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# SYMMETRY AND MONOTONICITY OF POSITIVE SOLUTIONS TO THE FRACTIONAL *p*-LAPLACIAN EQUATIONS

**Xiaoshan Wang and Ping Zhu**

Department of Mathematics
Luoyang Normal University
Luoyang, 471934, P. R. China
e-mail: xswang2017@126.com

## Abstract

This paper is devoted to study the fractional *p*-Laplacian equation with singular nonlinearities. We use the direct method of moving planes to derive the symmetry and monotonicity result of positive solutions to the fractional *p*-Laplacian equations with singular nonlinearities. Compared to the work proposed in Hu [13], we extend the results of fractional Laplacian to *p*-Laplacian in a bounded domain. In addition, we also consider a singular nonlinear elliptic equation with fractional *p*-Laplacian term in $\mathbb{R}^n \backslash \{0\}$.

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## 1. Introduction

In this paper, we consider the following nonlinear equations involving the fractional $p$-Laplacian:

$$\begin{cases} (-\Delta)_p^s u(x) = \dfrac{g(u(x))}{|x|^{\alpha}} + f(x, u(x)), & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \Omega^c, \end{cases} \tag{1}$$

and

$$(-\Delta)_p^s u(x) = \frac{g(u(x))}{|x|^{\alpha}}, \ x \in \mathbb{R}^n \setminus \{0\}, \tag{2}$$

where

$$0 < s < 1, \ \alpha > 0, \ p > 2.$$

In the special case when $p = 2$, $(-\Delta)_p^s$ becomes the well-known fractional Laplacian $(-\Delta)^s$. The nonlocal nature of these operators makes them difficult to study. Recently Chen et al. in [8] developed a direct method of moving planes that can deal with directly the fractional Laplacian with subcritical and critical Sobolev exponents and a series of fruitful results have been obtained [3, 5, 6, 7, 11, 15]. Results regarding fractional $p$-Laplacian can be found in [4, 9, 10, 12, 14, 16, 17, 18, 19, 20] and the reference therein.

In [2], Canino et al. studied the singular semilinear elliptic equation:

$$\begin{cases} -\Delta u(x) = \dfrac{1}{|x|^{\gamma}} + f(x, u(x)), & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{3}$$

They proved that the solution $u$ was symmetric with respect to $T_0^\lambda$ and non-decreasing with respect to the $\nu(\nu \in S^{n-1})$-direction in $\Omega_0^\nu$. In particular, if $\Omega$ is a ball centered at the origin of radius $R > 0$, then $u$ is radially symmetric with $\frac{\partial u}{\partial r}(r) < 0$ for $0 < r < R$.

Later, Barrios et al. [1] considered the following equation:

$$\begin{cases} (-\Delta)^s u(x) = \dfrac{g(u(x))}{|x|^{2s}} + f(x, u(x)), & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \Omega^c. \end{cases} \tag{4}$$

It is proved that $u$ is symmetric with respect to the $x_1$-variable and strictly increasing with respect to $x_1$-direction for $x_1 < 0$. Moreover, if $\Omega$ is a ball, then $u$ is radial and strictly radially decreasing. Later, Hu [13] studied equation (4) and used a different method to prove the same result under the weaker conditions in $f$.

Inspired by the ideas of [1, 2] and [13], our main concern in this paper is using a direct method of moving planes to prove the symmetry and monotonicity of positive solution for the nonlinear equations involving the fractional $p$-Laplacian in bounded domain and $\mathbb{R}^n \backslash \{0\}$.

