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APPLICATION OF MAXIMUM HAMILTONIAN OF PONTRYAGIN'S MAXIMUM PRINCIPLE TO SECTORAL LABOUR MARKET PERFORMANCE

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Abstract

This paper describes the application of maximum Hamiltonian of Pontryagin's Maximum Principle in determining the best sectoral labour market performance of a market split into three sectors namely; goods producing, service providing and agriculture sectors. The

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sectoral labour market performance is modeled as a time-independent optimal control problem where the optimal control is determined using the maximum Hamiltonian. The solution is sought using the property of the Hamiltonian that transforms the problem from time-independent to time-dependent problem. The transformation makes use of a switching function and takes note of the time when the control changes from one extreme value to the next. That is, when the labour market efforts switch from producing general worker population to the active sector participants, the sector with more switches within the duration of study indicates more chances of producing more sector participants. The study found that good producing sector had no switch time, service providing sector had three switches while agriculture sector had two switch times within the study period of ten years. Therefore, among the three sectors, the service providing sector had the best performance.

1. Introduction

The Hamiltonian system of Pontryagin's Maximum Principle (PMP) is characterized by its basic feature in the principle that indexes the parameters necessary for optimality. The systems solve an initial value problem by eliminating the control parameter while moving towards the extremals of the problem (Caillau et al. [2]). A number of applications of Hamiltonian systems in PMP to real life problems have been developed as a way of elaborating on Pontryagin Maximum Principle.

Ende et al. [3] explored extended Hamiltonian systems of dynamic problems with known initial and unknown final state. The solutions to the problems required a definition of Hamiltonian that would create room for at least a piecewise constant controlled dimensions regulating the control Hamiltonian. The summed up equation became an equation extended by controls.

Khakestari and Suleiman [4] conducted a research on application of PMP in a linear quadratic differential game. With a fixed duration of the game, the control functions of the players were subjected to integral constraints. By using PMP, the necessary condition of the maximum principle and the optimal control was sought using maximum Hamiltonian.

Tarasyev and Usova [7] did a study on stabilizing the Hamiltonian System for constructing optimal trajectories. The paper sought to increase the growth rate of the consumption level in an economic growth model by optimizing the mechanisms of investment in basic production assets. Consumption index was constructed as an optimal control problem on an infinite horizon interval whose solution was sought by use of maximized Hamiltonian of the Pontryagin's Maximum Principle. Necessary and sufficient optimality conditions were verified, a qualitative analysis of the Hamiltonian systems was performed, an algorithm for constructing optimal trajectories was proposed and its accuracy was estimated.

In this paper, we explore the application of the maximum Hamiltonian to a dynamic system of sectoral labour market performance model with fixed initial time and state but free final time. The sectoral labour market performance model takes the form of an optimal control problem. The aim is to identify the times when the labour market switches between producing general worker population or active sector workers' population in the three sectors of the labour market namely; goods producing sector, service providing and agriculture sectors. The sector with the highest number of switches within the study period would be the best performing sector.

The rest of this paper is organized as follows: Section 2 gives the preliminary and data processing. In Section 3, we highlight sectoral labour market performance model in use. Section 4 presents the theoretical solution to sectoral labour market performance. Section 5 checks the maximum Hamiltonian effect to sector labour market participation and finally Section 6 gives the conclusion.

2. Preliminary and Data Processing

The data utilized for analysis is from the International Labour Organization of between the year 2010 and 2019. The sectoral participants are considered as persons working in the various economic activities as summarized in Table 1.

Table 1. Classification of the sector based on economic activities

Sector	**Economic activities**
Goods producing sector	Participants in manufacturing, mining and quarrying, construction, electricity, gas and water production.
Service providing sector	Participants from; public administration, defense and compulsory social security, wholesale and retail traders, repair of motor vehicle and motorcycle, accommodation and food services, transport, storage and communication, finance and insurance, real estate, business and administration, education, human health and social work activities.
Agriculture sector	Participants from agricultural activities

Source: International Labour Organization, 2022.

Table 1 highlights the sector and the related economic activities. The Principal Component Analysis (PCA) is applied for dimensionality reduction while maintaining the features of the original data.

3. Sectoral Labour Market Performance Model

Sectoral labour market performance is a function of the optimal general labour market performance. That is, when maximum labour market efforts are applied to produce more active sector workers. This indicates that, if $W(t)$ is the number of workers in the general labour market, $Q(t)$ is the number of sector active workers, and $U(t)$ is the fraction of labour market effort directed towards generating more sector active workers, then sectoral labour market performance is measured as a function of maximum labour market efforts that is, when $U(t) = 1$.

