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A MIXED FINITE ELEMENT-CHARACTERISTIC MIXED VOLUME ELEMENT AND CONVERGENCE ANALYSIS OF DARCY-FORCHHEIMER DISPLACEMENT PROBLEM

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Abstract

A mixed finite element-characteristic mixed volume element is presented to solve three-dimensional incompressible Darcy-Forchheimer miscible displacement, and convergence analysis is shown in this paper. A mixed finite element approximation is applied to obtain the pressure and Darcy-Forchheimer velocity, and the accuracy of velocity is improved one order. The concentration is computed by a coupled scheme of characteristics and mixed volume element, where the diffusion is treated by the mixed volume element and the convection is treated by the method of characteristics. The method of characteristics has strong computation stability at sharp fronts and it can avoid numerical dispersion and nonphysical oscillation. Larger time-steps along the characteristics are shown to result in smaller time-truncation errors than those resulting from standard methods. More important in numerical simulation of seepage mechanics, mixed volume element has the property of conservation on each element and it can obtain numerical solution of the concentration and its adjoint vector function simultaneously. Using some techniques of priori estimates of differential equations, we show an optimal second order estimate in discrete L^2 norm. Numerical data are consistent with theoretical analysis, and the composite combination method could possibly become a powerful tool for solving the actual problems in porous media.

1. Introduction

To consider the miscible displacement of two phases in porous media, a pressure-velocity mathematical model of Darcy-Forchheimer flow is put forward. A nonlinear partial differential system with initial-boundary conditions is formulated as follows to interpret the incompressible displacement [3, 12, 14, 24].

$$\mu(c)\kappa^{-1}\mathbf{u} + \beta\rho(c)|\mathbf{u}|\mathbf{u} + \nabla p = r(c)\nabla d,$$

$$X = (x, y, z)^T \in \Omega, \quad t \in J = (0, \bar{T}], \tag{1.1a}$$

$$\nabla \cdot \mathbf{u} = q = q_I + q_p, \quad X \in \Omega, \quad t \in J, \tag{1.1b}$$

$$\phi \frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla c - \nabla \cdot (D(\mathbf{u}) \nabla c) = q_I c = q_I c_I, \quad X \in \Omega, \ t \in J, \tag{1.2}$$

where Ω is a bounded domain in $\mathcal{R}^3$.

To describe the displacement of highspeed flow in heterogeneous media especially nearby the wells [12], Forchheimer put forward one mathematical model of Darcy-Forchheimer flow and discussed its numerical simulation in [3]. As $\beta = 0$, Darcy-Forchheimer's law deduces into Darcy's law. The law of Darcy-Forchheimer was discussed in [24], and the regularity was analyzed in [14].

The conservation of mass is formulated by (1.1) and (1.2). $p(X, t)$ and $\mathbf{u}(X, t)$ denote the pressure and Darcy velocity, respectively. $c(X, t)$ denotes the concentration of one component. $\kappa(X)$, $\phi(X)$ and $\beta(X)$ mean the absolute permeability, porosity and Forchheimer coefficient, respectively. $r(X, c)$ is the gravity, and $d(X)$ is the vertical coordinate. $q(X, t)$ is the quantity, usually defined by a linear function of the production q_p and the injection q_I, i.e., $q(X, t) = q_I(X, t) + q_p(X, t)$. c_I, a known variable, is the concentration of injected fluid, and $c(X, t)$ is the concentration of the production well.

Suppose that two fluids are incompressible and the total volume of their mixture is not decreasing. Suppose that they have no chemical reaction during the mixture. Let ρ_1 and ρ_2 denote the densities of different fluids and let the density of their mixture be denoted by

$$\rho(c) = c\rho_1 + (1 - c)\rho_2. \tag{1.3}$$

The mixture's viscosity is determined by

$$\mu(c) = (c\mu_1^{-1/4} + (1 - c)\mu_2^{-1/4})^{-4}. \tag{1.4}$$

The diffusion coefficient $D(\mathbf{u})$ is defined by a tensor of molecular diffusion and engineering dispersion

$$D(\mathbf{u}) = \phi d_m I + |\mathbf{u}|(d_l E(\mathbf{u}) + d_t E^{\perp}(\mathbf{u})), \tag{1.5}$$

where d_m, d_l and d_t represent the molecular diffusion, the diffusion coefficients in the direction of the flow and the transverse direction to the flow, respectively. I denotes a 3×3 identity matrix, $E\mathbf{u} = \mathbf{u} \otimes \mathbf{u} / |\mathbf{u}|^2$, $E^{\perp}(\mathbf{u}) = I - E(\mathbf{u})$.

Initial conditions and boundary conditions:

$$\mathbf{u} \cdot \nu = 0, \, (D(\mathbf{u})\nabla c - \mathbf{u}c) \cdot \nu = 0, \quad X \in \partial\Omega, \, t \in J, \tag{1.6a}$$

$$c(X, 0) = \hat{c}_0(X), \quad X \in \Omega, \tag{1.6b}$$

where ν is the outer normal vector to the boundary surface, denoted by $\partial\Omega$.

For two-phase typical displacement problems of Darcy flow in porous media, the group of Douglas illustrated a series of research work in [9, 10, 29-32]. While they only studied numerical simulation of Darcy flow's displacement. A mixed finite element method was introduced to solve the Forchheimer equation by Girault and Wheeler in [15]. They adopted piece-wise defined constant functions to approximate the velocity and use Crouzeix-Raviart element to solve the pressure. This mixed approximation is called the prime mixed element. Raviart-Thomas mixed element was applied to solve Forchheimer equation and its theoretical analysis was discussed in [17, 19]. A block-centered finite difference method was presented to argue two miscible displacement problems (incompressible or compressible) in [20-22]. A semidiscrete scheme of the mixed element method and convergence analysis was argued to solve time-dependent problems by Douglas and Park in [11, 18]. The discussions of Forchheimer flow seepage displacement in porous media are developed and generalized from Douglas's research, and the mathematical model of Forchheimer flow can describe the displacement of highspeed flow in heterogeneous media.

Standard finite element method does produce strong numerical dispersion and nonphysical oscillation in solving convection-dominated diffusion problems. To overcome these numerical faults, many scholars put forward a variety of new approximation techniques such as the Eulerian-Lagrangian localized adjoint method (ELLAM) [5]. ELLAM can conserve mass locally but need large-scale computation to evaluate the resulting integrals. To improve the computational order, Arbogast and Wheeler presented a locally conservative method of characteristic mixed element in the time-space variation form in [2]. Using the postprocessing technique, Arbogast and Wheeler derived error estimates of $3/2$ order. While many mapping integrals of test functions were introduced, it made the computation more complicated and difficult. We extended the work of Arbogast and Wheeler [2], and presented a combination method of mixed finite element and characteristic mixed finite element [26]. Based on numerical experiments, it followed that computational work was reduced greatly and this method was feasible and effective to solve many actual problems. While we only obtained first-order error estimates and it was not applicable for three-dimensional problems. Finite volume element scheme has the simplicity of finite difference and the high-order computational accuracy of finite element method. Moreover, an important physical nature of local conservation of mass was shown in [26, 27]. So it was used efficiently to solve partial differential equations. Mixed finite element was used to obtain the pressure and Darcy velocity simultaneously, and the accuracy was improved by one order. A mixed finite volume element scheme was studied in [23, 28] based on combining finite volume element and mixed element. Its computational efficiency was testified experimentally in [4, 16, 33]. Convergence analysis was mainly stated for elliptic problems in [6-8], and a general theoretical framework was given.

Based on the above work, a combination method of mixed element and characteristic mixed volume element is presented to simulate three-dimensional two-phase displacement of Darcy-Forchheimer flow in this paper. The pressure and Darcy-Forchheimer velocity are computed simultaneously by a mixed element method improving the one-order computational accuracy. The characteristic mixed volume element is adopted to solve the concentration, where the method of characteristics is used for the convection and the scheme of mixed volume element is applied for the diffusion. The method of characteristics has strong stability and high accuracy at sharp fronts, and eliminates numerical dispersion. More important in actual computations, the large time steps can be adopted possibly with no loss of stability or accuracy. The mixed volume element can obtain the concentration and adjoint vector simultaneously, and it keeps the local conservation by taking piece-wise defined constants as test functions. The conservative nature plays an important role in numerical simulation of seepage mechanics. Considering the nature of mixed finite element and applying some theoretical techniques such as the variation, energy estimates, induction hypothesis, Sobolev embedding theorem, numerical theory and useful techniques of a priori estimates, we obtain optimal order estimates in L^2 norm without postprocessing, superior to $3/2$-order result of Arbogast and Wheeler [2]. In this paper, numerical data are illustrated for a three-dimensional system of elliptic-convection-diffusion equations to show the feasibility and the consistency with theoretical analysis. Thus this method may be taken as an efficient tool to solve the well-known problems successfully [9, 10, 25, 29].

Suppose that the coefficients of (1.1) and (1.2) are positive definite,

$$0 < a_* |X|^2 \le (\mu(c)\kappa^{-1}(X)X) \cdot X \le a^* |X|^2,\ 0 < \phi_* \le \phi(X) \le \phi^*,$$

$$\text{(C)}\ \ 0 < D_* |X|^2 \le (D(X, \mathbf{u})X) \cdot X \le D^* |X|^2,\ 0 < \rho_* \le \rho(c) \le \rho^*,$$

$$\left|\frac{\partial(\kappa/\mu)}{\partial c}(X, c)\right| + \left|\frac{\partial r}{\partial c}(X, c)\right| + |\nabla\phi| + \left|\frac{\partial D}{\partial \mathbf{u}}(X, \mathbf{u})\right|$$

$$+ |q_I(X, t)| + \left|\frac{\partial q_I}{\partial t}(X, t)\right| \le K^*, \tag{1.7}$$

where $a_*, a^*, \phi_*, \phi^*, D_*, D^*, \rho_*, \rho^*$ and K^* are positive constants.

