \in \mathbb{N}$, we can construct $x_n \in F(x_{n-1})$ with $x_n \prec_X x_{n-1}$, such that

$$p(x_{n-1}, x_n) < a^{n-1}.$$

Therefore,

$$H^p(F(x_{n-1}), F(x_n)) \leq \alpha p(x_{n-1}, x_n) < \alpha^n.$$

For $m, n \in \mathbb{N}$, $m > n$ such that $x_m \prec_X x_n$, the following holds:

$$\begin{aligned} H^p(F(x_n), F(x_m)) &\leq H^p(F(x_n), F(x_{n+1})) + H^p(F(x_{n+1}), F(x_m)) \\ &\quad - H^p(F(x_{n+1}), F(x_{n+1})) \\ &\leq H^p(F(x_n), F(x_{n+1})) + H^p(F(x_{n+1}), F(x_m)) \\ &\leq \alpha^{n+1} + H^p(F(x_{n+1}), F(x_m)). \end{aligned}$$

By repeating the similar argument, we derive

$$\begin{aligned} H^p(F(x_{n+1}), F(x_m)) &\leq H^p(F(x_{n+1}), F(x_{n+2})) \\ &\quad + H^p(F(x_{n+2}), F(x_m)) - H^p(F(x_{n+2}), F(x_{n+2})) \end{aligned}$$

$$\leq H^{p}(F(x_{n+1}), F(x_{n+2})) + H^{p}(F(x_{n+2}), F(x_{m}))$$

$$\leq \alpha^{n+2} + H^{p}(F(x_{n+2}), F(x_{m})).$$

Therefore,

$$\begin{aligned} H^{p}(F(x_{n}), F(x_{m})) &\leq \alpha^{n+1} + \alpha^{n+2} + \cdots + \alpha^{m-1} \\ &\quad + H^{p}(F(x_{m-1}), F(x_{m})) \\ &= \alpha^{n+1} + \alpha^{n+2} + \cdots + \alpha^{m-1} + \alpha^{m} \\ &= \alpha^{n+1}\left(\frac{1 - \alpha^{m-n-1}}{1 - \alpha}\right) \\ &\leq \alpha^{n+1}\left(\frac{1}{1 - \alpha}\right). \end{aligned}$$

Since $\alpha^{n+1} \to 0$ as $n \to \infty$, the sequence $F(x_n)$ is Cauchy. By completeness of $CB^{p}(X)$, the sequence $F(x_n)$ converges. We claim that $F(x_n) \to F(x)$, where x is the limit of the comparable sequence $(x_n) \subseteq X$. Now, let any sequence (x_n) in X be such that its consecutive terms are comparable and $x_n \to x$ as $n \to \infty$. Hence, $x \prec_X x_n$. Then, by condition (3),

$$H^{p}(F(x_n), F(x)) \leq \alpha p(x_n, x).$$

Since $x_n \to x$, for each $n \in \mathbb{N}$,

$$p(x_n, x) - p(x, x) < \frac{1}{n},$$

and thus,

$$H^{p}(F(x_n), F(x)) - H^{p}(F(x), F(x)) < \frac{\alpha}{n}.$$

Since $\frac{\alpha}{n} \to 0$, it follows that $F(x_n) \to F(x)$. Finally, since $x_n \in F(x_{n-1})$, Lemma 1.2 implies $x \in F(x)$. Hence, F has a fixed point. This concludes the proof of Theorem 3.1. □

Observe that replacing the comparability condition among the sequence terms in Theorem 3.1 with a monotonicity condition still ensures the existence of a fixed point for the set-valued mapping F. The corresponding result is presented in the following corollary:

Corollary 3.2. *Let* $(CB^p(X), H^p, \prec_{CB^p(X)})$ *be a partially ordered p-Pompeiu-Hausdorff metric space constructed from a complete partially ordered partial metric space* $(X, p, \prec_X)$. *Consider a set-valued mapping* $F : X \to CB^p(X)$ *satisfying the following conditions*:

(1) *There exist* $x_0 \in X$ *and* $x_1 \in F(x_0)$ *such that* $x_1 \prec_X x_0$ *and* $p(x_0, x_1) < 1$.