The model assumptions are: First, maximum sectoral labour market performance is measured by sector participation rate; second, a high performing sector has maximum effort directed towards generation of its participants; third, overall labour market performance is a function of the general labour market and the sectoral participation given by $X(t) = (W(t), Q(t))$; fourth, the marginal labour contribution per worker given by $S(t)$ is measurable; fifth, both the general labour market and sectoral labour markets have measurable growth rate per one worker; and finally, transition and exit rates for the general labour market and sectoral labour markets are constant (Mwangi et al. [5]).

The model dynamics are

$$\dot{W}(t) = -aw(t) + bu(t)s(t)w(t),$$

$$w(0) = w^0 = 1, \tag{1}$$

where $\dot{W}(t)$ is the dynamics of the labour market growth per one worker, a is the labour market exit rate and b is the general labour market transition constant,

$$\dot{Q}(t) = -cq(t) + d(1 - u(t))s(t)w(t),$$

$$q(0) = q^0 = 1, \tag{2}$$

where $\dot{Q}(t)$ is the dynamics of the sectoral labour market growth per one worker, c is the sector worker exit rate and d is the active sector workers transition constant.

To obtain the solution to the sectoral labour market performance model, we define the objective function. That is, let J be the sector participation function to be maximized by choice of a trajectory $\{Q(t)\}$ for active workers participating in a sector.

The objective function is derived by the integral of sector performance derived from per unit entry in the sector $q(t)$ exponentially discounted at some positive rate g from initial time through all time to time $t \in [0, t_i]$ within T.

The state variable is capital per worker X, that is $X = (W(t), Q(t))$ measured as a function of the labour market efforts towards maximum sector performance. That is

$$\max_{q(t_1)} J = \int_0^T e^{-gt} Q(t) dt \text{ for } t_1 \in [0, T] \tag{3}$$

subject to

$$\dot{W}(t) = -aw(t) + bu(t)s(t)w(t),$$

$$W(0) = w^0 = 1,$$

$$\dot{Q}(t) = -cq(t) + d(1 - u(t))s(t)w(t),$$

$$Q(0) = q^0 = 1.$$

4. Solution to Sectoral Labour Market Performance Model

The solution is sought by responding to the necessary conditions of the maximum Hamiltonian guided by Theorem 4.1. This is a modification of Theorem 5 in tutorial by Boscain et al. [1].

Theorem 4.1. *In order that the admissible control* $0 \le u(t) \le 1$ *and* $t_0 \le t \le t_1$, *and the corresponding state* $X(t) = (W(t), Q(t))$ *yield a solution to the free interval optimal control problem with fixed initial time and state and free endpoints at different times within* $t \in [0, t_1]$, *it is necessary that there exist a constant* λ_0 *and a continuous function* $\Lambda(t) = (\lambda_1(t_0), ..., \lambda_n(t_1))$ *such that* $\lambda_0(t) \neq 0$ *or* $\lambda_0(t) = -1$ *for every* $t_0 \le t \le t_1$, *and*

(i) $\dot{\lambda}_i = -\dfrac{\partial H}{\partial x_i}$,

(ii) $0 = \dfrac{\partial H}{\partial u}$,

(iii) $\dot{x}_i = \dfrac{\partial H}{\partial \lambda_i}$,

(iv) $\lambda_i(t_1)$ *is free*,

(v) $H(x(t), \Lambda(t), t) = 0$,

(vi) $H(x^*(t), u^*(t), \lambda(t), t) = Max\{H(x(t), u, \lambda(t), t)\}$,

where (i), (ii), (iii), (iv) *are the adjoint equation*, *optimality condition*, *state equation*, *and transversality condition*, *respectively*. *Condition* (v) *indicates that the maximum Hamiltonian is zero along the trajectory while condition* (vi) *is the maximum principle*. *The optimal control* $u^*(t)$ *is the value that maximizes the Hamiltonian* $H(x(t), u, \lambda(t), t)$ *at time* $t \in [0, T]$. *The Hamiltonian system is therefore*,

$$H(x(t), u, \lambda(t), t) = \lambda' f(x, u). \tag{4}$$

Solving the problem by meeting the optimality conditions

Maximum Hamiltonian to our problem therefore becomes

$$H = \lambda_1 \dot{W}(t) + \lambda_2 \dot{Q}(t). \tag{5}$$

Maximum Hamiltonian transforms the problem by use of a switching function that considers the time when the problem changes from one extreme value of the control to the next otherwise known as the evolving/response time (Mizuta and Fujii [6]).