**Theorem 1.1.** *Assume that* $0 < s < 1$, $p > 2$ *and* $u \in L_{2s}(\mathbb{R}^n) \cap C_{loc}^{1,1}(\Omega)$ *is a positive solution of the equation* (1). *Let* $\Omega$ *be a bounded domain in* $\mathbb{R}^n$ *convex in the* $x_1$ *direction and symmetric with respect to the plane* $T_0 = \{x \in \mathbb{R}^n \mid x_1 = 0\}$. Also, assume that $f(\cdot, \cdot)$, $g(\cdot)$ *satisfy*

(i) $f(x, t)$ *and* $g(t)$ *are locally Lipschitz continuous in t and* $g(t) \geq 0$;

(ii) $f(x, t) \leq f(x^\lambda, t)$ *if* $\lambda < 0$, $x \in \Omega \cap \{x_1 < \lambda\}$ *and* $t \in [0, \infty)$, *where* $x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n)$;

(iii) $f(x, t) = f(x_0, t)$ *if* $x \in \Omega \bigcap \{x_1 < 0\}$ *and* $t \in [0, \infty)$.

*Then u is symmetric in* $x_1$ *direction and monotone increasing for* $x_1 < 0$.

**Remark 1.1.** If the domain is $\Omega = B_1(0)$, then the above conclusion is still true under the same hypothesis. The positive solution of equation (1) is radially symmetric and monotone decreasing about origin.

**Theorem 1.2.** *Assume that* $0 < s < 1$, $p > 2$ *and*

$$u \in L_{2s}(\mathbb{R}^n) \bigcap C^{1,1}_{loc}(\mathbb{R}^n \backslash \{0\})$$

*is a positive solution of the equation* (2) *with* $\lim_{|x| \to \infty} u(x) = 0$ *and*

$$\liminf_{y \to 0} u(y) > u(x), \forall x \in \mathbb{R}^n \backslash \{0\}.$$

*Suppose* $g(t) \geq 0$ *is monotone non-increasing on* $(0, t_0]$ *for some small* $t_0$. *Then u must be radially symmetric and monotone decreasing with respect to origin*.

**Remark 1.2.** When $\alpha = 0$ in the equation (2),

$$(-\Delta)^s_p u(x) = g(u(x)), \quad x \in \mathbb{R}^n \tag{5}$$

which is obtained by Chen and Li in [10] under the assumption of $\lim_{|x| \to \infty} u(x) = 0$ and $g'(s) \leq 0$ as $s > 0$ sufficiently small.

We first introduce some notation. Choose $x_1$ direction as any direction. Let moving plane be

$$T_\lambda = \{x \in \mathbb{R}^n \mid x_1 = \lambda, \lambda \in \mathbb{R}\},$$

and the domain to the left hand of the moving plane be

$$\Sigma_\lambda = \{x \in \mathbb{R}^n \mid x_1 < \lambda\},$$

and

$$x^{\lambda} = (2\lambda - x_1, x_2, ..., x_n)$$

be the reflection of $x$ about the moving plane $T_{\lambda}$.

We denote $u(x^{\lambda}) = u_{\lambda}(x)$, then compare the values of $u(x)$ and $u_{\lambda}(x)$, let

$$w_{\lambda}(x) = u_{\lambda}(x) - u(x).$$

After a direct calculation, we derive that $w_{\lambda}(x^{\lambda}) = -w_{\lambda}(x)$.

We use $C$ for a various positive constant that may depend on $n$, $s$ and $p$, and the value may differ from line to line.

## 2. Proof of Theorem 1.1

We prove this theorem divided onto two steps:

**Step 1.** Set

$$m = \inf_{\Omega} x_1.$$

Move the plane along $x_1$ direction from near $m$, and denote $\Omega_{\lambda} = \{x \in \Sigma_{\lambda} \mid \lambda - m < x_1 < \lambda\}$. In this step, we prove that for $\lambda$ sufficiently closed to $m$,

$$w_{\lambda}(x) \geq 0, \ \forall x \in \Sigma_{\lambda}. \tag{6}$$

Apparently, $w_{\lambda}(x) = u_{\lambda}(x) - u(x) = u_{\lambda}(x) > 0$ when $x \in \Sigma_{\lambda} \backslash \Omega_{\lambda}$. So we only need to prove

