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# EXISTENCE AND UNIQUENESS OF THE SOLUTION OF A HYPERBOLIC PROBLEM WITH POLYNOMIAL NONLINEARITY

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## Abstract

In this work, we study a hyperbolic problem with polynomial nonlinearity and a homogeneous Neumann condition.

## 1. Introduction

Consider the following problem:

$$u'' - \Delta u + u^3 + \frac{\partial u}{\partial t} = f(x, t), \quad x \in \Omega, t \in \, ]0, T[, \qquad (1.1)$$

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$$\frac{\partial u}{\partial \vec{n}}(x, t)|_{\partial\Omega} = 0, \tag{1.2}$$

$$u(x, 0) = u_0(x), \quad u'(x, 0) = \frac{\partial u}{\partial t}(x, 0) = u_1(x), \tag{1.3}$$

in a cylinder

$$Q_T = \{x, t : x \in \Omega \subset \mathbb{R}^n, 0 < t \leq T < \infty\},$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ with the differentiable border $\partial\Omega$, $\vec{n}$ is the exterior normal to $\partial\Omega$, $u'' = \frac{\partial^2 u}{\partial t^2}$ and $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$.

Put

$$H^1(\Omega) = \left\{v/v \in L^2(\Omega), \frac{\partial v}{\partial x_i} \in L^2(\Omega), i = 1, ..., n\right\},$$

with the norm

$$\| v \|_{H^1(\Omega)} = \left(\int_\Omega \left[| v |^2 + \sum_{i=1}^{n} \left| \frac{\partial v}{\partial x_i} \right|^2\right] dx\right)^{1/2}$$

and

$$\| u \|_{H^1(\Omega)}^2 = \| u \|_{L^2(\Omega)}^2 + \| \nabla u \|_{L^2(\Omega)}^2,$$

where $f$, $u_0$ and $u_1$ are given functions.

## 2. Existence of the Solution of the Problem

### 2.1. Variational formulation

Multiplying equation (1.1) by the function $\eta(x, t)$ and integrating on $Q_T$, we obtain

$$\int_{Q_T} u''\eta(x, t)\,dxdt - \int_{Q_T} \Delta u\eta(x, t)\,dxdt$$

$$+ \int_{Q_T} u^3\eta(x, t)\,dxdt + \int_{Q_T} \frac{\partial u}{\partial t}\eta(x, t)\,dxdt$$

$$= \int_{Q_T} f(x, t)\eta(x, t)\,dxdt. \tag{2.1}$$

Using integration by parts, the first two integrals of the equality (2.1) give

$$\int_{Q_T} u''\eta(x, t)\,dxdt = \int_\Omega \left( \int_0^T u''\eta(x, t)\,dt \right) dx$$

$$= \int_\Omega u'(x, T)\eta(x, T)\,dx - \int_\Omega u_1(x)\eta(x, 0)\,dx$$

$$- \int_{Q_T} u_t\eta_t\,dxdt,$$

$$\int_{Q_T} \Delta u\eta(x, t)\,dxdt = -\int_{Q_T} (\nabla u \nabla \eta)\,dxdt + \int_{\partial Q_T} \frac{\partial u}{\partial \vec{n}}\eta(x, t)\,dxdt$$

$$= -\int_{Q_T} (\nabla u \nabla \eta)\,dxdt.$$

Then equality (2.1) takes the form:

$$\int_\Omega u'(x, T)\eta(x, T)\,dx - \int_\Omega u_1(x)\eta(x, 0)\,dx - \int_{Q_T} u_t\eta_t\,dxdt$$

$$+ \int_{Q_T} (\nabla u \nabla \eta)\,dxdt + \int_{Q_T} u^3\eta(x, t)\,dxdt + \int_{Q_T} \frac{\partial u}{\partial t}\eta(x, t)\,dxdt$$

$$= \int_{Q_T} f(x, t)\eta(x, t)\,dxdt.$$

**Definition.** The *distributional solution* of the problem (1.1)-(1.3), is any function $u(x, t)$, equal to $u_0(x)$ for $t = 0$ satisfying the following integral:

$$-\int_{\Omega} u_1(x)\eta(x, 0)dx - \int_{Q_T} u_t\eta_t dxdt + \int_{Q_T} (\nabla u \nabla \eta)dxdt$$

$$+\int_{Q_T} u^3\eta(x, t)dxdt + \int_{Q_T} \frac{\partial u}{\partial t}\eta(x, t)dxdt$$

$$= \int_{Q_T} f(x, t)\eta(x, t)dxdt \tag{2.2}$$

for all $\eta(x, t)$ whose trace for $t = T$ is equal to 0.

### 2.2. A priori estimates of these solutions

Multiplying (1.1) by $\frac{\partial u}{\partial t}$ and using integration by parts on $\Omega$:

$$\int_{\Omega} \frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial t}dx - \int_{\Omega} \Delta u \frac{\partial u}{\partial t}dx + \int_{\Omega} u^3 \frac{\partial u}{\partial t}dx + \int_{\Omega} \frac{\partial u}{\partial t}\frac{\partial u}{\partial t}dx$$

$$= \int_{\Omega} f(x, t)\frac{\partial u}{\partial t}dx. \tag{2.3}$$

By transforming the first three integrals of (2.3), we obtain

$$\int_{\Omega} \frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial t}dx = \frac{1}{2}\int_{\Omega} \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)^2 dx = \frac{1}{2}\frac{\partial}{\partial t}\int_{\Omega}\left(\frac{\partial u}{\partial t}\right)^2 dx = \frac{1}{2}\frac{\partial}{\partial t}\left\| \frac{\partial u}{\partial t}\right\|^2_{L^2(\Omega)}. \tag{2.4}$$

By applying Green's formula, we have

$$\int_{\Omega} \Delta u \frac{\partial u}{\partial t}dx = -\int_{\Omega} \nabla u.\nabla\left(\frac{\partial u}{\partial t}\right)dx = -\frac{1}{2}\frac{\partial}{\partial t}\int_{\Omega}(\nabla u)^2 dx = -\frac{1}{2}\frac{\partial}{\partial t}\| \nabla u \|^2_{L^2(\Omega)} \tag{2.5}$$

and

$$\int_{\Omega} u^3 \frac{\partial u}{\partial t}dx = \int_{\Omega} \frac{\partial u}{\partial t}.u^3 dx = \frac{1}{4}\frac{d}{dt}\int_{\Omega} u^4 dx = \frac{1}{4}\frac{\partial}{\partial t}\| u \|^4_{L^4(\Omega)}. \tag{2.6}$$

By substituting (2.4)-(2.6) in (2.3), this latter becomes

$$\frac{1}{2}\frac{\partial}{\partial t}\left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{\partial}{\partial t}\|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{4}\frac{\partial}{\partial t}\|u\|^4_{L^4(\Omega)} + \int_\Omega \frac{\partial u}{\partial t}\frac{\partial u}{\partial t}dx$$

$$= \int_\Omega f\frac{\partial u}{\partial t}dx$$

which implies

$$\frac{1}{2}\left[\frac{\partial}{\partial t}\left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(\Omega)} + \frac{\partial}{\partial t}\|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{\partial}{\partial t}\|u\|^4_{L^4(\Omega)}\right] + \left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(\Omega)}$$

$$= \int_\Omega f\frac{\partial u}{\partial t}dx. \tag{2.7}$$

Using inequalities of Hölder and Young, the second member of (2.7) gives

$$\int_\Omega f\frac{\partial u}{\partial t}dx \le \left(\int_\Omega |f|^2 dx\right)^{1/2}\cdot\left(\int_\Omega \left|\frac{\partial u}{\partial t}\right|^2 dx\right)^{1/2}$$

$$\le \|f\|_{L^2(\Omega)}\left\|\frac{\partial u}{\partial t}\right\|_{L^2(\Omega)}$$

$$\le \frac{1}{2}\left[\|f\|^2_{L^2(\Omega)} + \left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(\Omega)}\right]. \tag{2.8}$$

By using (2.8), (2.7) becomes

$$\frac{1}{2}\left[\frac{\partial}{\partial t}\left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(\Omega)} + \frac{\partial}{\partial t}\|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{\partial}{\partial t}\|u\|^4_{L^4(\Omega)}\right] + \left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(\Omega)}$$

$$\le \frac{1}{2}\left[\|f\|^2_{L^2(\Omega)} + \left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(\Omega)}\right]$$

which implies

$$\frac{\partial}{\partial t}\left[\left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 + \| \nabla u \|_{L^2(\Omega)}^2 + \frac{1}{2}\| u \|_{L^4(\Omega)}^4\right] + 2\left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2$$

$$\leq \| f \|_{L^2(\Omega)}^2 + \left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2. \tag{2.9}$$

Integrating (2.9) on $(0, t)$:

$$\int_0^t \frac{\partial}{\partial s}\left[\left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 + \| \nabla u \|_{L^2(\Omega)}^2 + \frac{1}{2}\| u \|_{L^4(\Omega)}^4\right] ds + 2\int_0^t \left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 ds$$

$$\leq \int_0^t \left( \| f \|_{L^2(\Omega)}^2 + \left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 \right) ds$$

which implies

$$\left[\left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 + \| \nabla u \|_{L^2(\Omega)}^2 + \frac{1}{2}\| u \|_{L^4(\Omega)}^4\right]_0^t + 2\int_0^t \left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 ds$$

$$\leq \int_0^t \left( \| f \|_{L^2(\Omega)}^2 + \left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 \right) ds.$$

This gives

$$\left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 + \| \nabla u \|_{L^2(\Omega)}^2 + \frac{1}{2}\| u \|_{L^4(\Omega)}^4 + 2\int_0^t \left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 ds$$

$$- \left( \left\| \frac{\partial u(x, 0)}{\partial t} \right\|_{L^2(\Omega)}^2 + \| \nabla u_0 \|_{L^2(\Omega)}^2 + \frac{1}{2}\| u_0 \|_{L^4(\Omega)}^4 \right)$$

$$\leq \int_0^t \left( \| f \|_{L^2(\Omega)}^2 + \left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 \right) ds$$

and

$$\left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 + \| \nabla u \|_{L^2(\Omega)}^2 + \frac{1}{2} \| u \|_{L^4(\Omega)}^4 + 2\int_0^t \left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 ds$$

$$- \left( \| u_1 \|_{L^2(\Omega)}^2 + \| \nabla u_0 \|_{L^2(\Omega)}^2 + \frac{1}{2} \| u_0 \|_{L^4(\Omega)}^4 \right)$$

$$\leq \int_0^t \left( \| f \|_{L^2(\Omega)}^2 + \left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 \right) ds.$$

Finally, we obtain

$$\left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 + \| \nabla u \|_{L^2(\Omega)}^2 + \frac{1}{2} \| u \|_{L^4(\Omega)}^4 + 2\int_0^t \left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 ds$$

$$\leq \| u_1 \|_{L^2(\Omega)}^2 + \| \nabla u_0 \|_{L^2(\Omega)}^2 + \frac{1}{2} \| u_0 \|_{L^4(\Omega)}^4$$

$$+ \int_0^t \left( \| f \|_{L^2(\Omega)}^2 + \left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 \right) ds. \tag{2.10}$$

Adding the term $\| u \|_{L^2(\Omega)}^2$ to the first and second members of (2.10), we have

$$\left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 + \| u \|_{L^2(\Omega)}^2 + \| \nabla u \|_{L^2(\Omega)}^2 + \frac{1}{2} \| u \|_{L^4(\Omega)}^4 + 2\int_0^t \left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 ds$$

$$\leq \| u \|_{L^2(\Omega)}^2 + \| u_1 \|_{L^2(\Omega)}^2 + \| \nabla u_0 \|_{L^2(\Omega)}^2 + \frac{1}{2} \| u_0 \|_{L^4(\Omega)}^4$$

$$+ \int_0^t \left( \| f \|_{L^2(\Omega)}^2 + \left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 \right) ds. \tag{2.11}$$

