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A STUDY OF ISOSPECTRAL FLOW ON BANDED MATRICES

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Abstract

An isospectral flow manipulates a matrix preserving its spectrum. In this paper, we study a flow, originally introduced by Arsie and Ebenbauer [2], of the form $\dot{P} = [[P', P]_{du}, P]$, where $\dot{P}$ is the time derivative of matrix P, P' is transpose of matrix P, and $[P', P]_{du}$ is same as matrix $[P', P]$ on the upper triangular elements and all elements below the diagonal are zero. We extend results in [2] to the case of real banded (band) matrices. If the initial condition, $P(0) = P_0$, is a banded matrix having lower bandwidth $p > 0$ with a simple and real spectrum, then $P(t)$ converges as $t \to \infty$ to a banded

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symmetric matrix having bandwidth p, isospectral to P_0. Also, the limit point has the same sign pattern in the pth subdiagonal elements as in P_0.

1. Introduction

An isospectral flow is a dynamical system of matrices that preserve the eigenvalues of the solution matrix. These flows appear in many applications [18, 11]. They are important in the study of direct and inverse eigenvalue problems in linear algebra, especially when we are interested in reconstructing a matrix with a special structure and a given spectrum because the eigenvalues are invariant of trajectories [8, 5]. Numerical analysts are interested in the flow because the eigenvalues of a certain matrix remain fixed throughout the computation. They widely appear in the field of integrable systems [6, 16, 7], the QR algorithm [17], representation theory via coadjoint orbits [13], control theory and many other fields [2, 3].

An isospectral flow, also known as Lax flow, is the differential equation:

$$\dot{P} = [Q, P], \quad P(0) = P_0,$$

where P is a square matrix of order n, Q is a matrix function, and $[P, Q] := PQ - QP$. The existence and uniqueness theorem for ordinary differential equations shows that the flow has the unique solution locally if Q has elements that are smooth functions of the elements of P. In the case we study here, the elements of $Q = [P', P]_{du}$ are quadratic polynomials of the elements of P.

A *banded matrix* is a special type of *sparse matrices* whose nonzero entries are arranged uniformly near the diagonal. Let $M = (m_{ij})$ be a square matrix of order n. If $a_{ij} = 0$ whenever $i - j > p$ for the smallest integer p, then p is the *lower bandwidth* of a matrix M [9]. Similarly, the *upper bandwidth* is defined.

We study the following nonlinear system of ordinary differential equations of the form

$$\dot{P} = [[P', P]_{du}, P]. \tag{1.1}$$

The flow (1.1) leaves the vector space of real banded matrices invariant. If the initial condition P_0 is banded matrix having lower bandwidth 2 (so that computations and notations follow easily), upper bandwidth 0 (so that the eigenvalues of P_0 are the diagonal elements of P_0), and a simple and real spectrum with nonzero second subdiagonal elements, then we prove that $P(t)$ converges as $t \to \infty$ and $\lim_{t \to +\infty} P(t)$ is a symmetric matrix isospectral to P_0. More specifically, $\lim_{t \to +\infty} P(t)$ is a symmetric pentadiagonal matrix isospectral to the initial condition P_0 having the same sign pattern in the second subdiagonal elements as in P_0.

2. Notations and Preliminary Results

We denote the vector space of real banded matrices with lower bandwidth 2 by B^+, i.e., $P \in B^+$ iff $P_{ij} = 0$ for all $i - j > 2$. The notations P_l, P_u and P_{du} are consistent with the notations in [2]. The given matrix P_0 has a simple and real spectrum, which is denoted by Λ. $\mathcal{S}_\Lambda$ is the compact manifold containing all symmetric matrices that are isospectral to P_0 [3]. Let a subset of $\mathcal{S}_\Lambda$ consisting all pentadiagonal symmetric matrices isospectral to P_0 be denoted by $\mathcal{P}_\Lambda$. The notations $\mathcal{G}_0$ and $\mathfrak{g}_0$ are also consistent with the notations in [2]. The matrix norm we use in this paper is the Frobenius norm defined as $\| P \| := \sqrt{\operatorname{tr}(PP')}$ for a matrix P.