By considering equations (1) and (2), and substituting $\dot{W}(t)$ and $\dot{Q}(t)$ in the Hamiltonian, the Hamiltonian becomes

$$H = \lambda_1[-aw(t) + bs(t)u(t)w(t)] + \lambda_2[-cq(t) + d(1 - u(t))s(t)w(t)].$$

Since maximum Hamiltonian system is defined by the control, collecting the terms together with the control, the Hamiltonian becomes

$$H = [\lambda_1 bs(t)w(t) - \lambda_2 ds(t)w(t)]u(t) + \lambda_2 ds(t)w(t) - \lambda_1 aw(t) - \lambda_2 cq(t).$$

The expression in the square bracket is the switching function, that is, the system switches from a time independent to time dependent system when the switching function is equal to zero.

Therefore, the Hamiltonian is maximized with respect to $u(t)$ at $u^*(t)$. That is,

$$u^*(t) = \begin{cases} 0 & \text{if } \lambda_1 bs(t)w(t) - \lambda_2 ds(t)w(t) < 0, \\ 1 & \text{if } \lambda_1 bs(t)w(t) - \lambda_2 ds(t)w(t) > 0. \end{cases}$$

By use of the optimality condition described in Theorem 4.1, we obtain

(i) $\dfrac{\partial H}{\partial w} = -\dot{\lambda}_1 = -\lambda_1(bs(t)u(t) - a) - \lambda_2 d(1 - u(t))s(t),$

(ii) $\dfrac{\partial H}{\partial q} = -\dot{\lambda}_2 = \lambda_2 c,$

(iii) $\dfrac{\partial H}{\partial u} = \lambda_1(bs(t)w(t)) + \lambda_2(-\, ds(t)w(t)) = 0,$

(iv) $\dfrac{\partial H}{\partial \lambda_1} = \dot{w}(t) = -aw(t) + bs(t)u(t)w(t),$

(v) $\dfrac{\partial H}{\partial \lambda_2} = \dot{q}(t) = -cq(t) + d(1 - u(t))s(t)w(t),$

(vi) $\lambda_1[-aw(t) + bs(t)u(t)w(t)] + \lambda_2[-cq(t) + d(1 - u(t))s(t)w(t)] = 0.$

Setting $\dfrac{\partial H}{\partial w} = \dfrac{\partial H}{\partial q} = \dfrac{\partial H}{\partial u} = 0,$ we need to determine the value of λ_1 and λ_2. Given that the problem is time-independent, it had to be transformed to a time-dependent with the introduction of a switching function. To obtain λ_2 using $\dot{\lambda}_2 = \lambda_2 c,$ we use the differential equations analogy that states that the derivative of a function is proportionate with the size of the function. In this case, the proportionality function is the constant c.

Therefore,

$$\lambda_2 = B\,Exp\,tc,$$

where B is the population size at time $t = 0$ and $q(0) = 1$ in the proposed model. This indicates that

$$\lambda_2 = e^{tc}.$$

To obtain λ_1 using $\dot{\lambda}_1 = [\lambda_2 ds(t) - \lambda_1 bs(t)]u(t) - [\lambda_2 ds(t) - \lambda_1 a]$ and this results to

$$u^*(t) = 1 \text{ when } \lambda_1 bs(t) \leq \lambda_1 a \text{ and } \dot{\lambda}_1 = \lambda_1 a - \lambda_1 bs(t),$$

$$u^*(t) = 0 \text{ when } \lambda_1 bs(t) > \lambda_1 a \text{ and } \dot{\lambda}_1 = \lambda_1 a - \lambda_2 ds(t).$$

For this model, at $t = 0$, $u(0) = 1$ since at the start, all the efforts of the labour market are directed to growing the general workers population that influences the growth of the active sector participants when the switch happens at time $t = t_1$ and $u(t_1) = 0$. There could be more than one switch/response times where the system switches back to $u(t_2) = 1$ at time $t = t_2$ and later to $u(t_3) = 0$ at time $t = t_3$. The cycle continues till when the system has no further switch times to time $t = T$.