Thus

$$\left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 + \| u \|_{H^1(\Omega)}^2 + \frac{1}{2} \| u \|_{L^4(\Omega)}^4 + 2\int_0^t \left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 ds$$

$$\leq \| u \|_{L^2(\Omega)}^2 + \| u_1 \|_{L^2(\Omega)}^2 + \| \nabla u_0 \|_{L^2(\Omega)}^2 + \frac{1}{2} \| u_0 \|_{L^4(\Omega)}^4$$

$$+ \int_0^t \left( \| f \|_{L^2(\Omega)}^2 + \left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 \right) ds. \tag{2.12}$$

For $t \in [0, T]$, we have

$$\left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 + \| u \|_{H^1(\Omega)}^2 + \frac{1}{2} \| u \|_{L^4(\Omega)}^4 + 2\int_0^T \left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 ds$$

$$\leq \| u \|_{L^2(\Omega)}^2 + \| u_1 \|_{L^2(\Omega)}^2 + \| \nabla u_0 \|_{L^2(\Omega)}^2 + \frac{1}{2} \| u_0 \|_{L^4(\Omega)}^4$$

$$+ \int_0^T \left( \| f \|_{L^2(\Omega)}^2 + \left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 \right) dt. \tag{2.13}$$

Putting $u(t) = u(x, t)$, we have

$$u(t) = \int_0^t \frac{\partial u}{\partial s}(s) ds + u(0),$$

$$| u(t) |^2 = \left| \int_0^t \frac{\partial u}{\partial s}(s) ds + u(0) \right|^2$$

$$\leq \left| \int_0^t \frac{\partial u}{\partial s}(s) ds + u(0) \right|^2 + \left| \int_0^t \frac{\partial u}{\partial s}(s) ds - u(0) \right|^2$$

and hence

$$| u(t) |^2 \leq 2\left[ \left( \int_0^t \left| \frac{\partial u}{\partial s}(s) \right| ds \right)^2 + | u(0) |^2 \right]. \tag{2.14}$$

Using Hölder's inequality, we obtain

$$\int_0^t \left|\frac{\partial u}{\partial s}(x, t)\right| ds \leq \left(\int_0^t |1| ds\right)^{1/2} . \left(\int_0^t \left|\frac{\partial u}{\partial s}(s)\right|^2 ds\right)^{1/2}$$

which implies

$$\left(\int_0^t \left|\frac{\partial u}{\partial s}(x, t)\right| ds\right)^2 \leq \left(\int_0^t ds\right) . \left(\int_0^t \left|\frac{\partial u}{\partial s}(s)\right|^2 ds\right)$$

$$\leq t \int_0^t \left|\frac{\partial u}{\partial s}(s)\right|^2 ds,$$

and (2.14) becomes

$$|u(t)|^2 \leq 2\left[t \int_0^t \left|\frac{\partial u}{\partial s}(s)\right|^2 ds + |u(0)|^2\right] \Leftrightarrow |u(t)|^2$$

$$\leq 2t \int_0^t \left|\frac{\partial u}{\partial s}(s)\right|^2 ds + 2|u(0)|^2.$$

Consider $t \in [0, 1] \subset [0, T]$. Then $1 - t \geq 0$. Adding the term $2(1 - t) \int_0^t \left|\frac{\partial u}{\partial s}(s)\right|^2 ds$ to the second member of (2.14), we get

$$|u(t)|^2 \leq 2t \int_0^t \left|\frac{\partial u}{\partial s}(s)\right|^2 ds + 2|u(0)|^2 + 2(1 - t) \int_0^t \left|\frac{\partial u}{\partial s}(s)\right|^2 ds$$

and, finally,

$$|u(t)|^2 \leq 2\left[|u(0)|^2 + \int_0^t \left|\frac{\partial u}{\partial s}(s)\right|^2 ds\right].$$

Integrating on $\Omega$, for $t \in [0, T]$, we obtain

$$\int_\Omega |u(t)|^2 dx \leq 2\int_\Omega \left[ |u(0)|^2 + \int_0^t \left| \frac{\partial u}{\partial s}(s) \right|^2 ds \right] dx$$

$$\leq 2\left[ \int_\Omega |u_0|^2 dx + \int_0^t \left( \int_\Omega \left| \frac{\partial u}{\partial s} \right|^2 dx \right) ds \right]$$

$$\leq 2\left[ \int_\Omega |u_0|^2 dx + \int_0^T \left( \int_\Omega \left| \frac{\partial u}{\partial t} \right|^2 dx \right) dt \right]$$

which implies

$$\| u(t) \|_{L^2(\Omega)}^2 \leq 2\left[ \| u_0 \|_{L^2(\Omega)}^2 + \int_0^T \left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 dt \right]. \tag{2.15}$$

By using (2.15), (2.13) becomes

$$\left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 + \| u \|_{H^1(\Omega)}^2 + \frac{1}{2} \| u \|_{L^4(\Omega)}^4 + 2\int_0^T \left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 dt$$

$$\leq 2\left[ \| u_0 \|_{L^2(\Omega)}^2 + \int_0^T \left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 dt \right]$$

$$+ \| u_1 \|_{L^2(\Omega)}^2 + \| \nabla u_0 \|_{L^2(\Omega)}^2 + \frac{1}{2} \| u_0 \|_{L^4(\Omega)}^4$$

$$+ \int_0^T \left( \| f \|_{L^2(\Omega)}^2 + \left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 \right) dt.$$

Thus

$$\left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 + \| u \|_{H^1(\Omega)}^2 + \frac{1}{2} \| u \|_{L^4(\Omega)}^4 + 2\int_0^T \left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 dt$$

$$\leq 2\| u_0 \|_{L^2(\Omega)}^2 + \| u_1 \|_{L^2(\Omega)}^2 + \| \nabla u_0 \|_{L^2(\Omega)}^2 + \frac{1}{2} \| u_0 \|_{L^4(\Omega)}^4$$

$$+ 3\int_0^T \left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 dt + \int_0^T \| f \|_{L^2(\Omega)}^2 dt. \tag{2.16}$$

By adding the term $\| \nabla u_0 \|^2_{L^2(\Omega)}$ to the second member, (2.16) becomes

$$\left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(\Omega)} + \| u \|^2_{H^1(\Omega)} + \frac{1}{2} \| u \|^4_{L^4(\Omega)} + 2\int_0^T \left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(\Omega)} ds$$

$$\leq 2\| u_0 \|^2_{L^2(\Omega)} + \| u_1 \|^2_{L^2(\Omega)} + 2\| \nabla u_0 \|^2_{L^2(\Omega)}$$

$$+ \frac{1}{2} \| u_0 \|^4_{L^4(\Omega)} + 3\int_0^T \left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(\Omega)} dt + \int_0^T \| f \|^2_{L^2(\Omega)} dt \tag{2.17}$$

and

$$\left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(\Omega)} + \| u \|^2_{H^1(\Omega)} + \frac{1}{2} \| u \|^4_{L^4(\Omega)} + 2\int_0^T \left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(\Omega)} dt$$

$$\leq 2\| u_0 \|^2_{H^1(\Omega)} + \| u_1 \|^2_{L^2(\Omega)} + \frac{1}{2} \| u_0 \|^4_{L^4(\Omega)}$$

$$+ 3\int_0^T \left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(\Omega)} dt + \int_0^T \| f \|^2_{L^2(\Omega)} dt. \tag{2.18}$$

By adding the term $\int_0^T \| u \|^2_{H^1(\Omega)} dt$ to the second member of (2.17), we obtain

$$\left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(\Omega)} + \| u \|^2_{H^1(\Omega)} + \frac{1}{2} \| u \|^4_{L^4(\Omega)}$$

$$\leq 2\| u_0 \|^2_{H^1(\Omega)} + \| u_1 \|^2_{L^2(\Omega)} + \frac{1}{2} \| u_0 \|^4_{L^4(\Omega)}$$

$$+ \int_0^T \left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(\Omega)} dt + \int_0^T \| f \|^2_{L^2(\Omega)} dt + \int_0^T \| u \|^2_{H^1(\Omega)} dt. \tag{2.19}$$

Putting

$$k = 2\| u_0 \|_{H^1(\Omega)}^2 + \| u_1 \|_{L^2(\Omega)}^2 + \frac{1}{2}\| u_0 \|_{L^4(\Omega)}^4 + \int_0^T \| f \|_{L^2(\Omega)}^2 dt,$$

(2.19) becomes

$$\left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 + \| u \|_{H^1(\Omega)}^2 + \frac{1}{2}\| u \|_{L^4(\Omega)}^4$$

$$\leq k + \int_0^T \left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 dt + \int_0^T \| u \|_{H^1(\Omega)}^2 dt \tag{2.20}$$

and

$$\left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 + \| u \|_{H^1(\Omega)}^2 \leq \left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 + \| u \|_{H^1(\Omega)}^2 + \frac{1}{2}\| u \|_{L^4(\Omega)}^4$$

$$\leq k + \int_0^T \left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 dt + \int_0^T \| u \|_{H^1(\Omega)}^2 dt.$$

Then we have the following relationship:

$$\left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 + \| u \|_{H^1(\Omega)}^2 \leq k + \int_0^T \left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 dt + \int_0^T \| u \|_{H^1(\Omega)}^2 dt. \tag{2.21}$$

Put

$$E(t) = \left\| \frac{\partial u}{\partial t} \right\|_{L^2(\Omega)}^2 + \| u(x, t) \|_{H^1(\Omega)}^2.$$

Then

$$E(t) \leq k + \int_0^T \left\| \frac{\partial u}{\partial t}(x, t) \right\|_{L^2(\Omega)}^2 dt + \int_0^T \| u(x, t) \|_{H^1(\Omega)}^2 dt.$$

According to Gronwall inequality,

$$E(t) \leq ke^{\int_0^T E(s) dt},$$

and hence $E(t) \leq ke^T$, or $E(t) \leq c$ with $c = ke^T$. Thus

$$E(t) = \left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(\Omega)} + \| u(x, t) \|^2_{H^1(\Omega)} \leq c \text{ p.p on } [0, T]. ,$$

and we have

$$\begin{cases} \left\| \dfrac{\partial u}{\partial t} \right\|^2_{L^2(\Omega)} \leq c_1 \\ \| u(x, t) \|^2_{H^1(\Omega)} \leq c_2 \end{cases} \quad \text{with } c = c_1 + c_2$$

$$\Rightarrow \begin{cases} \dfrac{\partial u}{\partial t} \in L^\infty(0, T; L^2(\Omega)) \\ u(x, t) \in L^\infty(0, T; H^1(\Omega)). \end{cases}$$

On the other hand, (2.18) gives

$$\left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(\Omega)} + \| u \|^2_{H^1(\Omega)} + \frac{1}{2} \| u \|^4_{L^4(\Omega)} + 2\int_0^T \left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(\Omega)} ds$$

$$\leq 2\| u_0 \|^2_{H^1(\Omega)} + \| u_1 \|^2_{L^2(\Omega)} + \frac{1}{2} \| u_0 \|^4_{L^4(\Omega)}$$

$$+ 3\int_0^T \left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(\Omega)} dt + \int_0^T \| f \|^2_{L^2(\Omega)} dt.$$