We discuss some important properties of the vector field (1.1):

(1) The product of a banded matrix having lower bandwidth 2, and an upper triangular matrix is a banded matrix having lower bandwidth 2. This implies that *if the initial condition* P_0 *in* (1.1) *is a banded matrix with lower bandwidth* 2, $P(t)$ *will remain banded matrix with lower bandwidth* 2 *for all t*.

(2) *For the flow* (1.1), $P(t)$ *exists for all* $t \geq 0$. We can show that $\| P \|$ is monotonically decreasing along (1.1) if $[P(t)', P(t)]$ is nonzero.

Consider a function V defined as

$$V(P) := \| P \|^2 := \mathrm{tr}(PP'). \tag{2.1}$$

Note that V is a positive definite function. After differentiating (2.1) with respect to (1.1) and simplifying, we get:

$$\dot{V}(P) = \mathrm{tr}(\dot{P}P') + \mathrm{tr}(P\dot{P}') = \mathrm{tr}(\dot{P}P') + \mathrm{tr}((\dot{P}P')') = 2\mathrm{tr}(\dot{P}P').$$

The flow (1.1) gives

$$\dot{V}(P) = 2\mathrm{tr}([[P', P]_{du}, P]P') = -2\mathrm{tr}([P, [P', P]_{du}]P').$$

Using the fact

$$\mathrm{tr}([M, N]P) = -\mathrm{tr}([M, P]N) = \mathrm{tr}([P, M]N),$$

we get

$$\dot{V}(P) = -2\mathrm{tr}([P', P][P', P]_{du}). \tag{2.2}$$

Note that $AA_{du} = ((A_{du})' + A_u)A_{du}$ and $\mathrm{tr}(A_u A_{du}) = 0$ is true for a symmetric matrix A.

Since $[P', P]$ is a symmetric matrix, (2.2) gives

$$\dot{V}(P) = -2\mathrm{tr}(([P', P]_{du})'[P', P]_{du}) = -2\| [P', P]_{du} \|^2 \leq 0. \tag{2.3}$$

Therefore, $\dot{V}(P) \leq 0$ if $[P(t)', P(t)]_{du}$ is nonzero; equivalently, $[P', P]$ is nonzero. So, $V(P)$ is non-increasing along (1.1). This proves that $\| P \|^2 = V(P) \leq V(P_0) = \mathrm{tr}(P_0 P_0') = \| P_0 \|^2$ and hence $\| P(t) \| \leq \| P_0 \|$ for all $t \geq 0$.

(3) Because $\dot{V}(P) = 2\mathrm{tr}(\dot{P}P'), \dot{V}(P) = 0$ at an equilibrium. From (2.3), we get $[P', P]_{du} = 0$. Since $[P', P]$ is symmetric, $[P', P] = 0$ identically.

This implies that P is normal. Also, P is symmetric because P has a real and simple spectrum [10]. Therefore, *equilibria of* (1.1) *are symmetric matrices*.

A *limit set* is a set that a dynamical system approaches infinitely often as time goes to positive infinity or negative infinity. Let $p \in \mathbb{R}^n$ and Φ_t be a flow on $\mathbb{R}^n$. An *omega limit point* of the orbit through p is a point $x \in \mathbb{R}^n$ if there is a non-decreasing sequence of numbers $\{t_j\}$ such that $\lim_{j\to\infty} t_j = \infty$ and $\lim_{j\to\infty} \Phi_{t_j}(p) = x$. The *omega limit set* of p is the set of all such omega limit points and is denoted by $\Omega(p)$ [4].

The following lemma describes an important property for the vector field (1.1) regarding the omega limit set.

Lemma 2.1. *If the initial condition* $P_0 \in B^+$ *has a simple and real spectrum* Λ *for* (1.1), *then* $\Omega(P_0) \subset \mathcal{S}_\Lambda$. *More precisely*, $\Omega(P_0) \subset \mathcal{P}_\Lambda$.