The regularity assumptions of (1.1)-(1.5) are given as follows:

$$\text{(R)}\ \begin{cases} p \in L^{\infty}(H^{k+1}), \\ \mathbf{u} \in L^{\infty}(H^{k+1}(\mathrm{div})) \cap L^{\infty}(W^{1,\infty}) \cap W^{1,\infty}(L^{\infty}) \cap H^{2}(L^{2}), \\ c \in L^{\infty}(H^{l+1}) \cap H^{1}(H^{l+1}) \cap L^{\infty}(W^{1,\infty}) \cap H^{2}(L^{2}). \end{cases}$$

To give numerical analysis of the composite combination scheme, we take $k \geq 1$. Common notations of Sobolev space are adopted in the present paper.

For simplicity, we assume that (1.1)-(1.5) is Ω-periodic [9, 10, 30, 31], i.e., all the functions are Ω-periodic. This is physically reasonable, since no-flow boundaries (1.6a) are generally treated by reflection, and because in general interior flow patterns are much more important than boundary effects. Thus, the no-flow boundary conditions above can be dropped [9, 10, 30, 31].

In the following discussions, the symbols K and ε denote a generic positive constant and a generic small positive number, respectively. They may have different definitions at different places.

2. Notation and Preparations

Two different partitions are given to construct the procedures. The pressure and Darcy-Forchheimer velocity are computed on the large-step nonuniform partition, and the concentration is obtained on the small-step nonuniform partition. The large-step partition is considered first.

For simplicity to consider a three-dimensional problem, take $\Omega = \{[0, 1]\}^3$ and let the boundary denoted by $\partial\Omega$. Ω is partitioned by $\delta_x \times \delta_y \times \delta_z$,

$$\delta_x : 0 = x_{1/2} < x_{3/2} < \cdots < x_{N_x-1/2} < x_{N_x+1/2} = 1,$$

$$\delta_y : 0 = y_{1/2} < y_{3/2} < \cdots < y_{N_y-1/2} < y_{N_y+1/2} = 1,$$

$$\delta_z : 0 = z_{1/2} < z_{3/2} < \cdots < z_{N_z-1/2} < z_{N_z+1/2} = 1.$$

For $i = 1, 2, ..., N_x$; $j = 1, 2, ..., N_y$; $k = 1, 2, ..., N_z$, define the following notation,

$$\Omega_{ijk} = \{(x, y, z) \mid x_{i-1/2} < x < x_{i+1/2}, y_{j-1/2} < y < y_{j+1/2},$$

$$z_{k-1/2} < z < z_{k+1/2}\},$$

$$x_i = (x_{i-1/2} + x_{i+1/2})/2, \quad y_j = (y_{j-1/2} + y_{j+1/2})/2,$$

$$z_k = (z_{k-1/2} + z_{k+1/2})/2, \quad h_{x_i} = x_{i+1/2} - x_{i-1/2},$$

$$h_{y_j} = y_{j+1/2} - y_{j-1/2}, \quad h_{z_k} = z_{k+1/2} - z_{k-1/2},$$

$$h_{x, i+1/2} = (h_{x_i} + h_{x_{i+1}})/2 = x_{i+1} - x_i,$$

$$h_{y, j+1/2} = (h_{y_j} + h_{y_{j+1}})/2 = y_{j+1} - y_j,$$

$$h_{z, k+1/2} = (h_{z_k} + h_{z_{k+1}})/2 = z_{k+1} - z_k,$$

$$h_x = \max_{1 \le i \le N_x} \{h_{x_i}\}, \quad h_y = \max_{1 \le j \le N_y} \{h_{y_j}\}, \quad h_z = \max_{1 \le k \le N_z} \{h_{z_k}\},$$

$$h_p = (h_x^2 + h_y^2 + h_z^2)^{1/2}.$$

The partition is regular, if there exist two positive constants α_1 and α_2 such that

$$\min_{1 \le i \le N_x} \{h_{x_i}\} \ge \alpha_1 h_x, \quad \min_{1 \le j \le N_y} \{h_{y_j}\} \ge \alpha_1 h_y, \quad \min_{1 \le k \le N_z} \{h_{z_k}\} \ge \alpha_1 h_z,$$

$$\min\{h_x, h_y, h_z\} \ge \alpha_2 \max\{h_x, h_y, h_z\},$$

where α_1 and α_2 depend on $\delta_x \times \delta_y \times \delta_z$, the partition of Ω. The simple case of $N_x = 4, N_y = 3, N_z = 3$ is illustrated in Figure 1. Define an experimental function space by $M_l^d(\delta_x) = \{f \in C^l[0, 1] : f|_{\Omega_i} \in p_d(\Omega_i),$ $i = 1, 2, ..., N_x\}$, where $p_d(\Omega_i)$ denotes a space consisting of polynomial functions of degree at most d constricted on $\Omega_i = [x_{i-1/2}, x_{i+1/2}]$. The function f may be discontinuous on $[0, 1]$ as $l = -1$. The spaces $M_l^d(\delta_y)$,

$M_l^d(\delta_z)$ are defined similarly. Let

$$S_h = M_{-1}^0(\delta_x) \otimes M_{-1}^0(\delta_y) \otimes M_{-1}^0(\delta_z),$$

$$\begin{aligned} V_h = \{\mathbf{w} \,|\, \mathbf{w} = (w^x, w^y, w^z),\ & w^x \in M_0^1(\delta_x) \otimes M_{-1}^0(\delta_y) \otimes M_{-1}^0(\delta_z), \\ & w^y \in M_{-1}^0(\delta_x) \otimes M_0^1(\delta_y) \otimes M_{-1}^0(\delta_z), \\ & w^z \in M_{-1}^0(\delta_x) \otimes M_{-1}^0(\delta_y) \otimes M_0^1(\delta_z),\ \mathbf{w} \cdot \gamma|_{\partial\Omega} = 0\}. \end{aligned}$$

For a grid function $v(x, y, z)$, let v_{ijk}, $v_{i+1/2, jk}$, $v_{i, j+1/2, k}$ and $v_{ij, k+1/2}$ denote the values at (x_i, y_j, z_k), $(x_{i+1/2}, y_j, z_k)$, $(x_i, y_{j+1/2}, z_k)$ and $(x_i, y_j, z_{k+1/2})$, respectively.

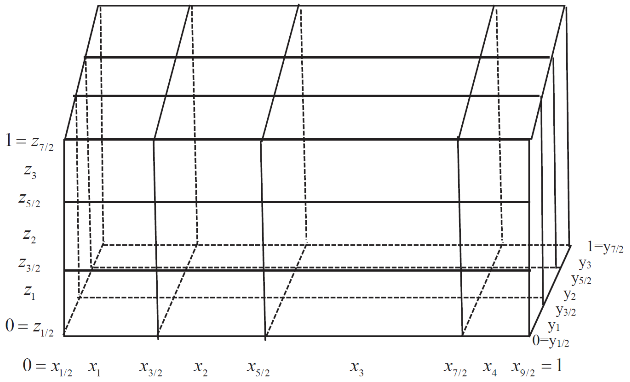

Figure 1. Nonuniform partition.

Define the inner products and norms as follows:

$$(v, w)_{\overline{m}} = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \sum_{k=1}^{N_z} h_{x_i} h_{y_j} h_{z_k} v_{ijk} w_{ijk},$$

$$(v, w)_x = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \sum_{k=1}^{N_z} h_{x_{i-1/2}} h_{y_j} h_{z_k} v_{i-1/2, jk} w_{i-1/2, jk},$$

$$(v, w)_y = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \sum_{k=1}^{N_z} h_{x_i} h_{y_{j-1/2}} h_{z_k} v_{i,\, j-1/2,\, k} w_{i,\, j-1/2,\, k},$$

$$(v, w)_z = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \sum_{k=1}^{N_z} h_{x_i} h_{y_j} h_{z_{k-1/2}} v_{ij,\, k-1/2} w_{ij,\, k-1/2},$$

$$\| v \|_s^2 = (v, v)_s, \; s = \overline{m}, \; x, \; y, \; z,$$

$$\| v \|_\infty = \max_{1 \le i \le N_x, 1 \le j \le N_y, 1 \le k \le N_z} | v_{ijk} |,$$

$$\| v \|_{\infty(x)} = \max_{1 \le i \le N_x, 1 \le j \le N_y, 1 \le k \le N_z} | v_{i-1/2,\, jk} |,$$

$$\| v \|_{\infty(y)} = \max_{1 \le i \le N_x, 1 \le j \le N_y, 1 \le k \le N_z} | v_{i,\, j-1/2,\, k} |,$$

$$\| v \|_{\infty(z)} = \max_{1 \le i \le N_x, 1 \le j \le N_y, 1 \le k \le N_z} | v_{ij,\, k-1/2} |.$$

For a vector function $\mathbf{w} = (w^x, w^y, w^z)^T$, let

$$|\| \mathbf{w} \|| = (\| w^x \|_x^2 + \| w^y \|_y^2 + \| w^z \|_z^2)^{1/2},$$

$$|\| \mathbf{w} \||_\infty = \| w^x \|_{\infty(x)} + \| w^y \|_{\infty(y)} + \| w^z \|_{\infty(z)},$$

$$\| \mathbf{w} \|_{\overline{m}} = (\| w^x \|_{\overline{m}}^2 + \| w^y \|_{\overline{m}}^2 + \| w^z \|_{\overline{m}}^2)^{1/2},$$

$$\| \mathbf{w} \|_\infty = \| w^x \|_\infty + \| w^y \|_\infty + \| w^z \|_\infty.$$