Adding the term $\int_0^T \| u \|^4_{L^4(\Omega)} dt$ to the second member of (2.18), we obtain

$$\left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(\Omega)} + \| u \|^2_{H^1(\Omega)} + \frac{1}{2} \| u \|^4_{L^4(\Omega)}$$

$$\leq 2\| u_0 \|^2_{H^1(\Omega)} + \| u_1 \|^2_{L^2(\Omega)} + \frac{1}{2} \| u_0 \|^4_{L^4(\Omega)}$$

$$+ \int_0^T \left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(\Omega)} dt + \int_0^T \| f \|^2_{L^2(\Omega)} dt + \int_0^T \| u \|^4_{L^4(\Omega)} dt.$$

Putting

$$k_1 = 2\| u_0 \|^2_{H^1(\Omega)} + \| u_1 \|^2_{L^2(\Omega)} + \frac{1}{2}\| u_0 \|^4_{L^4(\Omega)}$$

$$+ \int_0^T \| f \|^2_{L^2(\Omega)} + \int_0^T \left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(\Omega)} dt,$$

the inequality becomes

$$\left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(\Omega)} + \| u \|^2_{H^1(\Omega)} + \frac{1}{2}\| u \|^4_{L^4(\Omega)} \leq k_1 + \int_0^T \| u \|^4_{L^4(\Omega)} dt.$$

We have

$$\frac{1}{2}\| u \|^4_{L^4(\Omega)} \leq \left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(\Omega)} + \| u \|^2_{H^1(\Omega)} + \frac{1}{2}\| u \|^4_{L^4(\Omega)}$$

$$\leq k_1 + \int_0^T \| u \|^4_{L^4(\Omega)} dt. \tag{2.22}$$

Then

$$\frac{1}{2}\| u \|^4_{L^4(\Omega)} \leq k_1 + \int_0^T \| u \|^4_{L^4(\Omega)} dt \tag{2.23}$$

$$\Rightarrow \| u \|^4_{L^4(\Omega)} \leq 2k_1 + 2\int_0^T \| u \|^4_{L^4(\Omega)} dt$$

$$\Rightarrow \| u \|^4_{L^4(\Omega)} \leq k_2 + 2\int_0^T \| u \|^4_{L^4(\Omega)} dt.$$

Putting $F(t) = \| u \|^4_{L^4(\Omega)}$, we have

$$F(t) \leq k_2 + 2\int_0^T \| u \|^4_{L^4(\Omega)} dt.$$

Due to Gronwall inequality,

$$F(t) \leq k_2 e^{2\int_0^T F(s)\,dt},$$

and hence $F(t) \leq k_2 e^{2T}$, or $F(t) \leq c_3$ with $c_3 = k_2 e^{2T}$.

Noting that $\inf\limits_{t \in [0, T]} \| u \|^4_{L^4(\Omega)} \leq c_3$, we have

$$u \in L^\infty(0, T, L^4(\Omega)).$$

From the above, it follows that

$$\begin{cases} \dfrac{\partial u}{\partial t} \in L^\infty(0, T; L^2(\Omega)) \\ u(x, t) \in L^\infty(0, T; H^1(\Omega)) \end{cases} \text{and } u \in L^\infty(0, T, L^4(\Omega)) \tag{2.24}$$

$$\Rightarrow \begin{cases} \dfrac{\partial u}{\partial t} \in L^\infty(0, T; L^2(\Omega)) \\ u(x, t) \in L^\infty(0, T; H^1(\Omega)) \cap L^\infty(0, T, L^4(\Omega)). \end{cases}$$

Then

$$\begin{cases} \dfrac{\partial u}{\partial t} \in L^\infty(0, T; L^2(\Omega)) \\ u(x, t) \in L^\infty(0, T; H^1(\Omega)) \cap L^4((\Omega)). \end{cases}$$

Hence, we have the following theorem:

### 2.3. Theorem

*Suppose* $f(x, t)$, $u_0(x)$ *and* $u_1(x)$ *are given functions such that*

$$f(x, t) \in L^2(Q_T),$$

$$u_0(x) \in H^1(\Omega) \cap L^4(\Omega),$$

$$u_1(x) \in L^2(\Omega).$$

*Then problem* (1.1)-(1.3) *has a solution* $u(x, t)$ *satisfying the following*:

$$\begin{cases} \dfrac{\partial u}{\partial t} \in L^{\infty}(0, T; L^2(\Omega)), \\ u(x, t) \in L^{\infty}(0, T; H^1(\Omega) \cap L^4(\Omega)). \end{cases}$$

**Proof.** For the proof, we use the Faedo-Galerkin method which consists of three steps:

**Step 1.** Construction of approximate solutions;

**Step 2.** A priori estimates of these solutions;

**Step 3.** Transition to the boundary.

**Step 1.** Construction of approximate solutions

Put

$$V = \{u \in H^1(\Omega) \cap L^4(\Omega)\}.$$

Since the space $V$ is a separable Banach space, it has a basis, say, $(e_i)_{1 \leq i \leq m}$, $i \in \mathbb{N}^*$; $(e_i, e_j) = 0$, $\forall i \neq j$ and $\| e_j \| = 1$, $\forall j$, where functions $(e_j)$ are such that $(e_j) \in V$, $\forall j$ with the homogeneous Neumann condition, the operator $-\Delta$ has a sequence of eigenvalues $\{\lambda_i\}_{i \geq 0}$ whose associated functions $(e_j)$ are its own associated functions.

The problem is therefore to find in any subspace $V_m = \{e_1, e_2, ..., e_m\}$ of $V$ an approximate solution $u_m = u_m(t)$ in the form

$$u_m(t) = \sum_{i=1}^{m} u_{im}(t) e_i,$$

where $u_{im}$ is determined by the following conditions:

$$\begin{cases} \dfrac{\partial^2 u_m}{\partial t^2} - \Delta u_m + u_m^3 + \dfrac{\partial u_m}{\partial t} = f_m = p_m(f), & \\ \dfrac{\partial u_m}{\partial \vec{n}}|_{\partial\Omega} = 0, & t \in (0, T), \\ u_m|_{t=0} = u_{0m}(x), & x \in \Omega, \\ \dfrac{\partial u_m}{\partial t}|_{t=0} = u_{1m}(x), & x \in \Omega. \end{cases} \tag{2.25}$$

By putting $g(u_m(t)) = u_m^3$, for $e_j \in V$, we obtain

$$\left(\frac{\partial^2 u_m}{\partial t^2}, e_j\right) - (\Delta u_m, e_j) + (g(u_m(t)), e_j) + \left(\frac{\partial u_m}{\partial t}, e_j\right) = (f(t), e_j). \tag{2.26}$$

Replacing $u_m(t) = \sum_{i=1}^{m} u_{im}(t)e_i$ in equation (2.25), we obtain

$$\left(\frac{\partial^2 u_m}{\partial t^2}\sum_{i=1}^{m} u_{im}(t)e_i, e_j\right) - \left(\Delta u_m \sum_{i=1}^{m} u_{im}(t)e_i, e_j\right)$$

$$+ (g(u_m(t)), e_j) + \left(\frac{\partial u_m}{\partial t}\sum_{i=1}^{m} u_{im}(t)e_i, e_j\right) = (f(t), e_j),$$

$$\sum_{i=1}^{m} \frac{\partial^2}{\partial t^2}(u_{im}(t)e_i, e_j) + \sum_{i=1}^{m} u_{im}(t)(-\Delta e_i, e_j) + (g(u_m(t)), e_j)$$

$$+ \sum_{i=1}^{m} \frac{\partial}{\partial t}(u_{im}(t)e_i, e_j) = (f(t), e_j),$$

$$\sum_{i=1}^{m} (e_i, e_j)\frac{\partial^2}{\partial t^2} u_{im}(t) + \sum_{i=1}^{m} u_{im}(t)(\lambda_i e_i, e_j) + (g(u_m(t)), e_j)$$

$$+ \sum_{i=1}^{m} (e_i, e_j)\frac{\partial}{\partial t} u_{im}(t) = (f(t), e_j),$$

$$\sum_{i=1}^{m}(e_i, e_j)\frac{\partial^2}{\partial t^2}u_{im}(t) + \sum_{i=1}^{m}(e_i, e_j)\lambda_i u_{im}(t) + (g(u_m(t)), e_j)$$

$$+ \sum_{i=1}^{m}(e_i, e_j)\frac{\partial}{\partial t}u_{im}(t) = (f(t), e_j).$$

The base $(e_i)_{1\leq i\leq m}$ being orthonormal,

$$(e_i, e_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Therefore, we have

$$\frac{\partial^2}{\partial t^2}u_{im}(t) + \lambda_i u_{im}(t) + (g(u_m(t), e_j) + \frac{\partial}{\partial t}u_{im}(t) = (f(t), e_j).$$

Put $\frac{\partial^2}{\partial t^2}u_{im}(t) = u''_{im}$ and $\frac{\partial}{\partial t}u_{im}(t) = u'_{im}$. Then

$$u''_{im} + \lambda_i u_{im}(t) + (g(u_m(t), e_j) + u'_{im} = (f(t), e_j)$$

$$\Rightarrow u''_{im} + \lambda_i u_{im}(t) + (g(u_m(t), e_j) + u'_{im} - (f(t), e_j) = 0$$

$$\Rightarrow u''_{im} + \lambda_i u_{im}(t) + u'_{im} + (g(u_m(t) - f(t), e_j) = 0. \tag{2.27}$$

Equation (2.26) is a system of non-linear differential equations with the following initial conditions:

$$u_m(0) = u_{0m},\ u_{0m} = \sum_{i=1}^{m}\alpha_{im}e_i \underset{m\to\infty}{\to} u_0 \quad \text{in} \quad H^1(\Omega)\cap L^4(\Omega), \tag{2.28}$$

$$u'_m(0) = u_{1m},\ u_{1m} = \sum_{i=1}^{m}\beta_{im}e_i \underset{m\to\infty}{\to} u_1 \quad \text{in} \quad L^2(\Omega). \tag{2.29}$$

Equation (2.26) is a system of non-linear differential equations written in the following matrix form:

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}\begin{pmatrix} u''_{1m} \\ u''_{2m} \\ \vdots \\ u''_{mm} \end{pmatrix} + \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}\begin{pmatrix} u'_{1m} \\ u'_{2m} \\ \vdots \\ u'_{mm} \end{pmatrix}$$

$$+ \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_m \end{pmatrix}\begin{pmatrix} u_{1m} \\ u_{2m} \\ \vdots \\ u_{mm} \end{pmatrix} + \begin{pmatrix} (g(u_m(t) - f(t), e_1)) \\ (g(u_m(t) - f(t), e_2)) \\ \vdots \\ (g(u_m(t) - f(t), e_2)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$I_m X''(t) + I_m X'(t) + A_m X(t) + B_m = 0,$$

where

$$I_m = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \quad A_m = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_m \end{pmatrix},$$

$$B_m = \begin{pmatrix} (g(u_m(t) - f(t), e_1) \\ (g(u_m(t) - f(t), e_2) \\ \vdots \\ (g(u_m(t) - f(t), e_2) \end{pmatrix}, \quad X(t) = \begin{pmatrix} u_{1m} \\ u_{2m} \\ \vdots \\ u_{mm} \end{pmatrix},$$

$$X''(t) = \begin{pmatrix} u''_{1m} \\ u''_{2m} \\ \vdots \\ u''_{mm} \end{pmatrix} \quad \text{and} \quad X'(t) = \begin{pmatrix} u'_{1m} \\ u'_{2m} \\ \vdots \\ u'_{mm} \end{pmatrix}.$$

As $\det A_m = 1 \neq 0,$ the matrix $A_m$ is invertible. Therefore, the system has a unique solution defined in the interval $[0, t_m]$.

