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A NOTE ON ISOMORPHISMS BETWEEN GROUPS OF DIFFEOMORPHISMS OF COSYMPLECTIC MANIFOLDS

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Abstract

Given a cosymplectic structure (η, Ω) on a manifold M, we study its C^r-diffeomorphisms $(0 < r < \infty)$, on which the infinitesimal vector fields preserve η and Ω.

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1. Introduction

In this work, we revisit a theorem of Filipkiewicz (for more details, see [5]). Let M be a smooth manifold and $Diff(M)$ be the group of all diffeomorphisms of M. In [2], it is well known that $Diff(M)$ has a smooth manifold structure. Therefore, $Diff(M)$ is a Lie group. Denote by (η, Ω) the cosymplectic structure on M, i.e., η and Ω are d-closed. Let $Diff^r(M, \eta, \Omega)$ be the group of $\mathcal{C}^r$-diffeomorphisms preserving η and Ω, i.e., for all $X \in \mathfrak{X}(M)$, $\mathcal{L}_X\eta = \mathcal{L}_X\Omega = 0$. In [2], it is shown that for such vector fields, their infinitesimal transformations form a group of automorphisms of (M, η, Ω). If $f \in Diff^r(M, \eta, \Omega)$ and $X \in \mathfrak{X}(M)$, then

$$Supp(f) = \overline{\{x \in M \mid f(x) \neq x\}},$$

$$Fix(f) = \{x \in M \mid f(x) = x\},$$

$$Supp(X) = \overline{\{x \in M \mid X(x) \neq 0\}}.$$

If a function f has the compact support, so does X, hence X generates a flow φ_t on M such that $\varphi_t^*\Omega = \Omega$ and $\varphi_t^*\eta = \eta$ [3]. We denote by $Diff^r(M, \eta, \Omega)_c$, the group of all compactly supported diffeomorphisms of $Diff^r(M, \eta, \Omega)$, which is a normal subgroup and by $Diff^r(M, \eta, \Omega)_0$, the subgroup for which there exist a compact subset K of M and an isotopy $\{h_t\}$ connecting f and the identity, with h_t fixed outside K. We recall that an isotopy between two diffeomorphisms $\varphi_0, \varphi_1 \in Diff^r(M, \eta, \Omega)$ is a $\mathcal{C}^r$-map

$$H : [0, 1] \times M \to M$$

such that the mapping

$$h_t : M \to M, \quad x \mapsto h_t(x) = H(t, x)$$

for all $t \in [0, 1]$, is a $\mathcal{C}^r$-diffeomorphism and $h_0 = \varphi_0$, $h_1 = \varphi_1$.

Example 1.1. Let $\mathbb{S}^n$ be a unit-sphere. Then the antipodal map

$$A : \mathbb{S}^n \to \mathbb{S}^n, \quad x \mapsto -x$$

is a diffeomorphism on $\mathbb{S}^n$.

Consider the diffeomorphism

$$J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \quad (x_1, x_2, ..., x_{2n}) \mapsto (-x_2, x_1, ..., -x_{2n}, x_{2n+1}).$$

For all $t \in [0, 1]$, the mapping

$$H_t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \quad x \mapsto x\cos(\pi t) + J(x)\sin(\pi t)$$

induces a family of diffeomorphisms h_t of $\mathbb{S}^{2n-1}$ such that $h_0 = id$ and $h_1 = A$, where *id* denotes the identity diffeomorphism.

2. Some Axioms

The group $Diff^r(M, \eta, \Omega)$ is said to have $T(n)$ *property* if for any pairwise distinct *n*-tuples $x_1, ..., x_n$ and $y_1, ..., y_n \in M$, there is $f \in Diff^r(M, \eta, \Omega)$ such that $f(x_i) = y_i,\ i = \overline{1, n}.$

It is well known that an isotopy h_t of a manifold gives rise to a family of vector fields $\dot{h}_t$ defined by

$$\dot{h}_t(x) = \frac{dh_t}{dt}(h_t^{-1}(x)).$$

Conversely, a family of vector fields X_t with compact support gives an isotopy ϕ_t, via the existence and uniqueness theorem of solutions of ODE:

$$\frac{d\phi_t}{dt}(x) = X_t(\phi_t(x)), \quad \phi_0(x) = x.$$

A subgroup $G = G(M)$ of $Diff^r(M, \eta, \Omega)$ is called a *group of diffeomorphisms* of *M*. The subgroup consisting of all C^r-diffeomorphisms *f* belonging to G for each of which there exist a compact set K of M and an isotopy $\{f_t\}$ connecting f and the identity with each f_t fixed outside K is denoted by G_0.