Let

$$W_p^m(\Omega) = \left\{ v \in L^p(\Omega) \,\middle|\, \frac{\partial^n v}{\partial x^{n-l-r} \partial y^l \partial z^r} \in L^p(\Omega), \; n - l - r \ge 0, \right.$$

$$\left. l = 0, 1, \ldots, n; \; r = 0, 1, \ldots, n, \; n = 0, 1, \ldots, m; \; 0 < p < \infty \right\},$$

and $H^m(\Omega) = W_2^m(\Omega)$. Inner product and norm in $L^2(\Omega)$ are denoted by $(\cdot, \cdot)$, and $\| \cdot \|$. It holds obviously for $v \in S_h$,

$$\| v \|_{\overline{m}} = \| v \|. \tag{2.1}$$

Introduce the difference operators and other notation as follows:

$$[d_x v]_{i+1/2, jk} = \frac{v_{i+1, jk} - v_{ijk}}{h_{x, i+1/2}}, \; [d_y v]_{i, j+1/2, k} = \frac{v_{i, j+1, k} - v_{ijk}}{h_{y, j+1/2}},$$

$$[d_z v]_{ij, k+1/2} = \frac{v_{ij, k+1} - v_{ijk}}{h_{z, k+1/2}}, \; [D_x w]_{ijk} = \frac{w_{i+1/2, jk} - w_{i-1/2, jk}}{h_{x_i}},$$

$$[D_y w]_{ijk} = \frac{w_{i, j+1/2, k} - w_{i, j-1/2, k}}{h_{y_j}}, \; [D_z w]_{ijk} = \frac{w_{ij, k+1/2} - w_{ij, k-1/2}}{h_{z_k}},$$

$$\hat{w}_{ijk}^x = \frac{w_{i+1/2, jk}^x + w_{i-1/2, jk}^x}{2}, \; \hat{w}_{ijk}^y = \frac{w_{i, j+1/2, k}^y + w_{i, j-1/2, k}^y}{2},$$

$$\hat{w}_{ijk}^z = \frac{w_{ij, k+1/2}^z + w_{ij, k-1/2}^z}{2}, \; \overline{w}_{ijk}^x = \frac{h_{x, i+1}}{2h_{x, i+1/2}} w_{ijk} + \frac{h_{x, i}}{2h_{x, i+1/2}} w_{i+1, jk},$$

$$\overline{w}_{ijk}^y = \frac{h_{y, j+1}}{2h_{y, j+1/2}} w_{ijk} + \frac{h_{y, j}}{2h_{y, j+1/2}} w_{i, j+1, k},$$

$$\overline{w}_{ijk}^z = \frac{h_{z, k+1}}{2h_{z, k+1/2}} w_{ijk} + \frac{h_{z, k}}{2h_{z, k+1/2}} w_{ij, k+1},$$

and $\hat{\mathbf{w}}_{ijk} = (\hat{w}_{ijk}^x, \hat{w}_{ijk}^y, \hat{w}_{ijk}^z)^T$, $\overline{\mathbf{w}}_{ijk} = (\overline{w}_{ijk}^x, \overline{w}_{ijk}^y, \overline{w}_{ijk}^z)^T$. $d_s(s = x, y, z)$ and $D_s(s = x, y, z)$ are the difference quotient operators independent of the coefficient D in (1.2). Let L denote a positive integer, $\Delta t_c = T/L$, $t^n = n\Delta t$, and let v^n denote the value of v at t^n, $d_t v^n = (v^n - v^{n-1})/\Delta t$.

Using the above notation, we give four preparations for convergence analysis.

Lemma 1. *For* $v \in S_h$ *and* $\mathbf{w} \in V_h$,

$$(v, D_x w^x)_{\overline{m}} = -(d_x v, w^x)_x, \; (v, D_y w^y)_{\overline{m}} = -(d_y v, w^y)_y,$$

$$(v, D_z w^z)_{\overline{m}} = -(d_z v, w^z)_z. \tag{2.2}$$

Lemma 2. *For* $\mathbf{w} \in V_h$,

$$\| \hat{\mathbf{w}} \|_{\overline{m}} \le ||| \mathbf{w} |||. \tag{2.3}$$

Lemma 3. *For* $q \in S_h$,

$$\| \bar{q}^x \|_x \le M \| q \|_m, \; \| \bar{q}^y \|_y \le M \| q \|_m, \; \| \bar{q}^z \|_z \le M \| q \|_m, \tag{2.4}$$

where M is a constant independent on q and h.

Lemma 4. *For* $\mathbf{w} \in V_h$,

$$\| w^x \|_x \le \| D_x w^x \|_{\overline{m}}, \; \| w^y \|_y \le \| D_y w^y \|_{\overline{m}}, \; \| w^z \|_z \le \| D_z w^z \|_{\overline{m}}. \tag{2.5}$$

The small-step partition is refined usually by halving or quartering the large-step partition uniformly, i.e., $h_{\hat{c}} = h_p/2$ or $h_{\hat{c}} = h_p/4$. Other notation is defined above.

3. The Procedures of Mixed Element-Characteristic Mixed Volume Element

3.1. The procedures

In this subsection, we present the procedures to simulate Darcy-Forchheimer flow. The mixed element is considered first for the convection. Now, we have some notation in Sobolev space:

$$X = \{\mathbf{u} \in L^2(\Omega)^3, \nabla \cdot \mathbf{u} \in L^2(\Omega), \mathbf{u} \cdot \gamma = 0\}, \; \| \mathbf{u} \|_X = \| \mathbf{u} \|_{L^2} + \| \nabla \cdot \mathbf{u} \|_{L^2},$$

$$M = L_0^2(\Omega) = \left\{ p \in L^2(\Omega) : \int_\Omega p dX = 0 \right\}, \quad \| p \|_M = \| p \|_{L^2},$$

$$V = H^1(\Omega), \quad \| c \|_V = \| c \|_{H^1}. \tag{3.1}$$

Weak form of the pressure (1.1) is given to find $(\mathbf{u}, p) : (0, T] \to (X, M)$,

$$\int_\Omega (\mu(c)\kappa^{-1}\mathbf{u} + \beta\rho(c)| \mathbf{u} |\mathbf{u}) \cdot v dX - \int_\Omega p\nabla v dX$$

$$= \int_\Omega r(c)\nabla d \cdot v dX, \ \forall v \in X, \tag{3.2a}$$

$$-\int_\Omega w\nabla \cdot \mathbf{u} dX = -\int_\Omega wq dX, \quad \forall w \in M. \tag{3.2b}$$

For (1.2), since the concentration changes along the characteristics, so we use a modified method of characteristics to approximate the hyperbolic term. The computational algorithm can use large time step but has strong stability and high accuracy [9, 10, 29-32]. Let $\psi(X, \mathbf{u}) = [\phi^2(X) + | \mathbf{u} |^2]^{1/2}$, $\frac{\partial}{\partial\tau} = \psi^{-1}\left\{\phi\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right\}$. To argue the diffusion term by using mixed volume element method, we re-express (1.2) in a standard form

$$\psi\frac{\partial c}{\partial\tau} + \nabla \cdot g = f(X, c), \tag{3.3a}$$

$$\mathbf{g} = -D(\mathbf{u})\nabla c, \tag{3.3b}$$

where $f(X, c) = (c_I - c)q_I$.

Since the concentration changes more rapidly than the pressure and velocity, so we adopt two time steps, a large step, denoted by Δt_p, for the pressure (1.1) and a small step, denoted by Δt_c, for the concentration (1.2). For the pressure, time interval $[0, T]$ is divided by $0 = t_0 < t_1 < \cdots < t_M = T$, where the first time step is denoted by $\Delta t_{p,1} = t_1 - t_0$ and $t_i = \Delta t_{p,1} + (i-1)\Delta t_p, \ i \geq 1$. Similarly, another partition is given for the concentration, $0 = t^0 < t^1 < \cdots < t^N = T, \ t^n = n\Delta t_c$. Suppose that there exists a positive integer *n* such that $t_m = t^n$ for any number *m*, i.e., $\frac{\Delta t_p}{\Delta t_c}$ is a positive integer. Let $j^0 = \Delta t_{p,1}/\Delta t_c$, $j = \Delta t_p/\Delta t_c$. Let J_p denote a quasi-regular partition of Ω consisting of hexahedron elements, $\Omega_{ijk} = [x_{i-1/2}, x_{i+1/2}] \times [y_{j-1/2}, y_{j+1/2}] \times [z_{k-1/2}, z_{k+1/2}]$, with maximum diameter h_p (see Figure 1 in Section 2). $(X_h, M_h) \subset X \times M$ is a mixed element space of Rariart-Thomas type or Brezzi-Douglas-Marini type with index k. Similarly, J_c denotes the quasi-regular partition of hexahedron elements with maximum diameter h_c.

Let P, $\mathbf{U}$, C and $\mathbf{G}$ denote the approximation solutions of p, $\mathbf{u}$, c and $\mathbf{g}$, respectively. Based on the previous notations and the variation equation (3.2), a mixed element scheme is constructed for the pressure and velocity

$$\int_\Omega (\mu(\overline{C}_m)\kappa^{-1}\mathbf{U}_m + \beta\rho(\overline{C}_m)|\,\mathbf{U}_m\,|\mathbf{U}_m)\cdot v_h dX - \int_\Omega P_m \nabla v_h dX$$

$$= \int_\Omega r(\overline{C}_m)\nabla d \cdot v_h dX, \quad \forall v_h \in X_h, \tag{3.4a}$$

$$-\int_\Omega w_h \nabla\cdot \mathbf{U}_m dX = -\int_\Omega w_h q_m dX, \quad \forall w_h \in M_h. \tag{3.4b}$$

The derivative along the characteristics of (3.3a) is approximated by the backward difference quotient,

$$\frac{\partial c^{n+1}}{\partial \tau}(X) \approx \frac{c^{n+1} - c^n(X - \phi^{-1}\mathbf{u}^{n+1}(X)\Delta t)}{\Delta t(1 + \phi^{-2}|\mathbf{u}^{n+1}|^2)^{1/2}}.$$