Proof. We observed, in the properties of the flow (1.1) that $P(t)$ is bounded. Also, if $P(t)$ is not an equilibrium, then $\| P(t) \|$ is strictly decreasing. So, there exists a sequence of times $\{t_n\}$ such that $\lim_{t\to+\infty} P(t) = P_\infty \in \Omega(P_0)$, with $\| P_\infty \| = \inf_{n\in\mathbb{N}} \| P(t_n) \|$. Then, P_∞ is an equilibrium [2] and hence normal by the properties of the flow. Moreover, a normal matrix A is Hermitian if and only if its spectrum lies on the real line [10, 12]. In particular, $\Omega(P_0) \subset \mathcal{S}_\Lambda$.

The second claim follows from the properties of the flow, i.e., $\Omega(P_0) \subset \mathcal{S}_\Lambda \cap B^+$. □

The following computation helps to understand the evolution of second subdiagonal elements of a banded matrix P_0 with lower bandwidth 2, subject to the evolution of (1.1). The flow (1.1) gives

$$\dot{P}_{j+2,\,j} = [[P',\,P]_{du},\,P]_{j+2,\,j} = ([P',\,P]_{du}P)_{j+2,\,j} - (P[P',\,P]_{du})_{j+2,\,j}$$

$$= \sum_{l=1}^{n}([P',\,P]_{du})_{j+2,\,l}P_{l,\,j} - \sum_{l=1}^{n}P_{j+2,\,l}([P',\,P]_{du})_{l,\,j}.$$

Using the information $P_{l,\,j} = 0$ for $l > j+2$ and $([P',\,P]_{du})_{j+2,\,l} = 0$ for $j+2 > l$ in the first sum, we get nonzero terms only if $l \le j+2$ and $j+2 \le l$. This gives $l = j+2$. A similar argument in the second sum gives $l = j$. Therefore,

$$\dot{P}_{j+2,\,j} = P_{j+2,\,j}(([P',\,P]_{du})_{j+2,\,j+2} - ([P',\,P]_{du})_{j,\,j}),\; j = 1,\,\ldots,\,n-2. \quad (2.4)$$

From (2.4), if $(P_0)_{j+2,\,j} = 0$, then $P_{j+2,\,j}$ stays zero. The existence and uniqueness theorem for a system of ordinary differential equations gives that $(P(t))_{j+2,\,j}$ is nonzero when $(P_0)_{j+2,\,j}$ is nonzero.

We summarize this information in the following lemma:

Lemma 2.2. *Assume that* P_0 *is a banded matrix with lower bandwidth* 2. *Suppose that* P_0 *evolves according to the flow* (1.1). *Then*, *for each* $j = 1,\,\ldots,\,n-2$,

(a) *if* $(P_0)_{j+2,\,j}$ *is zero*, *then* $(P(t))_{j+2,\,j}$ *is zero for all* $t \ge 0$,

(b) *if* $(P_0)_{j+2,\,j}$ *is nonzero*, *then* $(P(t))_{j+2,\,j}$ *is nonzero for all* $t \ge 0$ *and* $P(t)_{j+2,\,j}$ *cannot change sign.*

3. The Omega Limit Set

In this section, we discuss the omega limit set of the initial data P_0 of the flow (1.1). We refer to an initial condition P_0 *admissible*, if it is a lower triangular, banded matrix with lower bandwidth 2, has simple and real spectrum Λ, and has all second subdiagonal entries to be nonzero.

Lemma 3.1. *There exists no T, an upper triangular matrix, having determinant one, belonging to $\mathcal{G}_0$, such that for an admissible initial condition P_0 for the flow* (1.1), $(TP_0T^{-1})_{j+2,\, j} = 0,\ j = 1, ..., n-2.$