$$w_{\lambda}(x) \geq 0, \ \forall x \in \Omega_{\lambda}.$$

If not, then there exists $x^0 \in \Omega_{\lambda}$ such that

$$w_{\lambda}(x^0) = \min_{\Omega_{\lambda}} w_{\lambda}(x) < 0. \tag{7}$$

From the equation (1), we have

$$
\begin{aligned}
&(-\Delta)_p^s u_\lambda(x^0) - (-\Delta)_p^s u(x^0) \\
&= \frac{g(u_\lambda(x^0))}{|(x^0)^\lambda|^\alpha} + f((x^0)^\lambda, u_\lambda(x^0)) - \frac{g(u(x^0))}{|x^0|^\alpha} - f(x^0, u(x^0)) \\
&\geq \frac{g(u_\lambda(x^0)) - g(u(x^0))}{|x^0|^\alpha} + f(x^0, u_\lambda(x^0)) - f(x^0, u(x^0)) \\
&= \left[\frac{g'(\zeta_\lambda(x^0))}{|x^0|^\alpha} + f'(x^0, \eta_\lambda(x^0))\right] w_\lambda(x^0). \qquad (8)
\end{aligned}
$$

Denote

$$
c(x^0) = -\left[\frac{g'(\zeta_\lambda(x^0))}{|x^0|^\alpha} + f'(x^0, \eta_\lambda(x^0))\right]
$$

which is bounded for $f(x, t)$. The function $g(t)$ is locally Lipschitz continuous in $t$ and $\Omega$ is a bounded domain. When $\lambda = m$, we have $w_m(x) = u_m(x) - w_m(x) = u_m(x) > 0$ in $\sum_\lambda$. Therefore, when $\lambda - m$ sufficiently small, $\Omega_\lambda$ is a narrow region, and the above equation satisfies the narrow region principle involving fractional $p$-Laplacian in [18]. Thus

$$
w_\lambda(x) \geq 0, \text{ in } \Omega_\lambda,
$$

which yields a contradiction to equation (7), and hence (6) must be true.

**Step 2.** Inequality (6) provides a starting point, from which we move the plane $T_\lambda$ towards the right as long as (6) holds to its limiting position to show that $u$ is monotone increasing along $x_1$ direction. More precisely, we define $\lambda_0 = \sup\{\lambda \mid w_\mu(x) \geq 0, x \in \sum_\mu, \forall \mu \leq \lambda\}$, and then show that

$$
\lambda_0 = 0. \qquad (9)
$$

If not, then $\lambda_0 < 0$, and we show that the plane can be moved further right to yield a contradiction to the definition of $\lambda_0$. More precisely, there exists a small $\varepsilon > 0$ such that for all $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$, $w_\lambda(x) \geq 0$, $x \in \Sigma_\lambda$.

First, by the definition of $\lambda_0$, we have $w_{\lambda_0}(x) \geq 0$ in $\Sigma_{\lambda_0}$.

We claim that, $w_{\lambda_0}(x) > 0$ in $\Sigma_{\lambda_0}$. If not, then there exists a point $\widetilde{x} \in \Sigma_{\lambda_0}$ such that

$$w_{\lambda_0}(\widetilde{x}) = \min_{\Sigma_{\lambda_0}} w_{\lambda_0}(x) = 0,$$

and then

$$\begin{aligned}
&(-\Delta)_p^s u_\lambda(\widetilde{x}) - (-\Delta)_p^s u(\widetilde{x}) \\
&= C_{n,sp} PV \int_{\mathbb{R}^n} \frac{G(u_{\lambda_0}(\widetilde{x}) - u_{\lambda_0}(y)) - G(u(\widetilde{x}) - u(y))}{|\widetilde{x} - y|^{n+sp}} dy \\
&= C_{n,sp} PV \int_{\Sigma_{\lambda_0}} \left[ \frac{1}{|\widetilde{x} - y|^{n+sp}} - \frac{1}{|\widetilde{x} - y^{\lambda_0}|^{n+sp}} \right] \\
&\quad \times [G(u_{\lambda_0}(\widetilde{x}) - u_{\lambda_0}(y)) - G(u(\widetilde{x}) - u(y))] dy < 0. \qquad (10)
\end{aligned}$$