Then the characteristics-mixed volume element scheme is constructed for the concentration,

$$\left(\phi\frac{C^{n+1} - \hat{C}^n}{\Delta t}, v\right)_{\overline{m}} + (D_x G^{x,n+1} + D_y G^{y,n+1} + D_z G^{z,n+1}, v)_{\overline{m}}$$

$$= (f(\hat{C}^n), v)_{\overline{m}}, \quad \forall v \in S_h, \tag{3.5a}$$

$$(D^{-1}(E\mathbf{U}^{n+1})G^{x,n+1},\ w^x)_x + (D^{-1}(E\mathbf{U}^{n+1})G^{y,n+1},\ w^y)_y$$

$$+ (D^{-1}(E\mathbf{U}^{n+1})G^{z,n+1},\ w^z)_z$$

$$- (C^{n+1},\ D_x w^x + D_y w^y + D_z w^z)_{\overline{m}} = 0, \quad \forall w \in V_h, \tag{3.5b}$$

where $\hat{C}^n = C^n(\hat{X}^n)$, $\hat{X}^n = X - \phi^{-1}E\mathbf{U}^{n+1}\Delta t_c$. Let the concentration in nonlinear functions μ, ρ and *r* at $t = t_m$ be assigned by C_m,

$$\overline{C}_m = \min\{1, \max(0, C_m)\} \in [0, 1]. \tag{3.6}$$

The value of the velocity at t^{n+1}, $t_{m-1} < t^n \le t_m$, is assigned by using the following extrapolation formula

$$E\mathbf{U}^{n+1}$$

$$= \begin{cases} \mathbf{U}_0, & t_0 < t^{n+1} \le t_1,\ m = 1 \\ \left(1 + \dfrac{t^{n+1} - t_{m-1}}{t_{m-1} - t_{m-2}}\right)\mathbf{U}_{m-1} - \dfrac{t^{n+1} - t_{m-1}}{t_{m-1} - t_{m-2}}\mathbf{U}_{m-2}, & t_{m-1} < t^{n+1} \le t_m,\ m \ge 2. \end{cases}$$

(3.7)

Initial approximations:

$$C^0 = \widetilde{C}^0, \quad \mathbf{G}^0 = \widetilde{\mathbf{G}}^0, \quad X \in \Omega, \tag{3.8}$$

where $(\widetilde{C}^0, \widetilde{\mathbf{G}}^0)$ is an elliptic projection of $(c_0, \mathbf{g}_0)$ (see the definitions in next subsection).

Based on the procedures of mixed element-characteristic mixed volume element, approximation solutions are computed as follows. Using c_0, $\mathbf{g}_0 = -D\nabla c_0$, and elliptic projection, we get $\{\widetilde{C}^0, \widetilde{\mathbf{G}}^0\}$, then take $C^0 = \widetilde{C}^0$, $\mathbf{G}^0 = \widetilde{\mathbf{G}}^0$. Using (3.4a), (3.4b) and the method of conjugate gradient, we get $\{\mathbf{U}_0, P_0\}$. Then, we get $\{C^1, C^2, ..., C^{j_0}\}$ from (3.5). For $m \ge 1$, the concentrations are defined by $C^{j_0+(m-1)j} = C_m$. $\{\mathbf{U}_m, P_m\}$ is obtained from (3.4a) and (3.4b). Then we get $C^{j_0+(m-1)j+1}$, $C^{j_0+(m-1)j+2}$, ..., C^{j_0+mj} from (3.5a) and (3.5b). The computations run step by step as above, then all the numerical solutions are obtained. From the positive definiteness (C), the solutions exist and are sole.

3.2. Local conservation of mass

Suppose that the problem of (1.1)-(1.5) has no source or sink, $q \equiv 0$, and suppose that there is no permeation across the boundary. For simplicity, we suppose that the large-step and small-step partitions are the same, i.e.,

$$J_c \equiv J_p = \Omega_{ijk} = [x_{i-1/2}, x_{i+1/2}] \times [y_{j-1/2}, y_{j+1/2}] \times [z_{k-1/2}, z_{k+1/2}].$$

The local conservation of mass is interpreted as follows. For (1.2), it holds

$$\int_{J_c} \psi \frac{\partial c}{\partial \tau} dX - \int_{\partial J_c} \mathbf{g} \cdot \gamma_{J_c} dS = 0, \tag{3.9}$$

where γ_{J_c} represents the outer normal vector to ∂J_c, the boundary of J_c. Discrete formulation of local conservation is stated in the following theorem for (3.5a).

Theorem 1. *If* $q \equiv 0$, *then on* $J_c = \Omega_{ijk}$, *it holds for the concentration* (3.5a):

$$\int_{J_c} \phi \frac{C^{n+1} - \hat{C}^n}{\Delta t} dX - \int_{\partial J_c} \mathbf{G}^{n+1} \cdot \gamma_{J_c} dS = 0. \tag{3.10}$$

Proof. Noting that $v \in S_h$ is equal to 1 on a given element $J_c = \Omega_{ijk}$, and equals 0 on the other elements, then we rewrite (3.5a) as follows:

$$\left(\phi \frac{C^{n+1} - \hat{C}^n}{\Delta t}, 1 \right)_{\Omega_{ijk}} + (D_x G^{x,n+1} + D_y G^{y,n+1} + D_z G^{z,n+1}, 1)_{\Omega_{ijk}} = 0. \tag{3.11}$$

Using the above notations, we have

$$\left(\phi\frac{C^{n+1}-\hat{C}^{n}}{\Delta t},\,1\right)_{\Omega_{ijk}} = \phi_{ijk}\left(\frac{C_{ijk}^{n+1}-\hat{C}_{ijk}^{n}}{\Delta t}\right)h_{x_i}h_{y_j}h_{z_k}$$

$$= \int_{\Omega_{ijk}}\phi\frac{C^{n+1}-\hat{C}^{n}}{\Delta t}dX, \tag{3.12a}$$

$$(D_xG^{x,n+1}+D_yG^{y,n+1}+D_zG^{z,n+1},\,1)_{\Omega_{ijk}}$$

$$= (G^{x,n+1}_{i+1/2,jk}-G^{x,n+1}_{i-1/2,jk})h_{y_j}h_{z_k}$$

$$+ (G^{y,n+1}_{i,j+1/2,k}-G^{y,n+1}_{i,j-1/2,k})h_{x_i}h_{z_k} + (G^{z,n+1}_{ij,k+1/2}-G^{z,n+1}_{ij,k-1/2})h_{x_i}h_{y_j}$$

$$= -\int_{\partial\Omega_{ijk}}\mathbf{G}^{n+1}\cdot\gamma_{J_c}dS. \tag{3.12b}$$

Then the proof is completed by substituting (3.12) into (3.11).

The conservation of mass on the whole domain is given as follows:

Theorem 2. *Suppose that* $q \equiv 0$ *and no permeation happens across the boundary. Then the whole conservation is formulated by*

$$\int_{\Omega}\phi\frac{C^{n+1}-\hat{C}^{n}}{\Delta t}dX = 0, \quad n \geq 0. \tag{3.13}$$

Proof. Considering the summation of (3.10) on the whole domain,

$$\sum_{i,j,k}\int_{\Omega_{ijk}}\phi\frac{C^{n+1}-\hat{C}^{n}}{\Delta t}dX - \sum_{i,j,k}\int_{\partial\Omega_{ijk}}\mathbf{G}^{n+1}\cdot\gamma_{J_c}dS = 0, \tag{3.14}$$

and noting that $-\sum_{i,j,k}\int_{\partial\Omega_{ijk}}\mathbf{G}^{n+1}\cdot\gamma_{J_c}dS = -\int_{\partial\Omega}\mathbf{G}^{n+1}\cdot\gamma dS = 0,$ we complete the proof.

4. Convergence Analysis

Suppose that there exists a constant K independent of the partition such that

$$(A_{p,\mathbf{u}})\begin{cases}\inf\limits_{v_h\in X_h}\| f-v_h\|_{L^q}\le K\| f\|_{W^{m,q}}h_p^m, & 1\le m\le k+1,\\ \inf\limits_{w_h\in M_h}\| g-w_h\|_{L^q}\le K\| g\|_{W^{m,q}}h_p^m, & 1\le m\le k+1,\\ \inf\limits_{v_h\in X_h}\|\operatorname{div}(f-v_h)\|_{L^2}\le K\|\operatorname{div} f\|_{H^m}h_p^m, & 1\le m\le k+1,\end{cases}\tag{4.1a}$$

$$(I_{p,\mathbf{u}})\,\| v_h\|_{L^\infty}\le Kh_p^{-3/2}\| v_h\|_{L^2},\quad v_h\in X_h,\tag{4.1b}$$

$$(A_c)\,\inf_{\chi_h\in V_h}[\| f-\chi_h\|_{L^2}+h_c\| f-\chi_h\|_{H^1}]\le Kh_c^m\| f\|_m,\ 2\le m\le k+1,\tag{4.1c}$$

$$(I_c)\,\|\chi_h\|_{W^{m,\infty}}\le Kh_c^{-3/2}\|\chi_h\|_{W^m},\quad \chi_h\in V_h.\tag{4.1d}$$

In the following convergence analysis we suppose that $D(\mathbf{u})\approx\phi d_m I = D(X)$ and introduce two elliptic projections.

Definition. Define Forchheimer operator $(\Pi_h, P_h):(\mathbf{u}, p)\to(\Pi_h\mathbf{u}, P_h p) = (\widetilde{\mathbf{U}}, \widetilde{P})$, as follows:

$$\int_\Omega[\mu(c)\kappa^{-1}(\mathbf{u}-\widetilde{\mathbf{U}})+\beta\rho(c)(|\mathbf{u}|\mathbf{u}-\|\widetilde{\mathbf{U}}\|\widetilde{\mathbf{U}})]\cdot v_h dX$$

$$-\int_\Omega(p-\widetilde{P})\nabla v_h dX=0,\quad \forall v_h\in X_h,\tag{4.2a}$$

$$-\int_\Omega w_h\nabla\cdot(\mathbf{u}-\widetilde{\mathbf{U}})dX=0,\quad \forall w_h\in M_h.\tag{4.2b}$$

From (3.2a) and (3.2b),

$$\int_\Omega(\mu(c)\kappa^{-1}\widetilde{\mathbf{U}}+\beta\rho(c)|\widetilde{\mathbf{U}}|\widetilde{\mathbf{U}})\cdot v_h dX-\int_\Omega\widetilde{P}\nabla v_h dX$$

$$= \int_\Omega r(c)\nabla d \cdot v_h dX, \quad \forall v_h \in X_h, \tag{4.3a}$$

$$-\int_\Omega w_h \nabla \cdot \widetilde{\mathbf{U}} dX = -\int_\Omega w_h \nabla \cdot \mathbf{u} dX = -\int_\Omega w_h q dX, \ \forall w_h \in M_h. \tag{4.3b}$$

According to the discussion in [19], we find a positive constant K independent of h_p such that

$$\| \mathbf{u} - \widetilde{\mathbf{U}} \|_{L^2}^2 + \| \mathbf{u} - \widetilde{\mathbf{U}} \|_{L^3}^3 + \| p - \widetilde{P} \|_{L^2}^2$$

$$\leq K\{\| \mathbf{u} \|_{W^{k+1,3}}^2 + \| p \|_{H^{k+1}}^2\} h_p^{2(k+1)}. \tag{4.4a}$$

For $k \geq 1$ and sufficiently small h_p,

$$\| \widetilde{\mathbf{U}} \|_{L^\infty} \leq \| \mathbf{u} \|_{L^\infty} + 1. \tag{4.4b}$$

Let $F = f - \psi \dfrac{\partial c}{\partial \tau}$. Then define an elliptic projection of mixed volume element.