Proof. We have

$$(TP_0T^{-1})_{j+2,\, j} = \sum_{m=1}^{n} \sum_{l=1}^{n} T_{j+2,\, l}(P_0)_{lm}(T^{-1})_{mj};$$

since $T_{j+2,\, l} = 0$ for $l < j+2$, i.e., possible nonzero $T_{j+2,\, l}$ for $l \geq j+2$ and $(T^{-1})_{mj} = 0$ for $m > j$, i.e., possible nonzero $(T^{-1})_{mj}$ for $m \leq j$, the sums above can be reduced to

$$\sum_{m=1}^{j} \sum_{l=j+2}^{n} T_{j+2,\, l}(P_0)_{lm}(T^{-1})_{mj}.$$

Moreover, since P_0 is an admissible initial condition, $(P_0)_{lm} = 0$ for $l > m + 2$, in the last sum, the only surviving terms correspond to $m = j$ and $l = j+2$. Therefore,

$$(TP_0T^{-1})_{j+2,\, j} = T_{j+2,\, j+2}(P_0)_{j+2,\, j}(T^{-1})_{jj}, \tag{3.1}$$

for $j = 1, ..., n-2$. Since P_0 is an admissible initial condition, $(P_0)_{j+2,\, j} \neq 0$ and the matrices T and T^{-1} have a determinant one. Therefore, all diagonal entries of T and T^{-1} are nonzero. This proves the claim. □

If $P(t)$ evolves following (1.1), then for $T(t) \in \mathcal{G}_0 \subset GL_n(\mathbb{R})$, $P(t) = T(t)P_0T^{-1}(t)$. Since $P(t)$ evolves following (1.1), $T(t)$ evolves following

$$\dot{T} = [P', P]_{du}T, \quad 0 \leq t < t_{\max}, \tag{3.2}$$

where $T(0)$ is the identity matrix. It is important to note the fact that the boundedness of $P(t)$ for all $t \geq 0$ does not imply the boundedness of $T(t)$ and $T(t)^{-1}$ for all $t \geq 0$.

Considering $\| T \|^2 := \mathrm{tr}(TT')$, we obtain the following estimates for $T(t)$ and $T^{-1}(t)$ (for detailed computation, see [14]):

$$\ln(\| T(t) \|) \leq \int_0^t \| [P(s)', P(s)]_{du} \| + \ln(\| T(0) \|) ds \tag{3.3}$$

and

$$\ln(\| T^{-1}(t) \|) \leq \int_0^t \| [P(s)', P(s)]_{du} \| + \ln(\| T(0) \|) ds. \tag{3.4}$$

Therefore, it is sufficient to prove that the integral on (3.3) (or (3.4)) converges as $t \to +\infty$ to show $\| T(t) \|$ and $\| T^{-1}(t) \|$ remain bounded for all $t \geq 0$. We prove that, along a solution of (1.1) for admissible initial condition P_0, $\| [P', P(t)]_{du} \|$ converges to zero exponentially fast. To achieve this, we show that the compact manifold $\mathcal{S}_\Lambda$ is *exponentially attracting* for the flow (1.1) for admissible initial condition P_0 and hence we prove that $\mathcal{S}_\Lambda$ is *exponentially stable in linear approximation* (see [15] for the definition).

The following result highlights the connection between exponentially attracting and exponentially stable in linear approximation.

Theorem 3.2 [15]**.** *An invariant compact manifold* $\mathcal{M}$, *of* $\dot{x} = F(x)$, $x, F \in \mathbb{R}^n$, *is exponentially attracting iff* $\mathcal{M}$ *is exponentially stable in linear approximation*.

Recall that $\mathcal{S}_\Lambda$ is the manifold $\mathcal{M}$ consisting of a manifold of equilibria. To prove $\mathcal{S}_\Lambda$ is exponentially stable in linear approximation, it is sufficient to linearize (1.1) at an equilibrium point $S_0 \in \mathcal{S}_\Lambda$, and show that the normal directions to the tangent space $T_{S_0}\mathcal{S}_\Lambda = \{[W, S_0] : W' = -W\}$ [1] are decreasing exponentially.