**Step 2.** A priori estimates of these solutions

Multiplying equation (2.30) with index *j* by $u'_{jm}(t)$ and summing up in *j*, we have

$$\left(\frac{\partial^2 u_m}{\partial t^2},\, u'_{jm}(t)\right) - (\Delta u_m,\, u'_{jm}(t)) + (g(u_m(t)),\, u'_{jm}(t)) + \left(\frac{\partial u_m}{\partial t},\, u'_{jm}(t)\right)$$

$$= (f(t),\, u'_{jm}(t)),$$

$$\left(\frac{\partial^2 u_m}{\partial t^2},\, \frac{\partial u_m}{\partial t}\right) - \left(\Delta u_m,\, \frac{\partial u_m}{\partial t}\right) + \left(u_m^2,\, \frac{\partial u_m}{\partial t}\right) + \left(\frac{\partial u_m}{\partial t},\, \frac{\partial u_m}{\partial t}\right)$$

$$= \left(f(t),\, \frac{\partial u_m}{\partial t}\right). \tag{2.30}$$

According to Green's formula and using Hölder and Young's inequalities, we obtain

$$\frac{1}{2}\left[\frac{\partial}{\partial t}\left\| \frac{\partial u_m}{\partial t} \right\|^2_{L^2(\Omega)} + \frac{\partial}{\partial t}\| \nabla u_m \|^2_{L^2(\Omega)} + \frac{1}{2}\frac{\partial}{\partial t}\| u_m \|^4_{L^4(\Omega)}\right] + \left\| \frac{\partial u_m}{\partial t} \right\|^2_{L^2(\Omega)}$$

$$\leq \frac{1}{2}\left[\| f_m \|^2_{L^2(\Omega)} + \left\| \frac{\partial u_m}{\partial t} \right\|^2_{L^2(\Omega)}\right].$$

Then, we have

$$\frac{\partial}{\partial t}\left[\left\| \frac{\partial u_m}{\partial t} \right\|^2_{L^2(\Omega)} + \| \nabla u_m \|^2_{L^2(\Omega)} + \frac{1}{2}\| u_m \|^4_{L^4(\Omega)}\right] + 2\left\| \frac{\partial u_m}{\partial t} \right\|^2_{L^2(\Omega)}$$

$$\leq \| p_m(f) \|^2_{L^2(\Omega)} + \left\| \frac{\partial u_m}{\partial t} \right\|^2_{L^2(\Omega)}$$

and

$$\frac{\partial}{\partial t}\left[\left\| \frac{\partial u_m}{\partial t} \right\|^2_{L^2(\Omega)} + \| \nabla u_m \|^2_{L^2(\Omega)} + \frac{1}{2}\| u_m \|^4_{L^4(\Omega)}\right] + 2\left\| \frac{\partial u_m}{\partial t} \right\|^2_{L^2(\Omega)}$$

$$\leq \| f \|^2_{L^2(\Omega)} + \left\| \frac{\partial u_m}{\partial t} \right\|^2_{L^2(\Omega)}$$

because $\| p_m(f) \|^2_{L^2(\Omega)} \leq \| f \|^2_{L^2(\Omega)}$.

Integrating on $(0, T)$, we have

$$\int_0^t \frac{\partial}{\partial s}\left[\left\| \frac{\partial u_m}{\partial t} \right\|_{L^2(\Omega)}^2 + \| \nabla u_m \|_{L^2(\Omega)}^2 + \frac{1}{2} \| u_m \|_{L^4(\Omega)}^4 \right] ds$$

$$+ 2\int_0^t \left\| \frac{\partial u_m}{\partial t} \right\|_{L^2(\Omega)}^2 ds$$

$$\leq \int_0^t \left( \| f \|_{L^2(\Omega)}^2 + \left\| \frac{\partial u_m}{\partial t} \right\|_{L^2(\Omega)}^2 \right) ds.$$

Furthermore, we have

$$\left\| \frac{\partial u_m}{\partial t} \right\|_{L^2(\Omega)}^2 + \| \nabla u_m \|_{L^2(\Omega)}^2 + \frac{1}{2} \| u_m \|_{L^4(\Omega)}^4 + 2\int_0^t \left\| \frac{\partial u_m}{\partial t} \right\|_{L^2(\Omega)}^2 ds$$

$$- \left( \left\| \frac{\partial u_{0m}}{\partial t} \right\|_{L^2(\Omega)}^2 + \| \nabla u_{0m} \|_{L^2(\Omega)}^2 + \frac{1}{2} \| u_{0m} \|_{L^4(\Omega)}^4 \right)$$

$$\leq \int_0^t \left( \| f \|_{L^2(\Omega)}^2 + \left\| \frac{\partial u_m}{\partial t} \right\|_{L^2(\Omega)}^2 \right) ds$$

with $\frac{\partial u_{0m}}{\partial t} = u_{1m}$, and

$$\left\| \frac{\partial u_m}{\partial t} \right\|_{L^2(\Omega)}^2 + \| \nabla u_m \|_{L^2(\Omega)}^2 + \frac{1}{2} \| u_m \|_{L^4(\Omega)}^4 + 2\int_0^t \left\| \frac{\partial u_m}{\partial t} \right\|_{L^2(\Omega)}^2 ds$$

$$- \left( \| u_{1m} \|_{L^2(\Omega)}^2 + \| \nabla u_{0m} \|_{L^2(\Omega)}^2 + \frac{1}{2} \| u_{0m} \|_{L^4(\Omega)}^4 \right)$$

$$\leq \int_0^t \left( \| f \|_{L^2(\Omega)}^2 + \left\| \frac{\partial u_m}{\partial t} \right\|_{L^2(\Omega)}^2 \right) ds$$

$$\leq \| u_{1m} \|_{L^2(\Omega)}^2 + \| \nabla u_{0m} \|_{L^2(\Omega)}^2 + \frac{1}{2} \| u_{0m} \|_{L^4(\Omega)}^4$$

$$+ \int_0^t \left( \| f \|_{L^2(\Omega)}^2 + \left\| \frac{\partial u_m}{\partial t} \right\|_{L^2(\Omega)}^2 \right) ds. \tag{2.31}$$

Adding $\| u_m \|^2_{L^2(\Omega)}$, member to member of the inequality (2.30), we obtain

$$\left\| \frac{\partial u_m}{\partial t} \right\|^2_{L^2(\Omega)} + \| u_m \|^2_{L^2(\Omega)} + \| \nabla u_m \|^2_{L^2(\Omega)} + \frac{1}{2} \| u \|^4_{L^4(\Omega)} + 2\int_0^t \left\| \frac{\partial u_m}{\partial t} \right\|^2_{L^2(\Omega)} ds$$

$$\leq \| u_m \|^2_{L^2(\Omega)} + \| u_{1m} \|^2_{L^2(\Omega)} + \| \nabla u_{0m} \|^2_{L^2(\Omega)} + \frac{1}{2} \| u_{0m} \|^4_{L^4(\Omega)}$$

$$+ \int_0^t \left( \| f \|^2_{L^2(\Omega)} + \left\| \frac{\partial u_m}{\partial t} \right\|^2_{L^2(\Omega)} \right) ds.$$

This gives us

$$\left\| \frac{\partial u_m}{\partial t} \right\|^2_{L^2(\Omega)} + \| u_m \|^2_{H^1(\Omega)} + \frac{1}{2} \| u_m \|^4_{L^4(\Omega)} + 2\int_0^t \left\| \frac{\partial u_m}{\partial t} \right\|^2_{L^2(\Omega)} ds$$

$$\leq \| u_m \|^2_{L^2(\Omega)} + \| u_{1m} \|^2_{L^2(\Omega)} + \| \nabla u_{0m} \|^2_{L^2(\Omega)} + \frac{1}{2} \| u_{0m} \|^4_{L^4(\Omega)}$$

$$+ \int_0^t \left( \| f \|^2_{L^2(\Omega)} + \left\| \frac{\partial u_m}{\partial t} \right\|^2_{L^2(\Omega)} \right) ds. \qquad (2.32)$$

For all $t \in [0, T]$, we have

$$\left\| \frac{\partial u_m}{\partial t} \right\|^2_{L^2(\Omega)} + \| u_m \|^2_{H^1(\Omega)} + \frac{1}{2} \| u_m \|^4_{L^4(\Omega)} + 2\int_0^T \left\| \frac{\partial u_m}{\partial t} \right\|^2_{L^2(\Omega)} ds$$

$$\leq \| u_m \|^2_{L^2(\Omega)} + \| u_{1m} \|^2_{L^2(\Omega)} + \| \nabla u_{0m} \|^2_{L^2(\Omega)} + \frac{2}{3} \| u_{0m} \|^3_{L^3(\Omega)}$$

$$+ \int_0^T \left( \| f \|^2_{L^2(\Omega)} + \left\| \frac{\partial um}{\partial t} \right\|^2_{L^2(\Omega)} \right) ds. \qquad (2.33)$$

Putting $u_m(t) = u_m(x, t)$,

$$u_m(t) = \int_0^t \frac{\partial u_m}{\partial s}(s) ds + u_m(0) = \int_0^t \frac{\partial u_m}{\partial s}(s) ds + u_{0m},$$

because $u_m(0) = u_{0m}$ from (2.27). Taking modulus, we have

$$|u_m(t)| = \left| \int_0^t \frac{\partial u_m}{\partial s}(s)ds + u_{0m} \right| \Leftrightarrow |u_m(t)|^2 = \left| \int_0^t \frac{\partial u_m}{\partial s}(s)ds + u_{0m} \right|^2,$$

$$|u_m(t)|^2 \le \left| \int_0^t \frac{\partial u_m}{\partial s}(s)ds + u_{0m} \right|^2 + \left| \int_0^t \frac{\partial u_m}{\partial s}(s)ds - u_{0m} \right|^2$$

$$\le 2\left[ \left| \int_0^t \frac{\partial u_m}{\partial s}(s)ds \right|^2 + |u_{0m}^2| \right]$$

$$\le 2\left[ \left( \int_0^t \left| \frac{\partial u_m}{\partial s}(s) \right| ds \right)^2 + |u_{0m}|^2 \right].$$

From Hölder’s inequality, we have

$$\int_0^t \left| \frac{\partial u_m}{\partial s}(x, t) \right| ds \le \left( \int_0^t |1| ds \right)^{1/2} \left( \int_0^t \left| \frac{\partial u_m}{\partial s}(s) \right|^2 ds \right)^{1/2},$$

$$\left( \int_0^t \left| \frac{\partial u_m}{\partial s}(x, t) \right| ds \right)^2 \le \left( \int_0^t ds \right) \cdot \left( \int_0^t \left| \frac{\partial u_m}{\partial s}(s) \right|^2 ds \right) \le t \int_0^t \left| \frac{\partial u_m}{\partial s}(s) \right|^2 ds.$$

Thus

$$|u_m(t)|^2 \le 2\left[ t \int_0^t \left| \frac{\partial u_m}{\partial s}(s) \right|^2 ds \right] + |u_{0m}|^2$$

$$\le 2t \int_0^t \left| \frac{\partial u_m}{\partial s}(s) \right|^2 ds + 2|u_{0m}|^2.$$

Let $t \in [0, 1] \subset [0, T]$. Then $1 - t \ge 0$. Adding the term $2(1 - t) \int_0^t \left| \frac{\partial u_m}{\partial s}(s) \right|^2 ds$ to the second member, we have

$$|u_m(t)|^2 \le 2t \int_0^t \left| \frac{\partial u_m}{\partial s}(s) \right|^2 ds + 2|u_{0m}|^2 + 2(1 - t) \int_0^t \left| \frac{\partial u_m}{\partial s}(s) \right|^2 ds.$$