(2) *For all* $x_0, x_1 \in X$ *with* $x_0 \prec_X x_1$, *it holds that* $F(x_0) \prec_{CB^p(X)} F(x_1)$.

(3) *There exists* $\alpha \in (0, 1)$ *such that*

$$H^p(F(x), F(y)) \leq \alpha p(x, y),$$

for all $y \prec_X x$.

If every decreasing sequence (x_n) *in X that converges* (*i.e.*, $x_n \to x$) *implies* $x \prec_X x_n$, *then there exists* $x \in X$ *such that* $x \in F(x)$.

Proof. The proof of Corollary 3.2 follows the same structure as the proof of Theorem 3.1. However, the claim here is that the sequence $F(x_n)$ in $CB^p(X)$ converges to $F(x)$, where x is the limit of a monotone decreasing sequence (x_n) with $x_n \in F(x_{n-1})$ and $x_n \to x$, such that

$x \prec_X x_n$. Then, according to condition (3),

$$H^p(F(x_n), F(x)) \leq \alpha p(x_n, x).$$

Since $x_n \to x$, for every $n \in \mathbb{N}$,

$$p(x_n, x) - p(x, x) < \frac{1}{n}.$$

Hence, it follows that

$$H^p(F(x_n), F(x)) - H^p(F(x), F(x)) < \frac{\alpha}{n}.$$

This shows that $F(x_n)$ converges to $F(x)$, and the claim is thus verified. Since $x_n \in F(x_{n-1})$, by Lemma 1.2, we conclude that $x \in F(x)$. Therefore, F has a fixed point. Hence, the proof of Corollary 3.2 is complete. □

In Theorem 3.1 and Corollary 3.2, the role of Lemma 1.1 is crucial in establishing the existence of fixed points of the set-valued mapping F. However, even without employing Lemma 1.1, the existence of fixed points can still be guaranteed by imposing an additional condition. This condition states that: for any $x, y \in X$ with $x \prec_X y$, the inequality

$$p(u, v) \leq H^p(F(x), F(y))$$

holds for $u \in F(x)$ and $v \in F(y)$. This result is presented in Theorem 3.3 below.

Theorem 3.3. *Let* $(CB^p(X), H^p, \prec_{CB^p(X)})$ *be a partially ordered p-Pompeiu-Hausdorff metric space constructed from a complete partially ordered partial metric space* $(X, p, \prec_X)$, *and let* $F : X \to CB^p(X)$ *be a set-valued mapping satisfying the following conditions*:

(1) *There exist* $x_0 \in X$ *and* $x_1 \in F(x_0)$ *with* $x_1 \prec_X x_0$ *such that* $p(x_0, x_1) < 1$.

(2) *For all* $x_0, x_1 \in X$ *with* $x_0 \prec_X x_1$, $F(x_0) \prec_{CB^p(X)} F(x_1)$, *and the inequality*

$$p(u, v) \leq H^p(F(x_0), F(x_1))$$

holds for any $u \in F(x_0)$ *and* $v \in F(x_1)$.

(3) *There exists* $\alpha \in (0, 1)$ *such that*

$$H^p(F(x), F(y)) \leq \alpha p(x, y),$$

for every $y \prec_X x$.

If (x_n) *is a sequence in X such that* $x_n \to x$ *and each consecutive term is comparable with the limit, i.e.,* $x \prec_X x_n$, *then there exists* $x \in X$ *such that* $x \in F(x)$.