Write $dS_0 = S_0 + P(t)$, a first order deformation of S_0, in order to linearize the flow (1.1) at a point $S_0 \in \mathcal{S}_\Lambda$. We use the following result for $P(t)$ obtained in [2]

$$\dot{P}(t) = [([P', S_0] + [S_0, P])_{du}, S_0] = [[S_0, P - P']_{du}, S_0]. \tag{3.5}$$

Also, it is shown in [2] that if $P(t)$ evolves following the linearization (3.5) then its normal component

$$\pi_N(P(t)) = \frac{1}{2}(P - P') \in N_{S_0}\mathcal{S}_\Lambda \tag{3.6}$$

converges to zero exponentially. Therefore, we concluded that $\mathcal{S}_\Lambda$ is exponentially stable on linear approximation because the quadratic form $\frac{d\|\pi_N(P(t))\|^2}{dt}$ is negative definite on $N_{S_0}\mathcal{S}_\Lambda$ for each $S_0 \in \mathcal{S}_\Lambda$, $\mathcal{S}_\Lambda$ is compact and $\frac{d\|\pi_N(P(t))\|^2}{dt}$ is continuous. Using these results along with Theorem 3.2 along with the definition of exponentially stable in linear approximation, we conclude $\mathcal{S}_\Lambda$ is exponentially attracting manifold for the flow (1.1).

The time evolution of $\|\pi_N(P(t))\|$ and $\|[P(t)', P(t)]_{du}\|$ are related in the linearization. If $\pi_N(P(t))$ evolve following (3.5) converges exponentially fast to zero, then $[P(t)', P(t)]_{du}$ converges exponentially fast to zero for a solution $P(t)$ of the flow (1.1) [2]. This proves $\int_0^t \|[P(s)', P(s)]_{du}\| ds$ converges along a solution of (1.1) for $t \to +\infty$ for any admissible initial condition P_0.

We use the results obtained so far to prove the following theorem:

Theorem 3.3. *Let the solution of the flow* (1.1) *starting from an admissible initial condition* P_0 *be* $P(t)$. *Then,* $\lim_{t \to +\infty} P(t)$ *converges* *exponentially fast to* $\mathcal{P}_\Lambda$ *with the same sign pattern for second subdiagonal elements as in* P_0.

Proof. Using the inequalities (3.3) and (3.4) and the fact $\| [P(t)', P(t)]_{du} \|$ is converging exponentially fast to zero, we see the boundedness of $\| T(t) \|$ and $\| T^{-1}(t) \|$. Therefore, the eigenvalues of $T(t)$ and $T^{-1}(t)$ are bounded and bounded away from zero. Lemma 3.1 implies that $P(t)$ can not converge to the set of diagonal matrices for $t \to +\infty$. In particular, $\lim_{t \to +\infty} P(t)_{j+2, j}$ is nonzero for $j = 1, \ldots, n-2$. Therefore, $P(t)$ converges to $\mathcal{P}_\Lambda$ by Lemmas 2.1 and 2.2 implying that the second subdiagonal elements of $P(t)$ cannot change sign. This proves that $P(t)$ has to converge to $\mathcal{P}_\Lambda$ with the same sign pattern for the second subdiagonal elements as P_0. □

Finally, we will show that the omega limit set for an admissible initial condition is a single point.

Theorem 3.4. *Let an admissible initial condition for the flow* (1.1) *be* P_0. *Then*

(a) $\Omega(P_0)$ *is a singleton set, and the only element is a pentadiagonal symmetric matrix isospectral to* P_0.

(b) $\Omega(P_0)$ *has the same sign pattern in the second subdiagonal elements as the second subdiagonal elements of* P_0.

Proof. In this theorem, we only need to prove that $\Omega(P_0)$ is a singleton.

The submultiplicative property of the norm implies

$$\lim_{t \to +\infty} \int_0^t \| [[P(s)', P(s)]_{du}, P(s)] \| ds$$

$$\leq 2 \lim_{t \to +\infty} \int_0^t \| [P(s)', P(s)]_{du} \| \| P(s) \| ds.$$

Due to the second point of properties of the flow, $\| P \|$ can be bounded by a constant α and using the convergence of $\| [P(t)', P(t)]_{du} \|$ for $t \to +\infty$ along a solution of (1.1), we get

$$\lim_{t \to +\infty} \int_0^t \| [P(s)', P(s)]_{du} \| \| P(s) \| ds$$

$$\leq \alpha \lim_{t \to +\infty} \int_0^t \| [P(s)', P(s)]_{du} \| ds < +\infty.$$

The convergence of $\lim_{t \to +\infty} \int_0^t \| [P(s)', P(s)]_{du} \| \| P(s) \| ds$ implies the length of the solution curve is finite. This is enough to conclude $\Omega(P_0)$ is a singleton. □

4. Simulations and Applications

In this section, we illustrate the convergence properties of the flow using some simulations. We use MatLab^{TM} ODE solvers ode15s for the simulations. We also show that by choosing arbitrary numbers different from zero in the second subdiagonal in the initial condition, we can generate an arbitrary number in the second subdiagonal that follows the sign pattern as the initial condition.