Finally, we obtain

$$|u_m(t)|^2 \leq 2\left[|u_{0m}|^2 + \int_0^t \left|\frac{\partial u_m}{\partial s}(s)\right|^2 ds\right].$$

Integrating on $\Omega$, for $t \in [0, T]$, we have

$$\int_\Omega |u_m(t)|^2 dx \leq 2\int_\Omega \left[|u_{0m}|^2 + \int_0^t \left|\frac{\partial u_m}{\partial s}(s)\right|^2 ds\right] dx$$

$$\leq 2\left[\int_\Omega |u_{0m}|^2 dx + \int_0^t \left(\int_\Omega \left|\frac{\partial u_m}{\partial s}\right|^2 dx\right) ds\right]$$

$$\leq 2\left[\int_\Omega |u_{0m}|^2 dx + \int_0^T \left(\int_\Omega \left|\frac{\partial u_m}{\partial t}\right|^2 dx\right) dt\right],$$

$$\| u_m(t) \|^2_{L^2(\Omega)} \leq 2\left[\| u_{0m} \|^2_{L^2(\Omega)} + \int_0^T \left\| \frac{\partial u_m}{\partial t} \right\|^2_{L^2(\Omega)} dt\right]. \tag{2.34}$$

By using the (2.33), (2.32) becomes

$$\left\| \frac{\partial u_m}{\partial t} \right\|^2_{L^2(\Omega)} + \| u_m \|^2_{H^1(\Omega)} + \frac{1}{2}\| u_m \|^4_{L^4(\Omega)} + 2\int_0^T \left\| \frac{\partial u_m}{\partial t} \right\|^2_{L^2(\Omega)} dt$$

$$\leq 2\| u_{0m} \|^2_{L^2(\Omega)} + 3\int_0^T \left\| \frac{\partial u_m}{\partial t} \right\|^2_{L^2(\Omega)} dt + \| u_{1m} \|^2_{L^2(\Omega)}$$

$$+ \| \nabla u_{0m} \|^2_{L^2(\Omega)} + \frac{1}{2}\| u_{0m} \|^4_{L^4(\Omega)} + \int_0^T \| f \|^2_{L^2(\Omega)} dt. \tag{2.35}$$

By adding the term $\| \nabla u_{0m} \|^2_{L^2(\Omega)}$ to the second member of (2.34), we obtain

$$\left\| \frac{\partial u_m}{\partial t} \right\|^2_{L^2(\Omega)} + \| u_m \|^2_{H^1(\Omega)} + \frac{1}{2} \| u_m \|^4_{L^4(\Omega)}$$

$$\leq 2\| u_{0m} \|^2_{H^1(\Omega)} + \int_0^T \left\| \frac{\partial u_m}{\partial t} \right\|^2_{L^2(\Omega)} dt$$

$$+ \| u_{1m} \|^2_{L^2(\Omega)} + \frac{1}{2} \| u_{0m} \|^4_{L^4(\Omega)} + \int_0^T \| f \|^2_{L^2(\Omega)} dt. \tag{2.36}$$

By adding the term $\int_0^T \| u_m \|^2_{H^1(\Omega)} dt$ to the second member of (2.35), we obtain

$$\left\| \frac{\partial u_m}{\partial t} \right\|^2_{L^2(\Omega)} + \| u_m \|^2_{H^1(\Omega)} + \frac{1}{2} \| u_m \|^4_{L^4(\Omega)}$$

$$\leq 2\| u_{0m} \|^2_{H^1(\Omega)} + \int_0^T \left\| \frac{\partial u_m}{\partial t} \right\|^2_{L^2(\Omega)} dt + \| u_{1m} \|^2_{L^2(\Omega)}$$

$$+ \frac{1}{2} \| u_{0m} \|^4_{L^4(\Omega)} + \int_0^T \| f \|^2_{L^2(\Omega)} dt + \int_0^T \| u_m \|^2_{H^1(\Omega)} dt. \tag{2.37}$$

Putting

$$k_3 = 2\| u_{0m} \|^2_{H^1(\Omega)} + \| u_{1m} \|^2_{L^2(\Omega)} + \frac{1}{2} \| u_{0m} \|^4_{L^4(\Omega)} + \int_0^T \| f \|^2_{L^2(\Omega)} dt,$$

(2.36) becomes

$$\left\| \frac{\partial u_m}{\partial t} \right\|^2_{L^2(\Omega)} + \| u_m \|^2_{H^1(\Omega)} + \frac{1}{2} \| u_m \|^4_{L^4(\Omega)}$$

$$\leq k_3 + \int_0^T \left\| \frac{\partial u_m}{\partial t} \right\|^2_{L^2(\Omega)} dt + \int_0^T \| u_m \|^2_{H^1(\Omega)} dt,$$

$$\left\| \frac{\partial u_m}{\partial t} \right\|_{L^2(\Omega)}^2 + \| u_m \|_{H^1(\Omega)}^2$$

$$\leq k_3 + \int_0^T \left\| \frac{\partial u_m}{\partial t} \right\|_{L^2(\Omega)}^2 dt + \int_0^T \| u_m \|_{H^1(\Omega)}^2 dt. \tag{2.38}$$

Setting $I_m(t) = \left\| \frac{\partial u_m}{\partial t} \right\|_{L^2(\Omega)}^2 + \| u_m \|_{H^1(\Omega)}^2$, we have

$$I_m(t) \leq k_3 + \int_0^T I_m(s) ds.$$

According to Gronwall's inequality, $I_m(t) \leq k_3 e^{\int_0^T ds}$, and hence $I_m(t) \leq k_4$ with $k_4 = k_3 e^T$.

Then we obtain

$$\left\| \frac{\partial u_m}{\partial t} \right\|_{L^2(\Omega)}^2 + \| u_m \|_{H^1(\Omega)}^2 \leq k_4, \tag{2.39}$$

$$\begin{cases} \left\| \frac{\partial u_m}{\partial t} \right\|_{L^2(\Omega)}^2 \leq c_4 \\ \| u_m \|_{H^1(\Omega)}^2 \leq c_5 \end{cases} \text{ with } k_4 = c_4 + c_5, \tag{2.40}$$

which implies

$$\begin{cases} \frac{\partial u_m}{\partial t} \in L^\infty(0, T; L^2(\Omega)), \\ u_m \in L^\infty(0, T; H^1(\Omega)). \end{cases} \tag{2.41}$$

Adding the term $\int_0^T \| u_m \|_{L^4(\Omega)}^4 dt$ to the second member of (2.35), we obtain

$$\left\| \frac{\partial u_m}{\partial t} \right\|^2_{L^2(\Omega)} + \| u_m \|^2_{H^1(\Omega)} + \frac{1}{2} \| u_m \|^4_{L^4(\Omega)}$$

$$\leq 2 \| u_{0m} \|^2_{H^1(\Omega)} + \int_0^T \left\| \frac{\partial u_m}{\partial t} \right\|^2_{L^2(\Omega)} dt + \| u_{1m} \|^2_{L^2(\Omega)}$$

$$+ \frac{1}{2} \| u_{0m} \|^4_{L^4(\Omega)} + \int_0^T \| f \|^2_{L^2(\Omega)} dt + \int_0^T \| u_m \|^4_{L^4(\Omega)} dt. \tag{2.42}$$

Putting

$$k_5 = 2 \| u_{0m} \|^2_{H^1(\Omega)} + \int_0^T \left\| \frac{\partial u_m}{\partial t} \right\|^2_{L^2(\Omega)} dt + \| u_{1m} \|^2_{L^2(\Omega)}$$

$$+ \frac{1}{2} \| u_{0m} \|^4_{L^4(\Omega)} + \int_0^T \| f \|^2_{L^2(\Omega)} dt,$$

(2.41) becomes

$$\left\| \frac{\partial u_m}{\partial t} \right\|^2_{L^2(\Omega)} + \| u_m \|^2_{H^1(\Omega)} + \frac{1}{2} \| u_m \|^4_{L^4(\Omega)} \leq k_5 + \int_0^T \| u_m \|^4_{L^4(\Omega)} dt,$$

$$\frac{1}{2} \| u_m \|^4_{L^4(\Omega)} \leq \left\| \frac{\partial u_m}{\partial t} \right\|^2_{L^2(\Omega)} + \| u_m \|^2_{H^1(\Omega)} + \frac{1}{2} \| u_m \|^4_{L^4(\Omega)}$$

$$\leq k_5 + \int_0^T \| u_m \|^4_{L^4(\Omega)} dt$$

$$\Rightarrow \frac{1}{2} \| u_m \|^4_{L^4(\Omega)} \leq k_5 + \int_0^T \| u_m \|^4_{L^4(\Omega)} dt$$

$$\Rightarrow \| u_m \|^4_{L^4(\Omega)} \leq 2k_5 + 2 \int_0^T \| u_m \|^4_{L^4(\Omega)} dt$$

$$\Rightarrow \| u_m \|^4_{L^4(\Omega)} \leq k_6 + 2 \int_0^T \| u_m \|^4_{L^4(\Omega)} dt. \tag{2.43}$$

According to Gronwall's inequality, we have

$$\| u_m \|^4_{L^4(\Omega)} \leq k_6 e^{2\int_0^T dt},$$

or $\| u_m \|^4_{L^4(\Omega)} \leq c_6$ with $c_6 = k_6 e^{2T}$.

Noting that $\inf\limits_{t \in [0, T]} \| u_m \|^4_{L^4(\Omega)} \leq c_6$, it follows that

$$u_m \in L^\infty(0, T, L^4(\Omega)). \tag{2.44}$$

According to (2.40) and (2.43), we have

$$\begin{cases} \dfrac{\partial u_m}{\partial t} \in L^\infty(0, T; L^2(\Omega)), \\ u_m \in L^\infty(0, T; H^1(\Omega) \cap L^4(\Omega)). \end{cases} \tag{2.45}$$

When $m \to \infty$, $u_m$ is still a bounded set of $L^\infty(0, T; H^1(\Omega) \cap L^4(\Omega))$ and $\dfrac{\partial u_m}{\partial t}$ of $L^\infty(0, T; L^2(\Omega))$.