Proof. Let $x_0 \in X$ and $x_1 \in F(x_0)$ with $x_1 \prec_X x_0$ such that $p(x_0, x_1) < 1$. According to condition (3), since $x_1 \prec_X x_0$, there exists $\alpha \in (0, 1)$ such that

$$H^p(F(x_0), F(x_1)) \leq \alpha p(x_0, x_1) < \alpha \cdot 1 = \alpha.$$

Furthermore, since $x_1 \prec_X x_0$, it follows from condition (2) that $F(x_1) \prec_{CB^p(X)} F(x_0)$. Since $x_1 \in F(x_0)$, by the definition of the order in $CB^p(X)$, there exists $x_2 \in F(x_1)$ such that $x_2 \prec_X x_1$ and

$$p(x_1, x_2) \leq H^p(F(x_0), F(x_1)) < \alpha.$$

Note that $x_2 \prec_X x_1$, so by condition (3),

$$H^p(F(x_1), F(x_2)) \leq \alpha p(x_1, x_2) < \alpha \cdot \alpha = \alpha^2.$$

Again, since $x_2 \prec_X x_1$, condition (2) implies that $F(x_2) \prec_{CB^p(X)} F(x_1)$. Given that $x_2 \in F(x_1)$, then by the definition of the order in $CB^p(X)$, there exists $x_3 \in F(x_2)$ such that $x_3 \prec_X x_2$ and

$$p(x_2, x_3) \leq H^p(F(x_1), F(x_2)) < \alpha^2.$$

Since $x_3 \prec_X x_2$, again by condition (3),

$$H^p(F(x_2), F(x_3)) \leq \alpha p(x_2, x_3) < \alpha^3.$$

Continuing in this process, for each $n \in \mathbb{N}$, there exist $x_n \in F(x_{n-1})$ with $x_n \prec_X x_{n-1}$ such that

$$H^p(F(x_{n-1}), F(x_n)) \leq \alpha p(x_{n-1}, x_n) < \alpha^n.$$

As shown in the proof of Theorem 3.1, the sequence $(F(x_n))$ is a Cauchy sequence in $CB^p(X)$, and converges to $F(x)$, where $x_n \to x$ and $x \prec_X x_n$ for all *n*. Since $x_n \in F(x_{n-1})$, it follows from Lemma 1.2 that $x \in F(x)$. Hence, it is proven that *F* has a fixed point. Hence, the proof of Theorem 3.3 is complete. ☐

It is worth noting that Corollary 3.2 arises from a modification of the conditions of the assumptions in Theorem 3.1. In a similar manner, a corresponding modification of the assumptions in Theorem 3.3 yields the following result, presented as Corollary 3.4.

Corollary 3.4. *Suppose that* $(CB^p(X), H^p, \prec_{CB^p(X)})$ *is a partially ordered p-Pompeiu-Hausdorff metric space*, *where X is a complete partially ordered partial metric space. Let* $F : X \to CB^p(X)$ *be a set-valued mapping that satisfies the following conditions*:

(1) *There exist* $x_0 \in X$ *and* $x_1 \in F(x_0)$ *such that* $x_1 \prec_X x_0$ *and* $p(x_0, x_1) < 1$.

(2) *For any* $x_0, x_1 \in X$ *with* $x_0 \prec_X x_1$, *it holds that* $F(x_0) \prec_{CB^p(X)} F(x_1)$ *and the inequality*

$$p(u, v) \leq H^p(F(x_0), F(x_1))$$

is satisfied for all $u \in F(x_0)$ *and* $v \in F(x_1)$.

(3) *There exists a constant* $\alpha \in (0, 1)$ *such that*

$$H^p(F(x), F(y)) \leq \alpha p(x, y),$$

for all $y \prec_X x$.

If a decreasing monotone sequence (x_n) *in X converges to* $x \in X$ *in such a way that* $x \prec_X x_n$ *for all n, then F admits at least one fixed point; that is, there exists* $x \in X$ *such that* $x \in F(x)$.