From the last inequality due to strictly monotone increasing nature of $G(\cdot)$ and $\lambda_0 < 0$, we have

$$\frac{1}{|\widetilde{x} - y|^{n+sp}} > \frac{1}{|\widetilde{x} - y^{\lambda_0}|^{n+sp}}$$

and

$$\begin{aligned}
u_{\lambda_0}(\widetilde{x}) - u_{\lambda_0}(y) - (u(\widetilde{x}) - u(y)) &= w_{\lambda_0}(\widetilde{x}) - w_{\lambda_0}(y) \\
&= -w_{\lambda_0}(y) \leq 0 (\not\equiv 0).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(-\Delta)_p^s u_\lambda(\tilde{x}) - (-\Delta)_p^s u(\tilde{x}) \\
&= \frac{g(u_\lambda(\tilde{x}))}{|(\tilde{x})^\lambda|^\alpha} + f((\tilde{x})^\lambda, u_\lambda(\tilde{x})) - \frac{g(u(\tilde{x}))}{|\tilde{x}|^\alpha} - f(x^0, u(\tilde{x})) \\
&\geq \frac{g(u_\lambda(\tilde{x})) - g(u(\tilde{x}))}{|\tilde{x}|^\alpha} + f(\tilde{x}, u_\lambda(\tilde{x})) - f(\tilde{x}, u(\tilde{x})) = 0. \qquad (11)
\end{aligned}$$

This leads to a contradiction to (10). Then we must have

$$w_{\lambda_0}(x) > 0, \; x \in \Sigma_{\lambda_0}.$$

By the definition of $\lambda_0$, there exists a sequence $\lambda_k \searrow \lambda_0$ and $x^k \in \Sigma_{\lambda_k}$ such that

$$\lambda_k < \frac{\lambda_0}{2}, \; w_{\lambda_k}(x^k) = \min_{\Sigma_{\lambda_k}} w_{\lambda_k} < 0 \text{ and } \nabla w_{\lambda_k}(x^k) = 0. \qquad (12)$$

Since $|x^k|$ is bounded, we may assume that $x^k \to x^0 \in \overline{\Sigma_{\lambda_k}}$. Therefore, from (12), we deduce that

$$w_{\lambda_0}(x^0) \leq 0, \text{ hence } x^0 \in T_{\lambda_0}; \text{ and } \nabla w_{\lambda_0}(x^0) = 0.$$

It follows that

$$\frac{w_{\lambda_k}(x^k)}{\delta_k} = \frac{w_{\lambda_k}(x^k)}{|\lambda_k - x_1^k|} \to \frac{w_{\lambda_0}(x^0)}{|\lambda_0 - x_1^0|} = 0, \text{ as } k \to \infty.$$

Therefore, we may use (8) to obtain

$$\begin{aligned}
&\frac{(-\Delta)_p^s u_{\lambda_k}(x^k) - (-\Delta)_p^s u(x^k)}{\delta_k} \\
&= \frac{1}{\delta_k}\left[\frac{g(u_\lambda(x^k))}{|(x^k)^\lambda|^\alpha} + f((x^k)^\lambda, u_\lambda(x^k)) - \frac{g(u(x^k))}{|x^k|^\alpha} - f(x^k, u(x^k))\right]
\end{aligned}$$

$$\geq \frac{1}{\delta_k}\left[\frac{g(u_\lambda(x^k)) - g(u(x^k))}{|x^k|^\alpha} + f(x^k, u_\lambda(x^k)) - f(x^k, u(x^k))\right]$$

$$= \frac{1}{\delta_k}\left[\frac{g'(\zeta_\lambda(x^k))}{|x^k|^\alpha} + f'(x^k, \eta_\lambda(x^k))\right] w_\lambda(x^k)$$

$$:= \frac{1}{\delta_k} c(x^k) w_{\lambda_k}(x^k) \to 0,$$

where $c(x^k)$ is bounded for $f(x, t)$, $g(t)$ is locally Lipschitz continuous about $t$ in a bounded domain.