**Step 3.** Transition to the boundary

The sequence $(U_n)$ is bounded in $L^\infty(0, T; H^1(\Omega) \cap L^4(\Omega))$, so it is bounded in $L^2(0, T; H^1(\Omega) \cap L^4(\Omega))$. Since $L^\infty(0, T; H^1(\Omega) \cap L^4(\Omega))$ is a separable Banach space, there is a subsequence $(u_\xi)$ extracted from $(U_n)$ such that

$$\begin{cases} u_\xi \overset{*}{\to} u & \text{in } L^\infty(0, T; H^1(\Omega) \cap L^4(\Omega)) \\ \dfrac{\partial u_\xi}{\partial t} \overset{*}{\to} \dfrac{\partial u}{\partial t} & \text{in } L^\infty(0, T; L^2(\Omega)). \end{cases} \tag{2.46}$$

So

$$\begin{cases} u_\xi \to u & \text{in } L^\infty(0, T; H^1(\Omega) \cap L^4(\Omega)) \\ \dfrac{\partial u_\xi}{\partial t} \to \dfrac{\partial u}{\partial t} & \text{in } L^\infty(0, T; L^2(\Omega)). \end{cases} \tag{2.47}$$

From the problem (1.1)-(1.3),

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + u^3 + \frac{\partial u}{\partial t} = f$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = \Delta u - u^3 - \frac{\partial u}{\partial t} + f.$$

As

$$\Delta : H^1(\Omega) \to H^{-1}(\Omega) \Rightarrow \Delta \in \mathcal{L}(H^1(\Omega), H^{-1}(\Omega)),$$

we have

$$u \in L^\infty(0, T; H^1(\Omega)) \Rightarrow \Delta u \in L^\infty(0, T; H^1(\Omega)).$$

Also,

$$u^3 \in L^\infty(0, T; L^{\frac{4}{3}}(\Omega)),$$

which implies

$$\frac{\partial^2 u}{\partial t^2} \in L^\infty(0, T; H^1(\Omega)) \cup L^\infty(0, T; L^2(\Omega))$$

$$\cup\, L^\infty(0, T; L^{\frac{4}{3}}(\Omega)) \cup L^2(0, T; L^2(\Omega)),$$

$$\frac{\partial^2 u}{\partial t^2} \in L^\infty(0, T; H^{-1}(\Omega) \cup L^2(\Omega) \cup L^{\frac{4}{3}}(\Omega)) \cup L^2(0, T; L^2(\Omega)).$$

In particular,

$$\frac{\partial^2 u}{\partial t^2} \in L^2(0, T; H^{-1}(\Omega) \cup L^{\frac{4}{3}}(\Omega)).$$

As $u_m^3$ is bounded in $L^\infty(0, T; L^{\frac{4}{3}}(\Omega))$, $u_\xi^3 \overset{*}{\to} w$ in $L^\infty(0, T; L^{\frac{4}{3}}(\Omega))$.

### 2.3.1. Lemma

*Let* $\Omega$ *be a bounded open set of* $\mathbb{R}_x^n$, $h_\xi$ *and* $h$ *be two functions of* $L^q(\Omega)$, $1 < q < \infty$, *such that*

$$\| h_\xi \|_{L^q(\Omega)} \leq c, \quad h_\xi \to h \ p.p \ in \ \Omega.$$

*Then* $h_\xi \to h$ *in* $L^q$ *weak.*

We will apply this lemma with

$$h_\xi = u_\xi, \quad q = \frac{4}{3}.$$

This means that

$$h_\xi \to w \text{ in } L^\infty(0, T; L^{\frac{4}{3}}(\Omega)).$$

Therefore, according to Lemma 2.3.1, $w = h = u^3$. Hence

$$g(u_\xi) = u_\xi^3 \to g(u) = u^3.$$

On the other hand, we have $f_m = p_m(f)$, and

$$\| f_m \|_{L^2(\Omega)} = \| p_m(f) \|_{L^2(\Omega)} \leq \| f \|_{L^2(\Omega)}, \ f_m \text{ is bounded in } L^2(Q_T).$$

Extract a subsequence $f_\xi$ from $f_m$ such that $f_\xi \to f$ in $L^2(Q_T)$. Now, we show that $u$ satisfies all the conditions of (2.24). First, note that for a fixed $j$, we have

$$\left(\frac{\partial^2 u_\xi}{\partial t^2}, e_j\right) - (\Delta u_\xi, e_j) + (g(u_\xi(t)), e_j) + \left(\frac{\partial u_\xi}{\partial t}, e_j\right) = (f_\xi, e_j). \quad (2.48)$$

Passing to the limit, (2.47) becomes

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}, e_j\right) - (\Delta u, e_j) + \left(\frac{\partial u}{\partial t}, e_j\right) + (g(u), e_j) = (f, e_j).$$

Since $V_m$ is dense in $V$, for $v \in V$, $e_j \to v$ when $j \to \infty$, we obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}, v\right) - (\Delta u, v) + \left(\frac{\partial u}{\partial t}, v\right) + (g(u), v) = (f, v),$$

$$\left(\frac{\partial^2 u}{\partial t^2}, v\right) - (\Delta u, v) + \left(\frac{\partial u}{\partial t}, v\right) + (g(u), v) = (f, v),$$

$$\left(\frac{\partial^2 u}{\partial t^2} - \Delta u + \frac{\partial u}{\partial t} + g(u), v\right) = (f, v).$$

Hence

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + \frac{\partial u}{\partial t} + g(u) = f.$$

Now, we check that $\frac{\partial u}{\partial t}(x, 0) = u_1(x)$ and $u(x, 0) = u_0(x)$.

Let $\theta(x, t) \in L^\infty(0, T; H^1(\Omega) \cap L^4(\Omega))$, such that $\theta(x, T) = 0$ and $\theta(x, 0) \neq 0$.

As $(u_{0m}(x))_{m \geq 1}$ is bounded in $H^1(\Omega) \cap L^4(\Omega)$, we can extract a subsequence $(u_{0\xi}(x))_{\xi \geq 1}$ of $(u_{0m}(x))_{m \geq 1}$ such that $u_{0\xi}(x) \to u_0(x)$ in $H^1(\Omega) \cap L^4(\Omega)$. In the same way, $(u_{1m}(x))_{m \geq 1}$ is bounded in $L^2(\Omega)$, and thus we can extract a subsequence $(u_{1\xi}(x))_{\xi \geq 1}$ of $(u_{1m}(x))_{m \geq 1}$ such that $u_{1\xi}(x) \to u_1(x)$ in $L^2(\Omega)$.

Multiplying the first equation in (2.24) by $\theta(x, t)$ and integrating on $(0, T)$, we have

$$\int_0^T \frac{\partial^2 u_m}{\partial t^2}\theta(x, t)dt - \int_0^T \Delta u_m \theta(x, t)dt$$

$$+\int_0^T \frac{\partial u_m}{\partial t}\theta(x, t)dt + \int_0^T g(u_m)\theta(x, t)dt$$

$$= \int_0^T f_m \theta(x, t)dt.$$

Using integration by parts, we obtain

$$\left[\frac{\partial u_m}{\partial t}\theta(x, t)\right]_0^T - \int_0^T \frac{\partial u_m}{\partial t}\theta'(x, t)dt - \int_0^T \Delta u_m \theta(x, t)dt$$

$$+\int_0^T g(u_m)\theta(x, t)dt + \int_0^T \frac{\partial u_m}{\partial t}\theta(x, t)dt$$

$$= \int_0^T f_m \theta(x, t)dt,$$

$$\frac{\partial u_m}{\partial t}\theta(x, T) - \frac{\partial u_m}{\partial t}\theta(x, 0) - \int_0^T \frac{\partial u_m}{\partial t}\theta'(x, t)dt - \int_0^T \Delta u_m \theta(x, t)dt$$

$$+\int_0^T g(u_m)\theta(x, t)dt + \int_0^T \frac{\partial u_m}{\partial t}\theta(x, t)dt$$

$$= \int_0^T f_m \theta(x, t)dt,$$

$$-\int_0^T \frac{\partial u_m}{\partial t}\theta'(x, t)dt - \int_0^T \Delta u_m \theta(x, t)dt$$

$$+\int_0^T g(u_m)\theta(x, t)dt + \int_0^T \frac{\partial u_m}{\partial t}\theta(x, t)dt$$

$$= \int_0^T f_m \theta(x, t)dt + \frac{\partial u_m}{\partial t}\theta(x, 0),$$

$m$ being fixed. For $m = \xi$, we have

$$-\int_0^T \frac{\partial u_\xi}{\partial t}\theta'(x, t)dt - \int_0^T \Delta u_\xi \theta(x, t)dt$$

$$+\int_0^T g(u_\xi)\theta(x, t)dt + \int_0^T \frac{\partial u_\xi}{\partial t}\theta(x, t)dt$$

$$= \int_0^T f_\xi \theta(x, t)dt + \frac{\partial u_\xi}{\partial t}\theta(x, 0).$$

Passing to the limit, we have

$$-\int_0^T \frac{\partial u}{\partial t}\theta'(x, t)dt - \int_0^T \Delta u \theta(x, t)dt$$

$$+\int_0^T g(u)\theta(x, t)dt + \int_0^T \frac{\partial u}{\partial t}\theta(x, t)dt$$

$$= \int_0^T f.\theta(x, t)dt + \frac{\partial u}{\partial t}\theta(x, 0). \tag{2.49}$$

On the other hand, considering the equation:

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + \frac{\partial u}{\partial t} + g(u) = f,$$

and using the same calculations, we obtain

$$-\int_0^T \frac{\partial u}{\partial t}\theta'(x, t)dt - \int_0^T \Delta u \theta(x, t)dt$$

$$+\int_0^T g(u)\theta(x, t)dt + \int_0^T \frac{\partial u}{\partial t}\theta(x, t)dt$$

$$= \int_0^T f.\theta(x, t)dt + u_1(x)\theta(x, 0). \tag{2.50}$$

Differentiating (2.49) and (2.50), we have

$$\frac{\partial u}{\partial t}(x, 0)\theta(x, 0) - u_1(x)\theta(x, 0) = 0,$$

$$\left(\frac{\partial u}{\partial t}(x, 0) - u_1(x)\right)\theta(x, 0) = 0,$$

$$\frac{\partial u}{\partial t}(x, 0) = u_1(x), \text{ because } \theta(x, 0) \neq 0.$$

Now, we verify that $u(x, 0) = u_0(x)$.

Multiplying the first equation in (2.24) by $\theta(x, t)$ and integrating by parts on $(0, T)$, we have

$$\int_0^T \frac{\partial^2 u_m}{\partial t^2}\theta(x, t)dt - \int_0^T \Delta u_m \theta(x, t)dt$$

$$+ \int_0^T \frac{\partial u_m}{\partial t}\theta(x, t)dt + \int_0^T g(u_m)\theta(x, t)dt$$

$$= \int_0^T f_m \theta(x, t)dt.$$

Using integration by parts, we obtain

$$\int_0^T \frac{\partial^2 u_m}{\partial t^2}\theta(x, t)dt - \int_0^T \Delta u_m \theta(x, t)dt + u_m\theta(x, T) - u_m(x, 0)\theta(x, 0)$$

$$- \int_0^T u_m \theta'(x, t)dt + \int_0^T g(u_m)\theta(x, t)dt$$

$$= \int_0^T f_m \theta(x, t)dt,$$

$$\int_0^T \frac{\partial^2 u_m}{\partial t^2}\theta(x, t)dt - \int_0^T \Delta u_m \theta(x, t)dt$$

$$- \int_0^T u_m \theta'(x, t)dt + \int_0^T g(u_m)\theta(x, t)dt$$

$$= \int_0^T f_m \theta(x, t)dt + u_m(x, 0)\theta(x, 0),$$

$m$ being fixed. For $m = \xi,$ we have

$$\int_0^T \frac{\partial^2 u_\xi}{\partial t^2}\theta(x, t)dt - \int_0^T \Delta u_\xi \theta(x, t)dt$$

$$-\int_0^T u_\xi \theta'(x, t)dt + \int_0^T g(u_\xi)\theta(x, t)dt$$

$$= \int_0^T f_\xi \theta(x, t)dt + u_\xi(x, 0)\theta(x, 0).$$

Passing to the limit, we have

$$\int_0^T \frac{\partial^2 u}{\partial t^2}\theta(x, t)dt - \int_0^T \Delta u \theta(x, t)dt$$

$$-\int_0^T u\theta'(x, t)dt + \int_0^T g(u)\theta(x, t)dt$$

$$= \int_0^T f\theta(x, t)dt + u(x, 0)\theta(x, 0). \tag{2.51}$$

On the other hand, considering the equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + \frac{\partial u}{\partial t} + g(u) = f,$$

and using the same calculations, we obtain

$$\int_0^T \frac{\partial^2 u}{\partial t^2}\theta(x, t)dt - \int_0^T \Delta u \theta(x, t)dt$$

$$-\int_0^T u.\theta'(x, t)dt + \int_0^T g(u)\theta(x, t)dt$$

$$= \int_0^T f\theta(x, t)dt + u_0(x)\theta(x, 0). \tag{2.52}$$

Differentiating (2.51) and (2.52), we obtain

$$u(x, 0)\theta(x, 0) - u_0(x)\theta(x, 0) = 0,$$

$$(u(x, 0) - u_0(x))\theta(x, 0) = 0,$$

$$u(x, 0) = u_0(x), \text{ because } \theta(x, 0) \neq 0.$$

Hence, $u$ is the solution of the problem (1.1)-(1.3).