$\widetilde{\mathbf{G}} \in V_h$ and $\widetilde{C} \in S_h$ are determined by

$$(D_x \widetilde{G}^x + D_y \widetilde{G}^y + D_z \widetilde{G}^z, v)_{\overline{m}} = (F, v)_{\overline{m}}, \quad \forall v \in S_h, \tag{4.5a}$$

$$(D^{-1}\widetilde{G}^x, w^x)_x + (D^{-1}\widetilde{G}^y, w^y)_y + (D^{-1}\widetilde{G}^z, w^z)_z$$

$$-(\widetilde{C}, D_x w^x + D_y w^y + D_z w^z)_{\overline{m}} = 0, \quad \forall w \in V_h. \tag{4.5b}$$

Let $\pi = P - \widetilde{P}$, $\eta = \widetilde{P} - p$, $\sigma = \mathbf{U} - \widetilde{\mathbf{U}}$, $\rho = \widetilde{\mathbf{U}} - \mathbf{u}$, $\xi = C - \widetilde{C}$, $\zeta = \widetilde{C} - c$, $\alpha = \mathbf{G} - \widetilde{\mathbf{G}}$, $\beta = \widetilde{\mathbf{G}} - \mathbf{g}$. Suppose that (1.1)-(1.2) is positive definite (C) and exact solutions are regular (R). $\{\widetilde{\mathbf{G}}\}$ and $\{\widetilde{C}\}$ exist and are defined uniquely by (4.5) shown by Weiser and Wheeler in [28].

Lemma 5. *If* (C) *and* (R) *hold, then there exist two positive constants* $\widetilde{C}_1$ *and* $\widetilde{C}_2$ *independent of* h, Δt *such that*

$$\| \zeta \|_{\overline{m}} + ||| \beta ||| + \left\| \frac{\partial \zeta}{\partial t} \right\|_{\overline{m}} \leq \overline{C}_1 \{ h_p^2 + h_c^2 \}, \tag{4.6a}$$

$$||| \widetilde{\mathbf{G}} |||_\infty \leq C_2. \tag{4.6b}$$

To complete the analysis, we should give the following properties. An LBB condition of mixed element is given in Lemma 6, discussed by Pan and Rui in [19].

Lemma 6. *There exists a constant* $\bar{r}$ *independent of h such that*

$$\inf_{w_h \in M_h} \sup_{v_h \in X_h} \frac{(w_h, \nabla \cdot v_h)}{\| w_h \|_{M_h} \| v_h \|_{X_h}} \geq \bar{r}. \tag{4.7}$$

Lemma 7 and Lemma 8 show the properties of nonlinear Darcy-Forchheimer operator.

Lemma 7. *Define* $f(v) = | v | v,$ *then we find three positive constants* K_1, K_2, K_3 *for* $u, v, w \in L^3(\Omega)^d$ *such that*

$$K_1 \int_\Omega (| u | + | v |) | v - u |^2 dX \leq \int_\Omega (f(v) - f(u)) \cdot (v - u) dX, \tag{4.8a}$$

$$K_2 \| v - u \|_{L^3}^2 \leq \int_\Omega (f(v) - f(u)) \cdot (v - u) dX, \tag{4.8b}$$

$$K_3 \int_\Omega | f(v) - f(u) | | v - u | dX \leq \int_\Omega (f(v) - f(u)) \cdot (v - u) dX, \tag{4.8c}$$

$$\int_\Omega (f(v) - f(u)) \cdot w dX$$

$$\leq \left[\int_\Omega (| v | + | u |) | v - u |^2 dX \right]^{1/2} [\| u \|_{L^3}^{1/2} + \| v \|_{L^3}^{1/2}] \| w \|_{L^3}. \tag{4.8d}$$

Since $c \in [0, 1],$ so we have

$$| \overline{C}_{h,m} - c_m | \leq | C_{h,m} - c_m |. \tag{4.9}$$

Considering (3.4a), (3.4b) and (4.4) together, we have a property in the following lemma.

Lemma 8. *There exists a number K independent of the partition such that*

$$\| \mathbf{u}_m - \mathbf{U}_m \|_{L^2}^2 + \| \mathbf{u}_m - \mathbf{U}_m \|_{L^3}^3 \le K\{h_p^{2(k+1)} + \| c_m - C_m \|_{L^2}^2\}. \quad (4.10)$$

Proof. From (3.4a), (3.4b) and Darcy-Forchheimer projection (4.2), we have

$$(\mu(\overline{C}_m)K^{-1}(\mathbf{u}_m - \mathbf{U}_m) + \beta\rho(\overline{C}_m)(| \mathbf{u}_m |\mathbf{u}_m - | \mathbf{U}_m |\mathbf{U}_m), v_h)$$

$$- (\widetilde{P}_m - P_m, \nabla_h v_h)$$

$$= -((\mu(c_m) - \mu(\overline{C}_m))K^{-1}\widetilde{\mathbf{U}}_m + \beta(\rho(c_m) - \rho(\overline{C}_m))K^{-1}| \widetilde{\mathbf{U}}_m |\widetilde{\mathbf{U}}_m, v_h)$$

$$+ ((r(c_m) - r(\overline{C}_m))\nabla d, v_h), \quad \forall v_h \in X_h, \quad (4.11a)$$

$$- (w_h, \nabla \cdot (\widetilde{\mathbf{U}}_m - \mathbf{U}_m)) = -(w_h, \nabla \cdot (\widetilde{\mathbf{U}}_m - \mathbf{u}_m)) = 0, \forall w_h \in M_h. \quad (4.11b)$$

Then we can get $(\widetilde{P}_m - P_m, \nabla \cdot (\widetilde{\mathbf{U}}_m - \mathbf{U}_m)) = 0.$ Take $v_h = \widetilde{\mathbf{U}}_m - \mathbf{U}_m$ in (4.11a),

$$(\mu(\overline{C}_m)K^{-1}(\mathbf{u}_m - \mathbf{U}_m) + \beta\rho(\overline{C}_m)(| \mathbf{u}_m |\mathbf{u}_m), \mathbf{u}_m - \mathbf{U}_m)$$

$$= (\mu(\overline{C}_m)K^{-1}(\mathbf{u}_m - \mathbf{U}_m) + \beta\rho(\overline{C}_m)(| \mathbf{u}_m |\mathbf{u}_m), \mathbf{u}_m - \widetilde{\mathbf{U}}_m)$$

$$- ((\mu(c_m) - \mu(\overline{C}_m))K^{-1}\widetilde{\mathbf{U}}_m + \beta(\rho(c_m) - \rho(\overline{C}_m))K^{-1}| \widetilde{\mathbf{U}}_m |\widetilde{\mathbf{U}}_m, \widetilde{\mathbf{U}}_m - \mathbf{U}_m)$$

$$+ ((r(c_m) - r(\overline{C}_m))\nabla d, \widetilde{\mathbf{U}}_m - \mathbf{U}_m). \quad (4.12)$$

The terms on the left-hand side (l.h.s) and the right-hand side (r.h.s) of (4.12) are estimated as follows:

$$\text{l.h.s} \ge K_0\left\{\| \mathbf{u}_m - \mathbf{U}_m \|_{L^2}^2 + \| \mathbf{u}_m - \mathbf{U}_m \|_{L^3}^3\right.$$

$$+\int_\Omega (|\mathbf{u}_m|+|\mathbf{U}_m|)|\mathbf{u}_m-\mathbf{U}_m|^2dx\Bigg\} \tag{4.13a}$$

$$\begin{aligned}
\text{r.h.s} \le K\Bigg\{ & \|\mathbf{u}_m-\mathbf{U}_m\|_{L^2}\|\mathbf{u}_m-\widetilde{\mathbf{U}}_m\|_{L^2} \\
& +\left[\int_\Omega (|\mathbf{u}_m|+|\mathbf{U}_m|)|\mathbf{u}_m-\mathbf{U}_m|^2dx\right]^{1/2} \\
& \times[\|\mathbf{u}_m\|_{L^3}^{1/2}+\|\mathbf{U}_m\|_{L^3}^{1/2}]\|\mathbf{u}_m-\widetilde{\mathbf{U}}_m\|_{L^2} \\
& + (1+\|\widetilde{\mathbf{U}}_m\|_{L^\infty})\|c_m-C_m\|_{L^2}\|\mathbf{u}_m-\widetilde{\mathbf{U}}_m\|_{L^2}\Bigg\} \\
\le \varepsilon\Bigg\{ & \|\mathbf{u}_m-\mathbf{U}_m\|_{L^2}^2+\int_\Omega (|\mathbf{u}_m|+|\mathbf{U}_m|)|\mathbf{u}_m-\mathbf{U}_m|^2dx\Bigg\} \\
& +K\{\|\mathbf{u}_m-\widetilde{\mathbf{U}}_m\|_{L^2}^2+\|\mathbf{u}_m-\mathbf{U}_m\|_{L^3}^3+(1+\|\widetilde{\mathbf{U}}_m\|_{L^\infty})\|c_m-C_m\|_{L^2}^2\},
\end{aligned} \tag{4.13b}$$

where K_0 is a positive constant.