Suppose that P_0 is a square matrix of order 7 with spectrum $\sigma(P_0) = \{1, 2, 3, -4, 5, -6, 7\}$ having the diagonal entries $[1, 2, 3, -4, 5, -6, 7]$ and arbitrary chosen first subdiagonal and nonzero second subdiagonal elements. Then

$$P_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & 2 & 0 & 0 & 0 & 0 & 0 \\ 4 & -10 & 3 & 0 & 0 & 0 & 0 \\ 0 & -5 & -10 & -4 & 0 & 0 & 0 \\ 0 & 0 & -3 & 10 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 & -10 & -6 & 0 \\ 0 & 0 & 0 & 0 & -1 & 10 & 7 \end{bmatrix}.$$

Using the flow under study, the corresponding omega limit point under the numerical approximation is

$$\Omega(P_0) = \begin{bmatrix} 2.2202 & -1.0801 & 0.1503 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ -1.0801 & 0.7339 & -3.8420 & -0.5683 & 0.0000 & 0.0000 & 0.0000 \\ 0.1503 & -3.8420 & 0.2371 & -3.4945 & -0.2872 & 0.0000 & 0.0000 \\ 0.0000 & -0.5683 & -3.4945 & 1.4257 & 3.3154 & 0.1253 & 0.0000 \\ 0.0000 & 0.0000 & -0.2872 & 3.3154 & -0.6737 & -4.4453 & -0.0616 \\ 0.0000 & 0.0000 & 0.0000 & 0.1253 & -4.4453 & 1.5483 & 1.5191 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0616 & 1.5191 & 2.5086 \end{bmatrix}.$$

The spectrum

$$\sigma(\Omega(P_0)) = \{-6.0001, -3.9990, 7.0004, 4.9997, 0.9993, 1.9985, 3.0013\}.$$

Comparing it to the spectrum of P_0, the spectrum of $\Omega(P_0)$ is within the third decimal place, and the sign pattern of the second subdiagonal elements is reproduced faithfully.

Now, we present the transient behavior of the second subdiagonal elements for a square matrix of order 5. Suppose that P_0 has spectrum $\sigma(P_0) = \{1, 3, 5, -4, -5\}$ with arbitrary chosen first subdiagonal and nonzero second subdiagonal elements $[-3, -5, -5]$. Then

$$P_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 0 \\ -3 & 5 & 5 & 0 & 0 \\ 0 & -5 & -1 & -4 & 0 \\ 0 & 0 & -5 & -1 & -5 \end{bmatrix}.$$

The corresponding omega limit point under the numerical approximation is

$$\Omega(P_0) = \begin{bmatrix} 1.8702 & 1.2228 & -0.4363 & -0.0000 & -0.0000 \\ 1.2228 & 2.2611 & 1.0567 & -2.9223 & 0.0000 \\ -0.4363 & 1.0567 & 0.0811 & -1.5660 & -3.8841 \\ -0.0000 & -2.9223 & -1.5660 & -2.6109 & 0.3448 \\ -0.0000 & 0.0000 & -3.8841 & 0.3448 & -1.6015 \end{bmatrix}.$$

The spectrum $\sigma(\Omega(P_0)) = \{5.0000, 3.0000, 0.9999, -4.0001, -4.9997\}$. We can see that the sign pattern of the second subdiagonal entries has reproduced faithfully. We show the trajectories of the second subdiagonal elements as functions of time to get an idea about their transient behavior. Figure 1 displays the transient behavior of the second subdiagonal element along (1.1).