## 3. Uniqueness of the solution

### 3.1. Theorem

*Let u and* $v \in L^\infty(0, T; H^1(\Omega) \cap L^4(\Omega))$ *be two solutions of the problem* (1.1)-(1.3) *under the assumptions of Theorem* 2.1. *Then the solution of the problem obtained in the theorem is unique.*

**Proof.** Put $w = u - v$. Then, we have

$$\begin{cases} \dfrac{\partial^2 w}{\partial t^2} - \Delta w + \dfrac{\partial u}{\partial t} + g(u) - g(v) = 0, & \\ \dfrac{\partial w}{\partial \vec{n}}|_{\partial\Omega} = 0, & t \in (0, T), \\ w_0(x) = u_0(x) - v_0(x), & x \in \Omega, \\ w_1(x) = u_1(x) - v_1(x), & x \in \Omega. \end{cases} \tag{3.1}$$

Let $s \in (0, T)$. Consider the auxiliary function defined on $\Omega \times ]0, T[$:

$$\phi(x, t) = \begin{cases} -\int_t^s w(\sigma)d\sigma & \text{if } 0 < t \leq s, \\ 0 & \text{if } s < t \leq T. \end{cases} \tag{3.2}$$

Then $\dfrac{\partial\phi(x, t)}{\partial t} = w(x, t) = w(t)$, and

$$w_1(x, t) = \int_0^t w(\sigma)d\sigma, \text{ so that } \phi(x, t) = \phi_1(x, t) - \phi_1(x, s) \text{ if } t \leq s.$$

Multiplying the first equation in (3.1) by $\phi(x, t) = \phi(t)$ and integrating on $]0, s[$, we have

$$\int_0^s \frac{\partial^2 w}{\partial t^2}\phi(t)dt - \int_0^s \Delta w\phi(t)dt$$

$$+ \int_0^s \frac{\partial w}{\partial t}\phi(t)dt + \int_0^s (g(u) - g(v))\phi(t)dt = 0,$$

$$\int_0^s \frac{\partial^2 w}{\partial t^2}\phi(t)dt - \int_0^s \Delta w\phi(t)dt + \int_0^s \frac{\partial w}{\partial t}\phi(t)dt$$

$$= \int_0^s (g(v) - g(u))\phi(t)dt. \tag{3.3}$$

Integrating by parts the first and last terms of the first member of (3.3), we obtain

$$\frac{\partial w}{\partial t}(x, s)\phi(x, s) - \frac{\partial w}{\partial t}(x, 0)\phi(x, 0) - \int_0^s \frac{\partial w}{\partial t}\frac{\partial \phi}{\partial t}dt$$

$$- \int_0^s \Delta w\phi(t)dt + w(x, s)\phi(x, s) - w(x, 0)\phi(x, 0) - \int_0^s w(t)\frac{\partial \phi(t)}{\partial t}dt$$

$$= \int_0^s (g(v) - g(u))\phi(t)dt \tag{3.4}$$

or

$$\phi(x, s) = -\int_s^s w(\sigma)d\sigma = 0,$$

$$-\frac{\partial w}{\partial t}(x, 0)\phi(x, 0) - \int_0^s \frac{\partial w}{\partial t}\frac{\partial \phi}{\partial t}dt$$

$$- \int_0^s \Delta w\phi(t)dt - w(x, 0)\phi(x, 0) - \int_0^s w(t)\frac{\partial \phi(t)}{\partial t}dt$$

$$= \int_0^s (g(v) - g(u))\phi(t)dt.$$

Integrating on $\Omega$, we obtain

$$-\int_\Omega \frac{\partial w}{\partial t}(x, 0)\phi(x, 0)dx - \int_\Omega \int_0^s \frac{\partial w}{\partial t}\frac{\partial \phi}{\partial t} dtdx - \int_\Omega \int_0^s \Delta u \phi(t) dtdx$$

$$- \int_\Omega w(x, 0)\phi(x, 0)dx - \int_\Omega \int_0^s w(t)\frac{\partial \phi(t)}{\partial t} dtdx$$

$$= \int_\Omega \int_0^s (g(v) - g(u))\phi(t) dtdx. \tag{3.5}$$

On the one hand,

$$-\int_\Omega \frac{\partial w}{\partial t}(x, 0)\phi(x, 0)dx = -\int_\Omega w_1(x)\phi(x, 0)dx$$

$$= -\int_\Omega (u_1(x) - v_1(x))\phi(x, 0)dx$$

with $\phi(x, 0) = w_1(x, 0) - w_1(x, s)$;

$$w_1(x, s) = \int_0^s u(\sigma)d\sigma \text{ and } w_1(x, 0) = \int_0^0 w(\sigma)d\sigma = 0$$

$$\Rightarrow -\int_\Omega \frac{\partial w}{\partial t}(x, 0)\phi(x, 0)dx = \int_\Omega (u_1(x) - v_1(x))w_1(x, s)dx. \tag{3.6}$$

Repeating the same for

$$-\int_\Omega w(x, 0)\phi(x, 0)dx,$$

we obtain

$$-\int_\Omega w(x, 0)\phi(x, 0)dx = \int_\Omega (u_0(x) - v_0(x))w_1(x, s)dx. \tag{3.7}$$

On the other hand,

$$\int_\Omega \int_0^s \frac{\partial w}{\partial t} \frac{\partial \phi}{\partial t} dtdx = \int_\Omega \int_0^s \frac{\partial w}{\partial t} \frac{\partial}{\partial t}\left( -\int_t^s w(x, \sigma) d\sigma \right) dtdx$$

$$\Rightarrow \int_\Omega \int_0^s \frac{\partial w}{\partial t} \frac{\partial \phi}{\partial t} dtdx = \int_\Omega \int_0^s \frac{\partial w}{\partial t}\left( -\frac{\partial}{\partial t} \int_t^s w(\sigma) d\sigma \right) dtdx$$

$$\Rightarrow \int_\Omega \int_0^s \frac{\partial w}{\partial t} \frac{\partial \phi}{\partial t} dtdx = \frac{1}{2} \int_\Omega [w^2(x, t)]_0^s dx$$

$$= \frac{1}{2} \int_\Omega [w^2(x, s) - w^2(x, 0)] dx$$

with $w(x, 0) = w_0(x) = u_0(x) - v_0(x)$

$$\Rightarrow \int_\Omega \int_0^s \frac{\partial w}{\partial t} \frac{\partial \phi}{\partial t} dtdx = \frac{1}{2} \int_\Omega [w^2(x, s) - (u_0(x) - v_0(x))^2] dx$$

$$\Rightarrow \int_\Omega \int_0^s \frac{\partial w}{\partial t} \frac{\partial \phi}{\partial t} dtdx = \frac{1}{2} \| w(x, s) \|^2_{L^2(\Omega)} - \frac{1}{2} \| u_0(x) - v_0(x) \|^2_{L^2(\Omega)}. \quad (3.8)$$

Similarly, for

$$\int_\Omega \int_0^s w(t) \frac{\partial \phi(t)}{\partial t} dtdx,$$

we obtain

$$\int_\Omega \int_0^s w(t) \frac{\partial \phi(t)}{\partial t} dtdx = \int_\Omega \int_0^s w^2 dtdx = \int_0^s \| w \|^2_{L^2(\Omega)} dt. \quad (3.9)$$

According to Green's formula,

$$\int_\Omega \int_0^s \Delta w \phi(t) dtdx = \int_\Omega \int_0^s \nabla w . \nabla \phi dtdx = \int_\Omega \int_0^s \nabla w . \nabla \left( -\int_t^s w(\sigma) d\sigma \right) dtdx$$

$$\Rightarrow \int_\Omega \int_0^s \Delta w \phi(t) dtdx = \int_\Omega \int_0^s \nabla w . \left( -\int_t^s \nabla w(\sigma) d\sigma \right) dtdx.$$

As

$$\phi(x, t) = -\int_t^s w(\sigma)d\sigma \text{ if } t \le s;$$

$$\frac{\partial\phi(x, t)}{\partial t} = \frac{\partial}{\partial t}\left(-\int_t^s w(\sigma)d\sigma\right) = w(x, t) = w(t)$$

$$\Rightarrow \nabla w(t) = \nabla\left[\frac{\partial}{\partial t}\left(-\int_t^s w(\sigma)d\sigma\right)\right] = \frac{\partial}{\partial t}\left(-\nabla\int_t^s w(\sigma)d\sigma\right)$$

$$= \frac{\partial}{\partial t}\left(-\int_t^s \nabla w(\sigma)d\sigma\right),$$

we finally have

$$\int_\Omega\int_0^s \nabla w\nabla\phi dtdx = \int_\Omega\int_0^s \frac{\partial}{\partial t}\left(-\int_t^s \nabla w(\sigma)d\sigma\right)\left(-\int_t^s \nabla w(\sigma)d\sigma\right)dtdx$$

$$= \frac{1}{2}\int_\Omega\int_0^s \frac{\partial}{\partial t}\left(\int_t^s \nabla w(\sigma)d\sigma\right)^2 dtdx$$

$$= -\frac{1}{2}\int_\Omega |\nabla w_1(x, s)|^2 dx$$

$$\Rightarrow \int_\Omega\int_0^s \nabla w\nabla\phi dtdx = -\frac{1}{2}\| \nabla w_1(x, s)\|^2_{L^2(\Omega)}. \quad (3.10)$$

Using (3.6)-(3.10), (3.5) becomes

$$\int_\Omega (u_1(x) - v_1(x))w_1(x)dx - \frac{1}{2}\| w(x, s)\|^2_{L^2(\Omega)} + \frac{1}{2}\| u_0(x) - v_0(x)\|^2_{L^2(\Omega)}$$

$$- \int_0^s \| w\|^2_{L^2(\Omega)} dt + \int_\Omega (u_0(x) - v_0(x))w_1(x)dx + \frac{1}{2}\| \nabla w(x, s)\|^2_{L^2(\Omega)}$$

$$= \int_\Omega\int_0^s (g(v) - g(u))\phi(t)dtdx. \quad (3.11)$$

We have the following relationship:

$$g(v) - g(u) = v^3 - u^3.$$

Integrating on $(0, s)$,

$$\int_0^s g(v) - g(u)dt = \int_0^s (v^3 - u^3)dt,$$

$$\left| \int_0^s (g(v) - g(u))\phi(t)dt \right| = \left| \int_0^s (v^3 - u^3)\phi(t)dt \right| \leq \int_0^s |(v^3 - u^3)|| \phi(t)|dt$$

$$\leq \int_0^s |(g(v) - g(u))|| \phi(t)|dt$$

$$\leq \int_0^s || v| - | u |||| v|^2 + | v || u | + | u |^2||| \phi(t)|dt$$

or

$$|| v|^2 + | v || u | + | u |^2| \leq | v|^2 + | v || u | + | u |^2.$$

Suppose that $| u | \leq | v |$. Then $| u || v | \leq | v |^2$. Hence

$$| v|^2 + | v || u | + | u |^2 \leq 3| v |^2.$$

Then we obtain

$$\int_0^s |(g(v) - g(u))|| \phi(t)|dt \leq 3\int_0^s || v | - | u ||| v |^2| \phi(t)|dt$$

$$\Rightarrow \int_0^s |(g(v) - g(u))|| \phi(t)|dt \leq 3\int_0^s | w || v |^2| \phi(t)|dt$$

with $| v |^2 \in L^\infty(0, T; L^n(\Omega))$, $w \in L^\infty(0, T; L^2(\Omega))$ and

$$\phi \in L^\infty(0, T; L^q(\Omega)).$$

By integrating on $\Omega$, we have

$$\int_\Omega \left( \int_0^s |(g(v) - g(u))| |\phi(t)| dt \right) dx \leq 3 \int_\Omega \left( \int_0^s |w| |v|^2 |\phi(t)| dt \right) dx. \quad (3.12)$$