Combining (4.12) with (4.13a), (4.13b), Lemma 7 and (4.4), we can get the error estimate of the velocity and complete the proof of Lemma 8.

The concentration equation (1.2) is discussed first. Subtracting (4.5a) and (4.5b) at $t = t^{n+1}$ from (3.5a) and (3.5b), respectively, taking $v = \xi^{n+1}$, $w = \alpha^{n+1}$, we have

$$\left(\phi\frac{C^{n+1}-\hat{C}^n}{\Delta t}, \xi^{n+1}\right)_{\overline{m}} + (D_x\alpha^{x,n+1}+D_y\alpha^{y,n+1}+D_z\alpha^{z,n+1}, \xi^{n+1})_{\overline{m}}$$

$$=\left(f(\hat{C}^n)-f(c^{n+1})+\psi^{n+1}\frac{\partial c^{n+1}}{\partial\tau}, \xi^{n+1}\right)_{\overline{m}}, \tag{4.14a}$$

$$(D^{-1}\alpha^{x,n+1}, \alpha^{x,n+1})_x+(D^{-1}\alpha^{y,n+1}, \alpha^{y,n+1})_y+(D^{-1}\alpha^{z,n+1}, \alpha^{z,n+1})_z$$

$$-(\xi^{n+1}, D_x\alpha^{x,n+1}+D_y\alpha^{y,n+1}+D_z\alpha^{z,n+1})_{\overline{m}}=0. \tag{4.14b}$$

Summing (4.14a) and (4.14b),

$$\left(\phi\frac{C^{n+1}-\hat{C}^{n}}{\Delta t},\ \xi^{n+1}\right)_{\overline{m}}+(D^{-1}\alpha^{x,n+1},\ \alpha^{x,n+1})_{x}$$

$$+(D^{-1}\alpha^{y,n+1},\ \alpha^{y,n+1})_{y}+(D^{-1}\alpha^{z,n+1},\ \alpha^{z,n+1})_{z}$$

$$=\left(f(\hat{C}^{n})-f(c^{n+1})+\psi^{n+1}\frac{\partial c^{n+1}}{\partial\tau},\ \xi^{n+1}\right)_{\overline{m}}. \tag{4.15}$$

Using (1.2) $t=t^{n+1}$, we get

$$\left(\phi\frac{\xi^{n+1}-\xi^{n}}{\Delta t},\ \xi^{n+1}\right)_{\overline{m}}+\sum_{s=x,y,z}(D^{-1}\alpha^{s,n+1},\ \alpha^{s,n+1})_{s}$$

$$=\left(\left[\phi\frac{\partial c^{n+1}}{\partial t}+\mathbf{u}^{n+1}\cdot\nabla c^{n+1}\right]-\phi\frac{c^{n+1}-\check{c}^{n}}{\Delta t},\ \xi^{n+1}\right)_{\overline{m}}+\left(\phi\frac{\zeta^{n+1}-\zeta^{n}}{\Delta t},\ \xi^{n+1}\right)_{\overline{m}}$$

$$+(f(\hat{C}^{n})-f(c^{n+1}),\ \xi^{n+1})+\left(\phi\frac{\hat{c}^{n}-\check{c}^{n}}{\Delta t},\ \xi^{n+1}\right)_{\overline{m}}$$

$$-\left(\phi\frac{\hat{\zeta}^{n}-\check{\zeta}^{n}}{\Delta t},\ \xi^{n+1}\right)_{\overline{m}}+\left(\phi\frac{\hat{\xi}^{n}-\check{\xi}^{n}}{\Delta t},\ \xi^{n+1}\right)_{\overline{m}}$$

$$-\left(\phi\frac{\check{\zeta}^{n}-\zeta^{n}}{\Delta t},\ \xi^{n+1}\right)_{\overline{m}}+\left(\phi\frac{\check{\xi}^{n}-\xi^{n}}{\Delta t},\ \xi^{n+1}\right)_{\overline{m}}, \tag{4.16}$$

where $\check{c}^{n}=c^{n}(X-\phi^{-1}\mathbf{u}^{n+1}\Delta t)$, $\hat{c}^{n}=c^{n}(X-\phi^{-1}E\mathbf{U}^{n+1}\Delta t)$, ...

Using $a(a-b)\geq\frac{1}{2}(a^{2}-b^{2})$ to estimate the left-hand term of (4.16), and denoting the right-hand terms by $T_1, T_2, \ldots, T_8$, we have

$$\frac{1}{2\Delta t}\{(\phi\xi^{n+1},\ \xi^{n+1})_{\overline{m}}-(\phi\xi^{n},\ \xi^{n})_{\overline{m}}\}+\sum_{s=x,y,z}(D^{-1}\alpha^{s,n+1},\ \alpha^{s,n+1})_{s}$$

$$\leq T_1 + T_2 + \cdots + T_8. \tag{4.17}$$

Noting that $\phi \dfrac{\partial c^{n+1}}{\partial t} + \mathbf{u}^{n+1} \cdot \nabla c^{n+1} = \psi^{n+1} \dfrac{\partial c^{n+1}}{\partial \tau}$, we have [7, 10]

$$\frac{\partial c^{n+1}}{\partial \tau} - \frac{\phi}{\psi^{n+1}} \frac{c^{n+1} - \check{c}^n}{\Delta t}$$

$$= \frac{\phi}{\psi^{n+1}\Delta t} \int_{(\check{X},\, t^n)}^{(X,\, t^{n+1})} [|X - \hat{X}|^2 + (t - t^n)^2]^{1/2} \frac{\partial^2 c}{\partial \tau^2} d\tau. \tag{4.18}$$

Multiplying both sides by ψ^{n+1} and expressing the resulting summation in $\overline{m}$ -norm, we have

$$\left\| \psi^{n+1} \frac{\partial c^{n+1}}{\partial \tau} - \phi \frac{c^{n+1} - \check{c}^n}{\Delta t} \right\|_{\overline{m}}^2$$

$$\leq \int_\Omega \left[\frac{\psi^{n+1}}{\Delta t} \right]^2 \left| \int_{(\check{X}, t^n)}^{(X, t^{n+1})} \frac{\partial^2 c}{\partial \tau^2} d\tau \right|^2 dX$$

$$\leq \Delta t \left\| \frac{(\psi^{n+1})^3}{\phi} \right\|_\infty \int_\Omega \int_{(\check{X}, t^n)}^{(X, t^{n+1})} \left| \frac{\partial^2 c}{\partial \tau^2} \right|^2 d\tau \, dX$$

$$\leq \Delta t \left\| \frac{(\psi^{n+1})^4}{\phi^2} \right\|_\infty \int_\Omega \int_{t^n}^{t^{n+1}} \int_0^1 \left| \frac{\partial^2 c}{\partial \tau^2} (\bar{\tau}\check{X} + (1 - \bar{\tau}) X,\, t) \right|^2 d\bar{\tau} \, dX \, dt. \tag{4.19}$$

Then, T_1 is bounded by

$$| T_1 | \leq K \left\| \frac{\partial^2 c}{\partial \tau^2} \right\|_{L^2(t^n, t^{n+1}; \overline{m})}^2 \Delta t + K \| \xi^{n+1} \|_{\overline{m}}^2. \tag{4.20a}$$

T_2 and T_3 are estimated by using Lemma 5

$$|T_2| \le K\left\{(\Delta t)^{-1}\left\|\frac{\partial \zeta}{\partial t}\right\|^2_{L^2(t^n,t^{n+1};\overline{m})} + \|\xi^{n+1}\|^2_{\overline{m}}\right\}, \tag{4.20b}$$

$$|T_3| \le K\{\|\xi^{n+1}\|^2_{\overline{m}} + \|\xi^n\|^2_{\overline{m}} + (\Delta t)^2 + h^4\}. \tag{4.20c}$$

T_4, T_5 and T_6 are estimated similarly. Let f be defined on Ω, denoting one of the functions c, ζ and ξ, and let Z denote the normal vector of $\mathbf{U}^{n+1} - \mathbf{u}^{n+1}$. Then,

$$\int_\Omega \phi \frac{\hat{f}^n - \check{f}^n}{\Delta t} \xi^{n+1} dX$$

$$= (\Delta t)^{-1} \int_\Omega \phi \left[\int_{\check{X}}^{\hat{X}} \frac{\partial f^n}{\partial Z} dZ\right] \xi^{n+1} dX$$

$$= (\Delta t)^{-1} \int_\Omega \phi \left[\int_0^1 \frac{\partial f^n}{\partial Z}((1-\overline{Z})\check{X} + \overline{Z}\hat{X}) d\overline{Z}\right] |\hat{X} - \check{X}| \xi^{n+1} dX$$

$$= \int_\Omega \left[\int_0^1 \frac{\partial f^n}{\partial Z}((1-\overline{Z})\check{X} + \overline{Z}\hat{X}) d\overline{Z}\right] |E(\mathbf{u} - \mathbf{U})^{n+1}| \xi^{n+1} dX, \tag{4.21}$$

where $\overline{Z} \in [0, 1]$, and $\hat{X} - \check{X} = \Delta t[E\mathbf{u}^{n+1}(X) - E\mathbf{U}^{n+1}(X)]/\phi(X)$. Let

$$g_f = \int_0^1 \frac{\partial f^n}{\partial Z}((1-\overline{Z})\check{X} + \overline{Z}\hat{X}) d\overline{Z}.$$

Then three estimates are derived from (4.21),

$$|T_4| \le \|g_c\|_\infty \|E(\mathbf{u} - \mathbf{U})^{n+1}\|_{\overline{m}} \|\xi^{n+1}\|_{\overline{m}}, \tag{4.22a}$$

$$|T_5| \le \|g_\zeta\|_{\overline{m}} \|E(\mathbf{u} - \mathbf{U})^{n+1}\|_{\overline{m}} \|\xi^{n+1}\|_\infty, \tag{4.22b}$$

$$|T_6| \le \|g_\xi\|_{\overline{m}} \|E(\mathbf{u} - \mathbf{U})^{n+1}\|_{\overline{m}} \|\xi^{n+1}\|_\infty. \tag{4.22c}$$