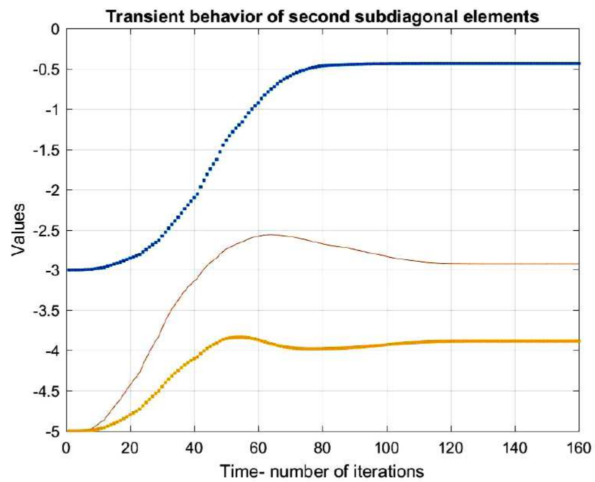

Figure 1. Transient behavior of the second subdiagonal elements.

In the following example, P_0 has been chosen in a more general form of a banded matrix with lower bandwidth p with a given sign pattern in the pth subdiagonal entries to reproduce a symmetric $(2p+1)$-diagonal matrix preserving the sign pattern in the pth subdiagonal entries as in P_0:

$$P_0 = \begin{bmatrix} -10 & -10 & 10 & 0 & 0 & 0 & 0 \\ -10 & -20 & -10 & -10 & 0 & 0 & 0 \\ -10 & -10 & 30 & -10 & -10 & 0 & 0 \\ 10 & -10 & -10 & -40 & 10 & 10 & 0 \\ 8 & 10 & -10 & 10 & 15 & -10 & 10 \\ 0 & -2 & -10 & 10 & -10 & 16 & 10 \\ 0 & 0 & -10 & 10 & -10 & 10 & 24 \end{bmatrix},$$

where

$$\sigma(P_0) = \{-48.6288, -23.7190, 38.0423, -1.8910, 26.8020, 6.5865, 17.8080\}.$$

The corresponding omega limit point under the numerical approximation is

$$\Omega(P_0) = \begin{bmatrix} -7.1316 & -5.5889 & -10.0849 & 7.2284 & 9.4857 & -0.0000 & 0.0000 \\ -5.5889 & -14.7562 & -6.3082 & -13.7612 & 13.3912 & -3.7925 & -0.0000 \\ -10.0849 & -6.3082 & 29.8806 & -0.5877 & -2.8564 & -7.9681 & -5.3284 \\ 7.2284 & -13.7612 & -0.5877 & -39.9472 & 4.8742 & 7.6116 & 4.2840 \\ 9.4857 & 13.3912 & -2.8564 & 4.8742 & 14.9688 & -3.6080 & -3.2357 \\ 0.0000 & -3.7925 & -7.9681 & 7.6116 & -3.6080 & 8.4163 & 1.6524 \\ 0.0000 & 0.0000 & -5.3284 & 4.2840 & -3.2357 & 1.6524 & 23.5693 \end{bmatrix},$$

where

$$\sigma(\Omega(P_0)) = \{-48.6289, -23.7192, 38.0423, -1.8909, 6.5864, 26.8021, 17.8083\}.$$

Figure 2 displays the transient behavior of the fourth subdiagonal element along (1.1).

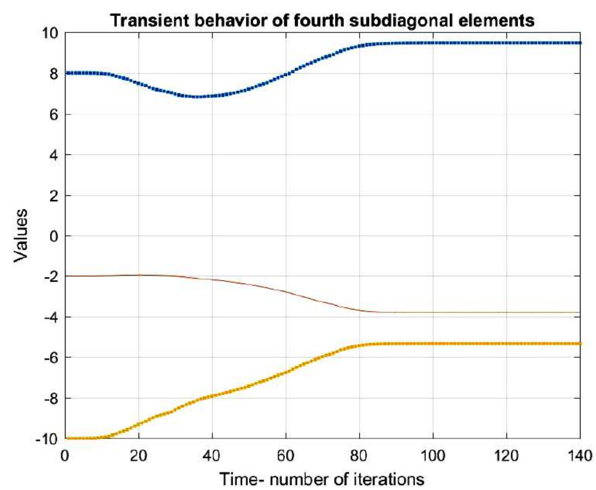

Figure 2. Transient behavior of the fourth subdiagonal elements.