To obtain t_i, consider that the time in between the switches is a small time interval $t_i = T - t$. So, when $u^*(t) = 1$, $\lambda_1 = e^{(a - bs(t))t_i}$ switches to $\lambda_1 = \lambda_2 ds(t) - e^{at_i}$ when $u^*(t) = 0$, considering the transversality condition where $\lambda_i(T)$ is free which means $\lambda_i(T)$ can take any value beginning from zero. Therefore, when $\lambda_1(T) = 0$ and the switch is at the point when $u^*(t) = 0$, then $\lambda_1 = \lambda_2 ds(t) - e^{at_i} = 0$. That gives

$$e^{at_i} = \lambda_2 ds(t),$$

where

$$t_i = \frac{1}{a}\log\frac{1}{\lambda_2 ds(t)}.$$

The solution to the optimal control problem is computed as a function of the switching function which determines the sector with the most switching times:

$$u^*(t) = \begin{cases} 1 & t_1 \\ 0 & t_2 \\ 1 & t_3 \\ 0 & t_4 \\ \text{and so forth.} & \end{cases}$$

Therefore, when the switching happens, the model dynamics change as:

When $U(t) = 0$,

$$\dot{w}(t) = -aw(t),$$

$$\dot{q}(t) = -cq(t) + ds(t)w(t). \tag{6}$$

When $U(t) = 1$,

$$\dot{w}(t) = -aw(t) + bs(t)w(t),$$

$$\dot{q}(t) = -cq(t). \tag{7}$$

5. The Maximum Hamiltonian Effect to Sector Labour Market Participation

To compare the sectors' optimal efforts allocated using maximum Hamiltonian, the sector participation is predicted by use of the obtained optimal control function $u^*(t)$. The consideration here is that the problem is depended on the control up to an unknown time when the control changes to an extreme value within the data period of ten years. The normalizing factor for the problem is that the optimal control has the bounds, $0 \leq u(t) \leq 1$ and switching times within the period. The model constants are estimated by considering the growth rate associated with the control at a particular switching time. The sector performance obtained by fitting the experimental data on the model dynamics in equations (6) and (7) is presented in Table 2.

Table 2. Sectoral labour market after solving using of maximum Hamiltonian

Parameters	Description	Parameter value		
		Goods producing sector	Service providing sector	Agriculture sector
Switching times	Times when the switching function changes sign	0	3	2
When $U = 1$	Adjoint response to the general labour market growth	$\lambda_1 = 1$ $\lambda_2 = 1$	$\lambda_1 = 1$ $\lambda_2 = 1$	$\lambda_1 = 1$ $\lambda_2 = 1$
When $U = 0$		No switch	t_3 $\lambda_1 = 0.3996$ $\lambda_2 = 0.5817$	t_7 $\lambda_1 = 97.18$ $\lambda_2 = 1.2794$
When $U = 1$		No switch	t_7 $\lambda_1 = 1122144$ $\lambda_2 = 0$	t_9 $\lambda_1 = 1.0673$ $\lambda_2 = 0.000318$
When $U = 0$		No switch	t_{10} $\lambda_1 = 10878905$ $\lambda_2 = 0$	No switch

Table 2 presents the switching times of the three sectors. The table shows that within the 10 years' period, the goods producing sector had zero switching times, indicating that labour market produced general worker population only. Service providing sector had three switching times at t_3, when the system switched from general labour market production to producing active workers in the sector, at t_7, when the systems switched from producing sector participants to producing general worker population and at t_{10}, when the system produced sector participants. Agriculture sector had two switching times at t_7, when the system switched from producing general worker population to producing agriculture sector participants and at t_9, when the systems switched to producing general worker population.

6. Conclusion

This paper splits labour market into three sectors being; goods producing, service providing and agriculture sector. The sectoral labour market performance is formulated as an optimal control problem whose solution is sought using maximum Hamiltonian. That is, when the labour market efforts are strictly directed towards production of optimal sector participants. By considering the property of the maximum Hamiltonian that allows working with switching functions, the switching times in each sector are determined. The solution indicates that there are zero, three, and two switching times in goods producing, service providing, and agriculture sectors, respectively. The fact that the general labour market in goods producing sector had no switching time means that the population in the sector did not grow within the study period. However, the general labour market developed service providing and agriculture sectors within the period. Therefore, among the three sectors, the service providing sector had a better performance followed by the agriculture sector.

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