From Lemma 1-Lemma 5, and (4.10),

$$\|E(\mathbf{u}-\mathbf{U})^{n+1}\|_{\overline{m}}^2 \le K\{\|\xi_{m-1}\|_{\overline{m}}^2 + \|\xi_{m-2}\|_{\overline{m}}^2 + h_p^{2(k+1)} + h_c^4 + (\Delta t_c)^2\}. \quad (4.23)$$

Since $g_c(X)$ is a mean value of the first partial derivatives of c^n, so it can be estimated by $\|c^n\|_{W_\infty^1}$. From (4.22a),

$$|T_4| \le K\{\|\xi_{m-1}\|_{\overline{m}}^2 + \|\xi_{m-2}\|_{\overline{m}}^2 + \|\xi^{n+1}\|_{\overline{m}}^2 + h_p^{2(k+1)} + h_c^4 + (\Delta t_c)^2\}. \quad (4.24)$$

In order to estimate $\|g_\zeta\|_{\overline{m}}$ and $\|g_\xi\|_{\overline{m}}$, we introduce the following hypothesis,

$$\sup_{0\le n\le L} |\!|\!|\sigma|\!|\!|_\infty \to 0, \quad \sup_{0\le n\le L} \|\xi^n\|_{\overline{m}} \to 0, \quad (h_c, h_p, \Delta t_c) \to 0. \quad (4.25)$$

The partition constraint is given by

$$\Delta t_c = O(h_c^2). \quad (4.26)$$

The function g_f is estimated by

$$\|g_f\|^2 \le \int_0^1\int_\Omega \left[\frac{\partial f^n}{\partial Z}((1-\overline{Z})\check{X} + \overline{Z}\hat{X})\right]^2 dX\,d\overline{Z}. \quad (4.27)$$

Define a transformation

$$\begin{aligned} G_{\overline{Z}}(X) &= (1-\overline{Z})\check{X} + \overline{Z}\hat{X} \\ &= X - [\phi^{-1}(X)E\mathbf{u}^{n+1}(X) + \overline{Z}\phi^{-1}(X)E(\mathbf{U}-\mathbf{u})^{n+1}(X)]\Delta t_c, \quad (4.28) \end{aligned}$$

and let $J_p = \Omega_{ijk} = [x_{i-1/2}, x_{i+1/2}]\times[y_{j-1/2}, y_{j+1/2}]\times[z_{k-1/2}, z_{k+1/2}]$ denote an partition element of the pressure equation. From (4.27), we get

$$\|g_f\|^2 \le \int_0^1 \sum_{J_p} \left|\frac{\partial f^n}{\partial Z}(G_{\overline{Z}}(X))\right|^2 dX\,d\overline{Z}. \quad (4.29)$$

It follows from (4.25) and (4.26)

$$\det DG_{\overline{Z}} = 1 + o(1).$$

Then g_f is estimated by

$$\|g_f\|^2 \le K\|\nabla f^n\|^2. \tag{4.30}$$

T_5 is argued by using (4.30), Lemma 5 and Sobolev embedding theorem [1],

$$\begin{aligned}|T_5| &\le K\|\nabla\zeta^n\| \cdot \|E(\mathbf{u}-\mathbf{U})^{n+1}\| \cdot h^{-(\varepsilon+1/2)}\|\nabla\xi^{n+1}\| \\ &\le K\{h_c^{2-(\varepsilon+1/2)}\|E(\mathbf{u}-\mathbf{U})^{n+1}\| \, \|\nabla\xi^{n+1}\|\} \\ &\le K\{\|\xi_{m-1}\|_{\overline{m}}^2 + \|\xi_{m-2}\|_{\overline{m}}^2 + h_p^{2(k+1)} + h_c^4 + (\Delta t_c)^2\} + \varepsilon|||\alpha^{n+1}|||^2.\end{aligned} \tag{4.31a}$$

It follows obviously from (4.23) that $\|E(\mathbf{u}-\mathbf{U})^{n+1}\|_{\overline{m}} = o(h_c^{-(\varepsilon+1/2)})$, then it is necessary to prove $\|\xi^n\|_{\overline{m}} = O(h_p^{k+1} + h_c^2 + \Delta t_c)$. Similar to the discussion in [10], we have

$$\begin{aligned}|T_6| &\le K\|\nabla\xi^n\| \cdot \|E(\mathbf{u}-\mathbf{U})^{n+1}\| \cdot h^{-(\varepsilon+1/2)}\|\nabla\xi^{n+1}\| \\ &\le \varepsilon\{|||\alpha^{n+1}|||^2 + |||\alpha^n|||^2\}.\end{aligned} \tag{4.31b}$$

T_7 and T_8 are discussed by using negative norm estimate

$$|T_7| \le Kh_c^4 + \varepsilon|||\alpha^{n+1}|||^2, \tag{4.32a}$$

$$|T_8| \le K\|\xi^n\|_{\overline{m}}^2 + \varepsilon|||\alpha^{n+1}|||^2. \tag{4.32b}$$

Considering (4.17), (4.20), (4.31) and (4.32) together, we have

$$\frac{1}{2\Delta t}\{(\phi\xi^{n+1}, \xi^{n+1})_{\overline{m}} - (\phi\xi^n, \xi^n)_{\overline{m}}\} + \sum_{s=x,y,z}(D_s\alpha^{s,n+1}, \alpha^{s,n+1})_s$$

$$\leq K\left\{\left\|\frac{\partial^2 c}{\partial\tau^2}\right\|^2_{L^2(t^n, t^{n+1}; \overline{m})}\Delta t + (\Delta t)^{-1}\left\|\frac{\partial\zeta}{\partial t}\right\|^2_{L^2(t^n, t^{n+1}; \overline{m})} + \|\xi^{n+1}\|^2_{\overline{m}}\right.$$

$$\left.+\|\xi^n\|^2_{\overline{m}} + \|\xi_{m-1}\|^2_{\overline{m}} + \|\xi_{m-2}\|^2_{\overline{m}} + h_p^{2(k+1)} + h_c^4 + (\Delta t_c)^2\right\}$$

$$+\varepsilon\{|||\alpha^{n+1}|||^2 + |||\alpha^n|||^2\}. \tag{4.33}$$

Multiplying both sides of (4.33) by $2\Delta t$, summing them on $t\,(0 \leq n \leq L)$, and noting that $\xi^0 = 0$, we have

$$\|\xi^{L+1}\|^2_{\overline{m}} + \sum_{n=0}^{L}|||\alpha^{n+1}|||^2\Delta t \leq K\left\{\sum_{n=0}^{L}\|\xi^{n+1}\|^2_{\overline{m}}\Delta t + h_p^{2(k+1)} + h_c^4 + (\Delta t_c)^2\right\}. \tag{4.34}$$

Applying Gronwall lemma,

$$\|\xi^{L+1}\|^2_{\overline{m}} + \sum_{n=0}^{L}|||\alpha^{n+1}|||^2\Delta t \leq K\{h_p^{2(k+1)} + h_c^4 + (\Delta t_c)^2\}. \tag{4.35a}$$

Continuing, we have

$$\sup_{0\leq n\leq L}|||\alpha^{n+1}|||^2 \leq K\{h_p^{2(k+1)} + h_c^4 + (\Delta t_c)^2\}. \tag{4.35b}$$

It remains to testify the induction hypothesis (4.25). From $\xi^0 = 0$, it is true for $n = 0$ obviously. Assume that (4.25) holds for $1 \leq n \leq L$. From (4.35) and (4.26), we have

$$\|\sigma^{L+1}\|_\infty \leq Kh_p^{-3/2}\{h_p^{k+1} + h_c^2 + \Delta t_c\} \leq Kh_p^{1/2} \to 0, \tag{4.36a}$$

$$\|\xi^{L+1}\|_\infty \leq Kh_c^{-3/2}\{h_p^{k+1} + h_c^2 + \Delta t_c\} \leq Kh_c^{1/2} \to 0. \tag{4.36b}$$

Then the induction hypothesis is proved for $n = L + 1$.

From (4.35) and Lemma 5, we conclude the following statement.

Theorem 3. *Suppose that the problem of* (1.1) *and* (1.2) *is regular* (*R*) *and the coefficients are positive definite* (*C*). *Numerical solutions are computed step by step from* (3.4), (3.5) *and* (3.8). *If the constraint* (4.26) *holds*, *then we have*

$$\begin{aligned}&\|\mathbf{u}-\mathbf{U}\|_{\bar{L}^{\infty}(J;L^{2})}+\|c-C\|_{\bar{L}^{\infty}(J;\bar{m})}+\|\mathbf{g}-\mathbf{G}\|_{\bar{L}^{2}(J;L^{2})}\\&\le M^{*}\{h_{p}^{k+1}+h_{c}^{2}+\Delta t\},\end{aligned}\tag{4.37}$$

where

$$\|g\|_{\bar{L}^{\infty}(J;X)}=\sup_{n\Delta t\le T}\|g^{n}\|_{X},\ \|g\|_{\bar{L}^{2}(J;X)}=\sup_{L\Delta t\le T}\left\{\sum_{n=0}^{L}\|g^{n}\|_{X}^{2}\Delta t\right\}^{1/2},$$

and the constant M^{*} depends on p, c and their derivatives.