This contradicts Narrow Region Principle in [10] and hence (9) is proved, i.e., $w_0 \geq 0$ in $\Sigma_0$. Therefore, $u$ is symmetric about the plane $T_0$. This completes the proof of Theorem 1.1.

## 3. Proof of Theorem 1.2

In order to prove that $u(x)$ is radially symmetric about some points in $\mathbb{R}^n$, we move the plane $T_\lambda$ along $x_1$ direction from $-\infty$ to the right.

The assumption of $\lim_{|x|\to\infty} u(x) = 0$ implies that, $\lim_{|x|\to\infty} w_\lambda(x) = 0$. Near the singular point $0^\lambda$ of $w_\lambda$, from our assumption

$$\liminf_{y\to 0} u(y) > u(x), \ \forall x \in \mathbb{R}^n \backslash \{0\},$$

and we have

$$\liminf_{x\to 0^\lambda} w_\lambda(x) = \liminf_{x\to 0^\lambda} (u_\lambda(x) - u(x)) = \liminf_{x\to 0} u(x) - u(0^\lambda) > 0.$$

Hence if $w_\lambda(x) < 0$ at some points in $\Sigma_\lambda$, then the negative minima must be attained in the interior of $\Sigma_\lambda \backslash \{0^\lambda\}$. We divided the proof into two steps.

**Step 1.** In this step, we prove that for $\lambda$ sufficiently negative,

$$w_\lambda(x) \geq 0, \ \forall x \in \Sigma_\lambda \backslash \{0^\lambda\}. \tag{13}$$

If not, then there exists $\bar{x} \in \Sigma_\lambda \backslash \{0^\lambda\}$ such that

$$w_\lambda(\bar{x}) = \min_{\Sigma_\lambda \backslash \{0^\lambda\}} w_\lambda(x) < 0, \tag{14}$$

from equation (2). Similar to the first part of the calculation, we have

$$(-\Delta)_p^s u_\lambda(\bar{x}) - (-\Delta)_p^s u(\bar{x}) \geq 0, \tag{15}$$

due to $\lambda$ being sufficiently negative. Since $u(\bar{x})$ is small enough, so is $\zeta_\lambda(\bar{x})$, and from the monotonicity of $g(\cdot)$, we have $g'(\zeta_\lambda(\bar{x})) \leq 0$.

On the other hand, by the definition of $p$-Laplacian and the monotonicity of $G(\cdot)$, we can derive that

$$\begin{aligned}
&(-\Delta)_p^s u_\lambda(\bar{x}) - (-\Delta)_p^s u(\bar{x}) \\
&= C_{n,sp} PV \int_{\mathbb{R}^n} \frac{G(u_\lambda(\bar{x}) - u_\lambda(y)) - G(u(\bar{x}) - u(y))}{|\bar{x} - y|^{n+sp}} dy \\
&= C_{n,sp} PV \int_{\Sigma_\lambda} \left[ \frac{1}{|\bar{x} - y|^{n+sp}} - \frac{1}{|\bar{x} - y^\lambda|^{n+sp}} \right] \\
&\quad \times [G(u_\lambda(\bar{x}) - u_\lambda(y)) - G(u(\bar{x}) - u(y))] dy \\
&\quad + C_{n,sp} \int_{\Sigma_\lambda} \frac{\begin{array}{c} G(u_\lambda(\bar{x}) - u(y)) - G(u(\bar{x}) - u_\lambda(y)) \\ + G(u_\lambda(\bar{x}) - u_\lambda(y)) - G(u(\bar{x}) - u(y)) \end{array}}{|\bar{x} - y^\lambda|^{n+sp}} dy \\
&= C_{n,sp} PV(I_1 + I_2).
\end{aligned} \tag{16}$$

We first estimate $I_1 \leq 0$. Due to the fact that $|\bar{x} - y| < |\bar{x} - y^\lambda|$, $\forall y \in \Sigma_\lambda \backslash \{0^\lambda\}$ and the monotonicity of $G(\cdot)$ for $u_\lambda(\bar{x}) - u_\lambda(y) - (u(\bar{x}) - u(y)) = w_\lambda(\bar{x}) - w_\lambda(y) \leq 0$.