Proposition 2.1. *Let M be a smooth manifold. Then the following assertions hold*:

(i) *Let $\{U_j\}$ be a finite open cover. If $f \in G_0$, then $f = f_1 \cdots f_s$ and $Supp(f_i) \subset \{U_{j(i)}\}$ for $i = 1, \ldots, s$.*

(ii) *For any $x \in U$, U open, there is $f \in G_0$ such that $f(x) \neq x$. Moreover, for any $x \in M$ and sufficiently small open ball U with the center at x, there exists a $g \in G_0$ with $Fix(g) = (M - U) \cup \{x\}$.*

(iii) *G satisfies $T(3)$.*

(iv) *There is $\mathfrak{X}(M)$, a Lie algebra of vector fields satisfying the (PS) property such that all elements of the one-parameter transformation group of any element of $\mathfrak{X}(M)$ belong to G.*

We recall that the Pursell-Shanks (PS) property states that *the Lie algebra of compactly supported vector fields of a manifold M determines completely its smooth structure*. For instance, see [2, 4, 6].

3. Isomorphisms on $Diff^r(M, \eta, \Omega)$

A group of diffeomorphisms $G(M)$ satisfies the path transitivity if for any $x, y \in M$ and for any path

$$c : [0, 1] \to M$$

such that $c(0) = x$, $c(1) = y$, there exists $f \in G(M)$ with its support contained in an arbitrarily small neighborhood of $im(c)$ such that $f(x) = y$.

Proposition 3.1. *Let* $G(M)$ *be a group of diffeomorphisms, and let* $dim(M) \geq 3$. *Then the path transitivity implies the* $T(n)$ *property.*

Proof. It is obvious. □

Let N be a smooth manifold.

Theorem 3.2 [1]**.** *The map*

$$\varphi_* : \mathfrak{X}(M) \to \mathfrak{X}(N), \quad X \mapsto \varphi \circ X \circ \varphi^{-1}$$

is simultaneously an isomorphism of $\mathcal{C}^{\infty}(M)$*-modules and of* $\mathbb{R}$*-Lie algebras. Moreover,*

$$\varphi_*(fX) = (f \circ \varphi^{-1}) \cdot \varphi_* X \text{ for all } f \in \mathcal{C}^{\infty}(M),\ X \in \mathfrak{X}(M).$$

In particular, if $N = M$, and for any $X \in \mathfrak{X}(M)$ with, compact support, then $\mathfrak{X}(M)$ satisfies the (PS) type property.

Moreover, the map in Theorem 3.2 is the unique isomorphism between $\mathfrak{X}(M)$ and $\mathfrak{X}(N)$.

In the same manner on Theorem 3.2, we have the following construction:

If $f \in G(M)$, and $\phi : M \to N$ is a homeomorphism, then $\phi f \phi^{-1} \in G(N)$. We denote $\Phi(f) = \phi f \phi^{-1}$.

Theorem 3.3. *Let M and N be the smooth manifolds such that* $G(M)$ *and* $G(N)$ *satisfy the assertions of Proposition* 2.1. *If there is a group isomorphism* $\Phi : G(M) \to G(N)$, *then there is a unique diffeomorphism* $\phi : M \to N$ *such that* $\Phi(f) = \phi f \phi^{-1}$ *for any* $f \in G(M)$.

Proof. The proof follows from the above construction. □

Remark 3.4. In the case where $M = N$, every automorphism of $G(M)$ is inner.

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