Proof. As established in the proof of Theorem 3.1, we obtain a convergent sequence $(F(x_n))$ in $CB^p(X)$. In particular, $F(x_n)$ converges to $F(x)$, where x is the limit of the monotonically decreasing sequence x_n in X, that is, $x_n \to x$ with $x \prec_X x_n$ for all n. Since $x_n \in F(x_{n-1})$ for each n, it follows from Lemma 1.2 that $x \in F(x)$. Therefore, it is concluded that F admits a fixed point. □

4. Fixed Point of Set-valued Mapping in Partially Ordered *p*-Pompeiu-Hausdorff Metric Spaces

Let (X, p) be a partial metric space, and let $F, G : X \to CB^p(X)$ be set-valued mappings. For any $x, y \in X$, we use the following notations:

$$K(F, G)$$

$$= \max\left\{p(x, y), p(x, F(x)), p(y, G(y)), \frac{1}{2} p(x, G(y)) + p(y, F(x))\right\}.$$

Theorem 4.1. *Suppose that* $(CB^p(X), H^p, \prec_{CB^p(X)})$ *be a partially ordered p-Pompeiu-Hausdorff metric space, where X is a complete* *partially ordered partial metric space. Consider two continuous mappings* $F, G : X \to CB^p(X)$ *satisfying the following conditions*:

(1) *There exist* $x_0 \in X$ *and* $x_1 \in F(x_0)$ *such that* $x_1 \prec_X x_0$.

(2) *For all* $x, y \in X$ *with* $y \prec_X x$, *either* $F(y) \prec_{CB^p(X)} G(x)$ *or* $G(y) \prec_{CB^p(X)} F(x)$.

(3) *There exists* $\alpha \in (0, 1)$ *such that*

$$H^p(F(x), G(y)) \leq \alpha K(F, G)$$

for each $y \prec_X x$.

If (x_n) *is any sequence in X with comparable consecutive terms such that* $x_n \to x$ *and* $x \prec_X x_n$, *then F and G have a common fixed point*.

Proof. Take $x_0 \in X$. Based on condition (1), we can choose $x_1 \in F(x_0)$ such that $x_1 \prec_X x_0$. Hence, from condition (2), we have $G(x_1) \prec_{CB^p(X)} F(x_0)$.

Since $x_1 \in F(x_0)$, by the definition of the partial order on $CB^p(X)$, there exists $x_2 \in G(x_1)$ with $x_2 \prec_X x_1$. Now, applying condition (3), we obtain

$$\begin{aligned}
& H^p(F(x_0), G(x_1)) \\
& \leq \alpha K(F, G) \leq \alpha \max\Big\{p(x_0, x_1), p(x_0, F(x_0)), p(x_1, G(x_1)), \\
& \qquad \frac{1}{2}[p(x_0, G(x_1)) + p(x_1, F(x_0))]\Big\} \\
& \leq \alpha \max\Big\{p(x_0, x_1), p(x_0, x_1), p(x_1, x_2) \frac{1}{2}[p(x_0, x_1) + p(x_1, x_2)]\Big\} \\
& \leq \alpha \max\{p(x_0, x_1), p(x_1, x_2)\}.
\end{aligned}$$

If $p(x_1, x_2)$ is the maximum value, then

$$p(x_1, x_2) \leq H^p(F(x_0), G(x_1)) \leq \alpha p(x_1, x_2),$$

which leads to a contradiction, since $\alpha \in (0, 1)$. Therefore, $p(x_0, x_1)$ must be the maximum value, and we conclude that

$$p(x_1, x_2) \leq H^p(F(x_0), G(x_1)) \leq \alpha p(x_0, x_1).$$

That is,

$$p(x_1, x_2) \leq \alpha p(x_0, x_1).$$

Since $x_2 \prec_X x_1$, by condition (2), it follows that $F(x_2) \prec_{CB^p(X)} G(x_1)$. Hence, we can choose $x_3 \in F(x_2)$ such that $x_3 \prec_X x_2$. Since $x_2 \prec_X x_1$, condition (3) again yields:

$$\begin{aligned}
&H^p(F(x_2), G(x_1)) \\
&\leq \alpha \max\Big\{p(x_2, x_1),\ p(x_2, F(x_2)),\ p(x_1, G(x_1)), \\
&\qquad \frac{1}{2}[p(x_2, G(x_1)) + p(x_1, F(x_2))]\Big\}.
\end{aligned}$$