Next we estimate $I_2$,

$$I_2 \le Cw_\lambda(\bar{x}) \int_{\Sigma_\lambda} \frac{|u(\bar{x}) - u(y)|^{p-2}}{|\bar{x} - y^\lambda|^{n+sp}} dy < 0.$$

Therefore,

$$(-\Delta)_p^s u_\lambda(\bar{x}) - (-\Delta)_p^s u(\bar{x}) < 0,$$

which contradicts (15). Therefore, (13) must be true.

**Step 2.** On the basis of inequality (13), the plane $T_\lambda$ must be moved to the right until the limit location of the (13) holds, then we show that $u$ is monotone increasing along $x_1$ direction. That is, we give the definition of $\lambda_0$:

$$\lambda_0 = \sup\{\lambda \,|\, w_\mu(x) \ge 0,\ x \in \Sigma_\mu \backslash \{0^\lambda\},\ \forall \mu \le \lambda\},$$

then we derive that

$$\lambda_0 = 0. \tag{17}$$

Otherwise, then $\lambda < 0$, we prove that the plane $T_\lambda$ can be moved further right to yield a contradiction with the definition of $\lambda_0$. More precisely, there exists a small $\varepsilon > 0$ such that for all $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$ such that

$$w_\lambda(x) \ge 0,\ x \in \Sigma_\lambda \backslash \{0^{\lambda_0}\}.$$

First, by the definition of $\lambda_0$, we have $w_{\lambda_0}(x) \ge 0$ in $\Sigma_{\lambda_0} \backslash \{0^{\lambda_0}\}$.

We will prove that, $w_{\lambda_0}(x) > 0$ in $\Sigma_{\lambda_0} \backslash \{0^{\lambda_0}\}$. If not, there exists a point $\tilde{x} \in \Sigma_{\lambda_0} \backslash \{0^{\lambda_0}\}$ such that

$$w_{\lambda_0}(\tilde{x}) = \min_{\Sigma_{\lambda_0} \backslash \{0^{\lambda_0}\}} w_{\lambda_0}(x) = 0,$$

by the definition of $p$-Laplacian, we can derive the same estimate as (10).

On the other hand, from equation (2)

$$
\begin{aligned}
&(-\Delta)_p^s u_{\lambda_0}(\tilde{x}) - (-\Delta)_p^s u(\tilde{x}) \\
&= \frac{g(u_{\lambda_0}(\tilde{x}))}{|(\tilde{x})^{\lambda_0}|^\alpha} - \frac{g(u(\tilde{x}))}{|\tilde{x}|^\alpha} \\
&\geq \frac{g(u_{\lambda_0}(\tilde{x})) - g(u(\tilde{x}))}{|\tilde{x}|^\alpha} \\
&= 0. \qquad (18)
\end{aligned}
$$

This leads to a contradiction to equation (10). Then we deduce that

$$w_{\lambda_0}(x) > 0, \ \ x \in \Sigma_{\lambda_0} \setminus \{0^{\lambda_0}\}.$$

From the definition of $\lambda_0$, there exists a sequence $\lambda_k \searrow \lambda_0$ and $x^k \in \Sigma_{\lambda_k} \setminus \{0^{\lambda_k}\}$ such that

$$\lambda_k < \frac{\lambda_0}{2}, \ w_{\lambda_k}(x^k) = \min_{\Sigma_{\lambda_k} \setminus \{0^{\lambda_k}\}} w_{\lambda_k} < 0 \text{ and } \nabla w_{\lambda_k}(x^k) = 0. \qquad (19)$$