As an application, the flow provides a solution to the inverse eigenvalue problem for pentadiagonal matrices with a simple and real spectrum and a given sign pattern for the nonzero second subdiagonal elements. More specifically, we can solve the following inverse problem:

Given a simple and real spectrum $\{\lambda_1, ..., \lambda_n\}$ *and an assigned sign pattern of* $n-2$ *ordered elements, reconstruct a real pentadiagonal symmetric square matrix of order n with the given spectrum and the sign pattern for second subdiagonal elements.*

It is worth mentioning that the flow (1.1) is the solution to an optimal control problem over the infinite time horizon.

Theorem 4.1 [2]**.** *Let* $A(t)$ *be a smooth function in the Lie algebra of upper triangular matrices. The optimal value function of the following control problem on the infinite time horizon*

$$\min_A \int_0^{+\infty} \mathrm{tr}(([P', P]_{du})'([P', P]_{du})) + \mathrm{tr}(A'A)ds, \text{ subject to } \dot{P} = [A, P], \quad (4.1)$$

is given by $\mathrm{tr}(P'P)$ *and the flow* (1.1) *is the solution of* (4.1), *i.e.*, *the optimal feedback is given by* $A = [P', P]_{du}$.

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References

[1] P.-A. Absil, R. Mahony and R. Sepulchre, Optimization Algorithms on Matrix Manifolds, Princeton University Press, 2008.

[2] A. Arsie and C. Ebenbauer, A Hessenberg-Jacobi isospectral flow, Nonlinear Differ. Equ. Appl. 22 (2015), 87-103.

[3] A. Arsie and K. Pokharel, A normalizing isospectral flow on complex Hessenberg matrices, J. Math. Anal. Appl. 432(2) (2015), 787-805.

[4] C. Chicone, Ordinary Differential Equations with Applications, 2nd ed., Springer, 2006.

[5] M. Chu, Inverse eigenvalue problem, SIAM Rev. 40(1) (1998), 1-39.

[6] P. Deift, L.-C. Li and C. Tomei, Matrix factorizations and integrable systems, Comm. Pure Appl. Math. 42(4) (1989), 443-521.

[7] H. Flaschka, The Toda lattice, II, Phys. Rev. B 9 (1974), 1924-1925.

[8] S. Friedland, J. Nocedal and M. Overton, The formulation and analysis of numerical methods for inverse eigenvalue problems, SIAM J. Numer. Anal. 24(3) (1987), 634-667.

[9] G. H. Golub and C. F. Van Loan, Matrix Computations, 3rd ed., Johns Hopkins, Baltimore, 1996.

[10] R. Horn and C. Johnson, Matrix Analysis, Cambridge University Press, 1985.

[11] A. Kaur, On solving an isospectral flow, J. Comput. Appl. Math. 308 (2016), 263-275.

[12] P. Lancaster and M. Tismenetsky, The Theory of Matrices: With Applications, 2nd ed., Academic Press, 1985.

[13] A. M. Perelomov, Integrable Systems of Classical Mechanics and Lie Algebras, Birkhäuser Verlag, Vol. 1, 1990.

[14] K. Pokharel, An isospectral flow for complex upper Hessenberg matrices, Doctoral dissertation, University of Toledo, OhioLINK Electronic Theses and Dissertations Center, 2015.

[15] L. B. Ryashko and E. E. Shnol, On exponentially attracting invariant manifolds of ODEs, Nonlinearity 16 (2003), 147-160.

[16] W. Symes, Hamiltonian group actions and integrable systems, Phys. D 1 (1980), 339-374.

[17] W. Symes, The QR algorithm and scattering for the finite nonperiodic Toda lattice, Phys. D 4 (1982), 275-280.

[18] M. D. Webb, Isospectral algorithms, Toeplitz matrices and orthogonal polynomials, PhD thesis, University of Cambridge, 2017.