Since $x_3 \in F(x_2)$ and $x_2 \in G(x_1)$,

$$\begin{aligned}
&H^p(F(x_2), G(x_1)) \\
&\leq \alpha \max\Big\{p(x_1, x_2),\ p(x_2, x_3),\ p(x_1, x_2),\ \frac{1}{2}[p(x_2, x_2) + p(x_1, x_3)]\Big\}.
\end{aligned}$$

Since $p(x_1, x_3) \leq p(x_1, x_2) + p(x_2, x_3) - p(x_2, x_2)$, we obtain

$$\begin{aligned}
&H^p(F(x_2), G(x_1)) \\
&\leq \alpha \max\{p(x_1, x_2),\ p(x_2, x_3)\}, \\
&\quad \frac{1}{2}[p(x_2, x_2) + p(x_1, x_2) + p(x_2, x_3) - p(x_2, x_2)].
\end{aligned}$$

Thus,

$$\begin{aligned} H^p(F(x_2), G(x_1)) &\leq \alpha \max\{p(x_1, x_2), p(x_2, x_3), \\ &\qquad \frac{1}{2}[p(x_1, x_2) + p(x_2, x_3)]\} \\ &\leq \alpha \max\{p(x_1, x_2), p(x_2, x_3)\}. \end{aligned}$$

If $p(x_2, x_3)$ is the maximum, then

$$p(x_2, x_3) \leq H^p(F(x_2), G(x_1)) \leq \alpha p(x_2, x_3),$$

which is again a contradiction. Hence,

$$p(x_2, x_3) \leq \alpha p(x_1, x_2) \leq \alpha^2 p(x_0, x_1).$$

By continuing this process, for each $n \in \mathbb{N}$, with $x_{2n+1} \in F(x_{2n})$ and also $x_{2n+2} \in G(x_{2n+1})$, we obtain the inequality

$$p(x_n, x_{n+1}) \leq \alpha^n p(x_0, x_1).$$

Next, we show that the sequence (x_n) is a Cauchy sequence in X.

Let $m, n \in \mathbb{N}$ with $n > m$. Then

$$\begin{aligned} p(x_n, x_m) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_m) - p(x_{n+1}, x_{n+1}) \\ &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_m) \\ &\leq \alpha^n p(x_0, x_1) + p(x_{n+1}, x_m). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} p(x_{n+1}, x_m) &\leq p(x_{n+1}, x_{n+2}) + p(x_{n+2}, x_m) - p(x_{n+2}, x_{n+2}) \\ &\leq p(x_{n+1}, x_{n+2}) + p(x_{n+2}, x_m) \\ &\leq \alpha^{n+1} p(x_0, x_1) + p(x_{n+2}, x_m). \end{aligned}$$

By continuing this pattern, we obtain

$$\begin{aligned} p(x_n, x_m) &\leq (\alpha^n + \alpha^{n+1} + \alpha^{n+2} + \cdots + \alpha^{m-1}) p(x_0, x_1) \\ &= \alpha^n (1 + \alpha + \alpha^2 + \cdots + \alpha^{m-n-1}) p(x_0, x_1) \\ &= \alpha^n p(x_0, x_1) \sum_{i=0}^{m-n-1} \alpha^i \\ &= \alpha^n p(x_0, x_1) \left(\frac{1 - \alpha^{m-n}}{1 - \alpha} \right) \\ &< \alpha^n p(x_0, x_1) \left(\frac{1}{1 - \alpha} \right). \end{aligned}$$

Since $\alpha \in (0, 1)$, it follows that $p(x_n, x_m) \to 0$ as $n \to \infty$.