5. Numerical Example

In this section, we apply the method of mixed element-characteristic mixed volume element to solve a system of elliptic-convection-diffusion equations:

$$\begin{cases}-\Delta p=\nabla\cdot\mathbf{u}=c+F, & X\in\partial\Omega,\ 0\le t\le T,\\ \dfrac{\partial c}{\partial t}+\mathbf{u}\cdot\nabla c-\varepsilon\Delta c=f, & X\in\Omega,\ 0<t\le T,\\ c(X,0)=c_{0}, & X\in\Omega,\\ \dfrac{\partial c}{\partial\nu}=0, & X\in\partial\Omega,\ 0<t\le T,\\ -\dfrac{\partial p}{\partial\nu}=\mathbf{u}\cdot\nu=0, & X\in\partial\Omega,\ 0<t\le T,\end{cases}\tag{5.1}$$

where p, $\mathbf{u}$ and c denote the pressure, Darcy velocity and the concentration, respectively. Let $\Omega=(0,1)\times(0,1)\times(0,1)$ and let ν denote the outer normal vector to the boundary surface $\partial\Omega$. We choose the definitions of F, f and c_{0} properly such that exact solutions are

$$p = e^{12t}(x_1^4(1-x_1)^4 x_2^4(1-x_2)^4 x_3^4(1-x_3)^4$$
$$- x_1^2(1-x_1)^2 x_2^2(1-x_2)^2 x_3^2(1-x_3)^2/21^3),$$

$$c = -e^{12t}\sum_{i=1}^{3}(12x_i^2(1-x_i)^4 - 32x_i^3(1-x_i)^3$$
$$+ 12x_i^4(1-x_i)^2)x_{i+1}^4(1-x_{i+1})^4 x_{i+2}^4(1-x_{i+2})^4,$$

where $x_4 = x_1$ and $x_5 = x_2$.

Table 1. Numerical data

	$h = 1/4$	$h = 1/8$	$h = 1/16$
$\| p - P \|_m$	$1.82852e-4$	$1.17235e-4$	$3.30572e-5$
$\|\| \mathbf{u} - \mathbf{U} \|\|$	$6.95898e-3$	$1.86974e-3$	$4.74263e-4$
$\| c - C \|_m$	$1.39414e-1$	$8.76624e-2$	$4.46468e-2$
$\|\| \mathbf{g} - \mathbf{G} \|\|$	$1.78590e-3$	$8.88468e-4$	$4.85070e-4$

Numerical data are shown in Table 1 for $\varepsilon = 10^{-3}$. $\Delta t = 0.01$, $T = 1$. From Table 1, the conservation statements of Theorem 1, Theorem 2 and convergence statements Theorem 3, we find that numerical method of this type can solve two-phase seepage displacement problem (1.1)-(1.5) effectively and precisely.

6. Conclusions and Discussions

In the present paper, a combination scheme of mixed element and characteristic mixed volume element is discussed to approximate three-dimensional oil-water Darcy-Forchheimer flow displacement in porous media, and convergence analysis is shown. In Section 1 Introduction, mathematical model is stated, and physical background and some related international study are introduced. In Section 2, some notations and preparations are introduced, and two different partitions (large-step and small-step) are defined. In Section 3, the procedures are constructed to compute the pressure, velocity and concentration. The flow equation is treated by a conservative mixed element scheme and the accuracy of Darcy-Forchheimer velocity is improved one order. The concentration is solved by the method of characteristics-mixed volume element, where the convection term is treated by the method of characteristics and the diffusion is discretized by the scheme of mixed volume element. The combination scheme develops the stability and accuracy greatly and has the nature of conservation, most important in numerical simulation of energy science. In Section 4, we show convergence analysis by using the priori estimates of differential equations and special techniques, and derive second order error estimates in the discrete L^2 norm. Moreover, convergence analysis shows the development of the work of Arbogast and Wheeler. In Section 5, one numerical example is illustrated to support theoretical analysis, and numerical data confirm that the scheme is feasible and efficient. In this paper, several interesting conclusions are stated as follows:

(I) The method has the physical nature of conservation, and it is most important in numerical simulation of underground seepage mechanics especially in chemical oil discovery.

(II) The method combines mixed volume element and the characteristics, and it has the advantages of strong stability and high accuracy. Therefore, it gives a useful tool to solve large-scale engineering computation on three-dimensional complicated region.

(III) This method improves the famous work of Ewing, Russell and Wheeler on classical Darcy flow essentially, and confirms the physical conservation [13].

(IV) The method improves convergence rate of $3/2$-order presented by Arbogast and Wheeler to second order, and it is able to solve the well-known problem better [9, 10, 25, 29] interest.

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References

[1] R. A. Adams, Sobolev Spaces, Academic Press, New York, 1975.

[2] T. Arbogast and M. F. Wheeler, A characteristics-mixed finite element methods for advection-dominated transport problems, SIAM J. Numer. Anal. 32(2) (1995), 404-424.

[3] K. Aziz and A. Settari, Petroleum reservoir simulation, Applied Science Publishers, London, 1979.

[4] Z. Cai, J. E. Jones, S. F. Mccormilk and T. F. Russell, Control-volume mixed finite element methods, Comput. Geosci. 1 (1997), 289-315.

[5] M. A. Cella, T. F. Russell, I. Herrera and R. E. Ewing, An Eulerian-Lagrangian localized adjoint method for the advection-diffusion equations, Adv. Water Resour. 13(4) (1990), 187-206.

[6] S. H. Chou, D. Y. Kawk and P. Vassileviki, Mixed volume methods for elliptic problems on triangular grids, SIAM J. Numer. Anal. 35 (1998), 1850-1861.

[7] S. H. Chou, D. Y. Kawk and P. Vassileviki, Mixed volume methods on rectangular grids for elliptic problem, SIAM J. Numer. Anal. 37 (2000), 758-771.

[8] S. H. Chou and P. Vassileviki, A general mixed covolume frame work for constructing conservative schemes for elliptic problems, Math. Comp. 12 (2003), 150-161.

[9] J. Douglas Jr., Finite difference method for two-phase incompressible flow in porous media, SIAM J. Numer. Anal. 4 (1983), 681-696.

[10] J. Douglas Jr. and Y. R. Yuan, Numerical simulation of immiscible flow in porous media based on combining the method of characteristics with mixed finite element procedure, Numerical Simulation in Oil Recovery, Springer-Verlag, New York, 1986, pp. 119-132.

[11] J. Douglas Jr., P. J. Paes-Leme and T. Giorgi, Generalized Forchheimer flow in porous media, Boundary Value Problems for Partial Differential Equations and Applications, RMA Res. Notes Appl. Math., Masson, Paris 29 (1993), 99-111.

[12] R. E. Ewing, R. D. Lazarov, S. L. Lyons, D. V. Papavassiliou, J. Pasciak and G. Qin, Numerical well model for non-Darcy flow through isotropic porous media, Comput. Geosci. 3(3-4) (1999), 185-204.

[13] R. E. Ewing, T. F. Russell and M. F. Wheeler, Convergence analysis of an approximation of miscible displacement in porous media by mixed finite elements and a modified method of characteristics, Comput. Methods Appl. Mech. Engrg. 47(1-2) (1984), 73-92.

[14] P. Fabrie, Regularity of the solution of Darcy-Forchheimer's equation, Nonlinear Anal. Theory. Methods Appl. 13(9) (1989), 1025-1049.

[15] V. Girault and M. F. Wheeler, Numerical discretization of a Darcy-Forchheimer model, Numer. Math. 110(2) (2008), 161-198.

[16] J. E. Jones, A mixed volume method for accurate computation of fluid velocities in porous media, Ph. D. Thesis, University of Colorado, Denver, Co., 1995.

[17] H. Lopez, B. Molina and J. J. Salas, Comparison between different numerical discretization for a Darcy-Forchheimer model, Electron. Trans. Numer. Anal. 34 (2009), 187-203.

[18] E. J. Park, Mixed finite element method for generalized Forchheimer flow in porous media, Numer. Methods Partial Differential Equations 21(2) (2005), 213-228.

[19] H. Pan and H. X. Rui, Mixed element method for two-dimensional Darcy-Forchheimer model, J. Sci. Comput. 52 (2012), 563-587.

[20] H. Pan and H. X. Rui, A mixed element method for Darcy-Forchheimer incompressible miscible displacement problem, Comput. Methods Appl. Mech. Engrg. 264 (2013), 1-11.

[21] H. X. Rui and H. Pan, A block-centered finite difference method for the Darcy-Forchheimer model, SIAM. J. Numer. Anal. 50(5) (2012), 2612-2631.

[22] H. X. Rui and W. Liu, A two-grid block-centered finite difference methods for Darcy-Forchheimer flow in porous media, SIAM J. Numer. Anal. 53(4) (2015), 1941-1962.

[23] T. F. Russell, Rigorous block-centered discretization on irregular grids: Improved simulation of complex reservoir systems, Project Report, Research Corporation, Tulsa, 1995.

[24] D. Ruth and H. Ma, On the derivation of the Forchheimer equation by means of the averaging theorem, Transport in Porous Media 7(3) (1992), 255-264.

[25] P. P. Shen, M. X. Liu and L. Tang, Mathematical model of petroleum exploration and development, Science Press, Beijing, 2002.

[26] T. J. Sun and Y. R. Yuan, An approximation of incompressible miscible displacement in porous media by mixed finite element method and characteristics-mixed finite element method, J. Comput. Appl. Math. 228(1) (2009), 391-411.

[27] T. J. Sun and Y. R. Yuan, Mixed finite method and characteristics-mixed finite element method for a slightly compressible miscible displacement problem in porous media, Math. Comput. Simulat. 107 (2015), 24-45.

[28] A. Weiser and M. F. Wheeler, On convergence of block-centered finite difference for elliptic problems, SIAM J. Numer. Anal. 25(2) (1988), 351-375.

[29] Y. R. Yuan, Theory and Application of Reservoir Numerical Simulation, Science Press, Beijing, 2013.

[30] Y. R. Yuan, A. J. Cheng, D. P. Yang and C. F. Li, Applications, theoretical analysis, numerical method and mechanical model of the polymer flooding in porous media, Special Topics and Reviews in Porous Media - An International Journal 6(4) (2015), 383-401.

[31] Y. R. Yuan, A. J. Cheng and D. P. Yang, Theory and Actual Applications of Numerical Simulation of Oil Reservoir, Science Press, Beijing, 2016.

[32] Y. R. Yuan, A. J. Cheng, D. P. Yang and C. F. Li, Numerical simulation method, theory and applications of three-phase (oil, gas, water) chemical-agent oil recovery in porous media, Special Topics and Reviews in Porous Media - An International Journal 7(3) (2016), 1-28.

[33] Y. R. Yuan, Y. X. Liu, C. F. Li, T. J. Sun and L. Q. Ma, Analysis on block-centered finite differences of numerical simulation of semiconductor device detector, Appl. Math. Comput. 279 (2016), 1-15.