Applying Hölder's inequality, (3.12) becomes

$$\int_\Omega \left( \int_0^s |(g(v) - g(u))| |\phi(t)| dt \right) dx$$

$$\leq 3 \left[ \left( \int_\Omega |w|^2 dx \right)^{\frac{1}{2}} \left( \int_\Omega ||v|^2|^n dx \right)^{\frac{1}{n}} \left( \int_\Omega |\phi(t)|^q dx \right)^{\frac{1}{q}} \right]$$

$$\leq 3 \int_0^s \| w \|_{L^2(\Omega)} \| |v|^2 \|_{L^n(\Omega)} \| \phi(t) \|_{L^q(\Omega)} dt,$$

where

$$\frac{1}{2} + \frac{1}{n} + \frac{1}{q} = 1.$$

As

$$\phi(x, t) = w_1(x, t) - w_1(x, s),$$

we have

$$\| \phi(x, t) \|_{L^q(\Omega)} = \| w_1(x, t) - w_1(x, s) \|_{L^q(\Omega)}$$

$$\leq \| w_1(x, t) \|_{L^q(\Omega)} + \| w_1(x, s) \|_{L^q(\Omega)}.$$

Since $H^1(\Omega) \subset L^q(\Omega)$, there exists $c_7 > 0$, such as

$$\| \phi(x, t) \|_{L^q(\Omega)} \leq c_7 \| \phi(x, t) \|_{H^1(\Omega)}$$

$$\leq c_7 (\| w_1(x, t) \|_{H^1(\Omega)} + \| w_1(x, s) \|_{H^1(\Omega)}). \quad (3.13)$$

By using (3.13), (3.12) becomes

$$\int_\Omega \left( \int_0^s |(g(v) - g(u))||\phi(t)|dt \right) dx$$

$$\leq c_8 \int_0^s [\| w \|_{L^2(\Omega)} \| |v|^2 \|_{L^n(\Omega)} \times (\| w_1(x, t) \|_{H^1(\Omega)} + \| w_1(x, s) \|_{H^1(\Omega)})] dt.$$

Note also that

$$\| |v|^2 \|_{L^n(\Omega)} \leq c_9 \| |v|^2 \|_{H^1(\Omega)}$$

and

$$v \in L^\infty(0, T; H^1(\Omega)) \Rightarrow \| |v|^2 \|_{H^1(\Omega)} \leq c_{10} \text{ p.p on } \Omega,$$

$$\Rightarrow \| |v|^2 \|_{L^n(\Omega)} \leq c_{11}.$$

Then we have

$$\int_\Omega \left( \int_0^s |(g(v) - g(u))||\phi(t)|dt \right) dx$$

$$\leq c_{12} \int_0^s \| w \|_{L^2(\Omega)} \times (\| w_1(x, t) \|_{H^1(\Omega)} + \| w_1(x, s) \|_{H^1(\Omega)}) dt$$

$$\leq c_{12} \int_0^s \| w \|_{L^2(\Omega)} \| w_1(x, t) \|_{H^1(\Omega)} dt$$

$$+ c_{12} \int_0^s \| w \|_{L^2(\Omega)} \| w_1(x, s) \|_{H^1(\Omega)} dt$$

$$\Rightarrow \int_\Omega \left( \int_0^s |(g(v) - g(u))||\phi(t)|dt \right) dx$$

$$\leq c_{12} \int_0^s \| w \|_{L^2(\Omega)} \| w_1(x, t) \|_{H^1(\Omega)} dt$$

$$+ c_{12} \| w_1(x, s) \|_{H^1(\Omega)} \int_0^s \| w \|_{L^2(\Omega)} dt. \tag{3.14}$$

Applying Young's inequality, (3.14) becomes

$$\int_{\Omega}\left(\int_0^s |(g(v)-g(u))||\phi(t)|dt\right)dx$$

$$\leq c_{12}\left[\int_0^s \frac{\varepsilon}{2}\| w \|_{L^2(\Omega)}^2 + \frac{1}{2\varepsilon}\| w_1(x,t) \|_{H^1(\Omega)}^2 dt\right]$$

$$+ c_{12}\left[\frac{\varepsilon}{2}T\| w_1(x,s) \|_{H^1(\Omega)}^2 + \int_0^s \frac{1}{2\varepsilon}\| w \|_{L^2(\Omega)}^2 dt\right]$$

$$\Rightarrow \int_{\Omega}\left(\int_0^s |(g(v)-g(u))||\phi(t)|dt\right)dx$$

$$\leq c_{12}\int_0^s\left[\left(\frac{\varepsilon}{2}+\frac{1}{2\varepsilon}\right)\| w \|_{L^2(\Omega)}^2 + \frac{1}{2\varepsilon}\| w_1(x,t) \|_{H^1(\Omega)}^2\right]dt$$

$$+ \frac{\varepsilon}{2}Tc_{12}\| w_1(x,s) \|_{H^1(\Omega)}^2. \tag{3.15}$$

By using (3.15), (3.11) becomes

$$\int_{\Omega}(u_1(x)-v_1(x))w_1(x)dx - \frac{1}{2}\| w(x,s) \|_{L^2(\Omega)}^2 + \frac{1}{2}\| u_0(x)-v_0(x) \|_{L^2(\Omega)}^2$$

$$+ \frac{1}{2}\| \nabla w(x,s) \|_{L^2(\Omega)}^2 + \int_{\Omega}(u_0(x)-v_0(x))w_1(x)dx - \int_0^s \| w \|_{L^2(\Omega)}^2 dt$$

$$\leq c_{12}\int_0^s\left[\left(\frac{\varepsilon}{2}+\frac{1}{2\varepsilon}\right)\| w \|_{L^2(\Omega)}^2 + \frac{1}{2\varepsilon}\| w_1(x,t) \|_{H^1(\Omega)}^2\right]dt$$

$$+ \frac{\varepsilon}{2}Tc_{12}\| w_1(x,s) \|_{H^1(\Omega)}^2$$

$$\Rightarrow -\frac{1}{2}\| w(x,s) \|_{L^2(\Omega)}^2 + \frac{1}{2}\| u_0(x)-v_0(x) \|_{L^2(\Omega)}^2 + \frac{1}{2}\| \nabla w(x,s) \|_{L^2(\Omega)}^2$$

$$\leq \int_0^s \| w \|_{L^2(\Omega)}^2 dt + c_{12}\int_0^s\left[\left(\frac{\varepsilon}{2}+\frac{1}{2\varepsilon}\right)\| w \|_{L^2(\Omega)}^2 + \frac{1}{2\varepsilon}\| w_1(x,t) \|_{H^1(\Omega)}^2\right]dt$$

$$+\frac{\varepsilon}{2}Tc_{12}\| w_1(x, s)\|^2_{H^1(\Omega)} - \int_\Omega (u_1(x) - v_1(x))w_1(x)dx$$

$$-\int_\Omega (u_0(x) - v_0(x))w_1(x)dx. \tag{3.16}$$

Applying Hölder's inequality to the second member of (3.16), we have

$$-\| w(x, s)\|^2_{L^2(\Omega)} + \| u_0(x) - v_0(x)\|^2_{L^2(\Omega)} + \| \nabla w(x, s)\|^2_{L^2(\Omega)}$$

$$\leq c_{12}\int_0^s \left[\left(\varepsilon + \frac{1}{\varepsilon} + \frac{2}{c_{12}}\right)\| w\|^2_{L^2(\Omega)} + \frac{1}{\varepsilon}\| w_1(x, t)\|^2_{H^1(\Omega)}\right]dt$$

$$+\varepsilon Tc_{12}\| w_1(x, s)\|^2_{H^1(\Omega)} - 2\| u_1(x) - v_1(x)\|^2_{L^2(\Omega)}\| w_1(x, s)\|^2_{L^2(\Omega)}$$

$$-2\| u_0(x) - v_0(x)\|^2_{L^2(\Omega)}\| w_1(x, s)\|^2_{L^2(\Omega)}. \tag{3.17}$$

Reducing (3.17) by appropriate expressions on both sides, we have

$$\| w(x, s)\|^2_{L^2(\Omega)} \leq c_{12}\int_0^s \left[\left(\varepsilon + \frac{1}{\varepsilon} + \frac{2}{c_{12}}\right)\| w\|^2_{L^2(\Omega)} + \frac{1}{\varepsilon}\| w_1(x, t)\|^2_{H^1(\Omega)}\right]dt$$

$$+\varepsilon Tc_{12}\| w_1(x, s)\|^2_{H^1(\Omega)}. \tag{3.18}$$

By adding the term $\| w_1(x, s)\|^2_{H^1(\Omega)}$ to the first and second members of (3.18), we obtain

$$\| w(x, s)\|^2_{L^2(\Omega)} + \| w_1(x, s)\|^2_{H^1(\Omega)}$$

$$\leq c_{12}\int_0^s \left[\left(\varepsilon + \frac{1}{\varepsilon} + \frac{2}{c_{12}}\right)\| w\|^2_{L^2(\Omega)} + \frac{1}{\varepsilon}\| w_1(x, t)\|^2_{H^1(\Omega)}\right]dt$$

$$+\varepsilon Tc_{12}\| w_1(x, s)\|^2_{H^1(\Omega)} + c_{13}\| w_1(x, s)\|^2_{H^1(\Omega)}$$

$$\leq c_{12}\int_0^s\left[\left(\varepsilon+\frac{1}{\varepsilon}+\frac{2}{c_{12}}\right)\| w\|^2_{L^2(\Omega)}+\frac{1}{\varepsilon}\| w_1(x, t)\|^2_{H^1(\Omega)}\right]dt$$

$$+(\varepsilon Tc_{12}+c_{13})\| w_1(x, s)\|^2_{H^1(\Omega)}.$$

By choosing $\varepsilon > 0$ infinitely small $(\varepsilon <<< 1)$ such that

$$\varepsilon+\frac{1}{\varepsilon}+\frac{2}{c_{12}} = \text{const} > 0$$

and

$$1-(\varepsilon Tc_{12}+c_{13}) = \text{const} > 0$$

with

$$\| w\|^2_{L^2(\Omega)} = \| w(x, t)\|^2_{L^2(\Omega)},$$

we have

$$\| w(x, s)\|^2_{L^2(\Omega)}+\| w_1(x, s)\|^2_{H^1(\Omega)}$$

$$\leq \text{const}\int_0^s\left[\| w(x, t)\|^2_{L^2(\Omega)}+\| w_1(x, t)\|^2_{H^1(\Omega)}\right]dt. \tag{3.19}$$

According to Gronwall inequality, we have

$$w = 0 \Leftrightarrow u - v = 0$$

$$\Leftrightarrow u = v.$$

Hence, problem (1.1)-(1.3) has a unique solution.

## References

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