This implies that (x_n) is a Cauchy sequence in X. Because X is complete, there exists $x \in X$ such that $x_n \to x$ as $n \to \infty$.

Therefore,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} x_{2n+1} = x.$$

Since F is continuous, it follows that $F(x_{2n}) \to F(x)$.

Because $x_{2n+1} \in F(x_{2n})$, by Lemma 1.2, we obtain $x \in F(x)$. On the other hand,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} x_{2n+2} = x.$$

Since G is continuous, it follows that $G(x_{2n+1}) \to G(x)$.

As $x_{2n+2} \in G(x_{2n+1})$, by Lemma 1.2, we obtain $x \in G(x)$. Hence, x is a common fixed point of F and G. $\square$

Acknowledgements

This work is supported by Directorate of Research and Community Service, Ministry of Higher Education, Science and Technology, Indonesia (Grant numbers: 624/UN10.A0501/B/PT.01.03.2/2025 dated 2 June 2025).

References

[1] H. Aydi, M. Abbas and C. Vetro, Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces, Topology Appl. 159 (2012), 3234-3242.

[2] I. Beg and A. Butt, Common fixed point for generalized set valued contractions satisfying implicit relation in partially ordered metric spaces, Math. Commun. 15(1) (2010), 65-76.

[3] I. Beg and A. Butt, Fixed point theorems for set valued mappings in partially ordered metric spaces, Int. J. Math. Sci. 7 (2013), 66-68.

[4] G. Bouligand, Introduction à la géométrie infinitésimale directe, Vuibert, Paris, 1932.

[5] S. Carl and S. Heikkila, Fixed Point Theory in Ordered Sets and Applications: From Differential and Integral Equations to Game Theory, Springer, New York, 2011.

[6] N. Chandra, M. Arya and M. Joshi, Coincidence point theorem for generalized contraction in partial metric spaces, Recent Advances in Fixed Point Theory and Applications, Nova Science Publishers, Inc (USA), 2017.

[7] P. Dhawan and J. Kaur, Some common fixed point theorem in ordered partial metric spaces via *f*-generalization contractive type mappings, Mathematics 7 (2019), 193.

[8] A. Ekayanti, Marjono, M. Muslikh and S. Fitri, Set-valued mapping's properties on metric space induced by partial metric, AIP Conf. Proc. 3235(1) (2024), 020007.

[9] Y. Feng and S. Liu, Fixed point theorems for multi-valued increasing operators in partial ordered spaces, Soochow J. Math. 30(4) (2004), 461-469.

[10] R. O. Gregorio and P. S. Macansantos, Fixed point theorems and stability of fixed point sets of multivalued mappings, Advances in Fixed Point Theory 3(4) (2013), 735-746.

[11] F. Hausdorff, Grundzüge der Mengenlehre, Leipzig Veit, Leipzig, Germany, 1914.

[12] S. Kakutani, A generalization of Brouwer’s fixed point theorem, Duke Math. J. 8(3) (1941), 457-459.

[13] K. Kuratowski, Topology, Academic Press, New York, Vol. II, 1966.

[14] S. G. Matthews, Partial metric topology, Ann. New York Acad. Sci. 806 (1994), 304-315.

[15] S. G. Matthews, Partial Metric Topology, Dept. of Computer Science, University of Warwick, 1994.

[16] M. Muslikh, Partial metric on space of subsets, Glob. J. Pure Appl. Math. 11 (2015), 2719-2734.

[17] M. Muslikh, S. Fitri Fungsi bernilai himpunan, UB Press, Malang, Indonesia, 2022.

[18] S. B. Nadler, Multi-valued contraction mappings, Pacific J. Math. 30(2) (1969), 457-488.

[19] P. Painlevé, Sur les équations différentielles du second ordre à points critiques fixes, CR Acad. Sci. Paris 143 (1906), 1111-1117.

[20] N. S. Rao and K. Kalyani, Unique fixed point theorems in partially ordered metric spaces, Heliyon 6 (2020), e05563.