The assumption $g(t) \geq 0$ is monotone non-increasing on $(0, t_0]$ for some small $t_0$ guarantees that there is a subsequence of $\{x^k\}$ that converges to some point $x^0$. In reality, from equation (2), we derive that

$$
\begin{aligned}
&(-\Delta)_p^s u_{\lambda_k}(x^k) - (-\Delta)_p^s u(x^k) \\
&= \frac{g(u_{\lambda_k}(x^k))}{|(x^k)^{\lambda_k}|^\alpha} - \frac{g(u(x^k))}{|x^k|^\alpha} \\
&\geq \frac{g'(\zeta_{\lambda_k}(x^k)) w_{\lambda_k}(x^k)}{|x^k|^\alpha}, \qquad (20)
\end{aligned}
$$

when $|x^k|$ is sufficiently large, we can see that $u(x^k)$ is sufficiently small, hence $\zeta_{\lambda_k}(x^k)$ is also sufficiently small. Under the assumption of $g(\cdot) \leq 0$ and monotone non-increasing on $(0, t_0]$ for some small $t_0$, from inequality of (20), we have

$$(-\Delta)_p^s u_{\lambda_k}(x^k) - (-\Delta)_p^s u(x^k) \geq 0.$$

However, if $x^k$ is a negative minimum of $w_{\lambda_k}$, from inequality (16), we have

$$(-\Delta)_p^s u_{\lambda_k}(x^k) - (-\Delta)_p^s u(x^k) < 0.$$

This derives a contradiction with the fact $x^k$ is a negative minimum of $w_{\lambda_k}$. Therefore the sequence $\{x^k\}$ must be bounded, then there exists a subsequence of $\{x^k\}$ that convergences to some point, and the subsequence still denoted by $\{x^k\}$. We may assume that $x^k \to x^0 \in \overline{\Sigma_{\lambda_k} \setminus \{0^{\lambda_k}\}}$. Therefore, from (19), we deduce that

$$w_{\lambda_0}(x^0) \leq 0, \text{ hence } x^0 \in T_{\lambda_0} \text{ and } \nabla w_{\lambda_0}(x^0) = 0.$$

It follows that

$$\frac{w_{\lambda_k}(x^k)}{\delta_k} = \frac{w_{\lambda_k}(x^k)}{|\lambda_k - x_1^k|} \to \frac{w_{\lambda_0}(x^0)}{|\lambda_0 - x_1^0|} = 0, \text{ as } k \to \infty.$$

Therefore, we may use (15) to obtain

$$\frac{(-\Delta)_p^s u_{\lambda_k}(x^k) - (-\Delta)_p^s u(x^k)}{\delta_k}$$

$$= \frac{1}{\delta_k}\left[\frac{g(u_{\lambda_k}(x^k))}{|(x^k)^{\lambda_k}|^{\alpha}} - \frac{g(u(x^k))}{|x^k|^{\alpha}}\right]$$

$$\geq \frac{1}{\delta_k} \frac{g'(\zeta_{\lambda_k}(x^k))}{|x^k|^{\alpha}} w_{\lambda_k}(x^k)$$

$$:= \frac{1}{\delta_k} c(x^k) w_{\lambda_k}(x^k) \to 0,$$

where $c(x^k)$ is bounded, $g(t)$ is locally Lipschitz continuous in $t$.

This will make a contradiction and hence (17) is proved, i.e., $w_0(x) = u_0(x) - u(x) \geq 0$ in $\sum_0$. By moving the plane along $x_1$ direction from $+\infty$, we also have $w_0(x) = u_0(x) - u(x) \leq 0$ in $\sum_0$. Therefore, $u$ is symmetric and monotone decreasing about the plane $T_0$. Since we can choose any direction as $x_1$ direction, we can derive $u$ is radially symmetric and monotone decreasing about the origin. This completes the proof of Theorem 1.2.

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