Far East Journal of Dynamical Systems
© 2023 Pushpa Publishing House, Prayagraj, India
http://www.pphmj.com
http://dx.doi.org/10.17654/0972111823003
Volume 36, Number 1, 2023, Pages 77-92 **P-ISSN: 0972-1118**

RIESZ BASIS PROPERTY AND EXPONENTIAL STABILITY OF A SECOND ORDER SYSTEM IN TIME WITH VARIABLE COEFFICIENTS

Bomisso Gossrin Jean-Marc[1], Koffi Claude Jean Joris[2], Touré Kidjégbo Augustin[2] and Coulibaly Adama[3]

[1]Université Nangui Abrogoua
D'Abobo-Adjamé and UFR Sciences Fondamentales et Appliquées
BP 801 Abidjan 02, Côte D'Ivoire
e-mail: bogojm@yahoo.fr

[2]Institut National Polytechnique Félix Houphouët-Boigny De Yamoussoukro
BP 1093 Yamoussoukro, Côte D'Ivoire
e-mail: koffijoris@gmail.com
latoureci@gmail.com

[3]Université Félix Houphouët-Boigny De Cocody
BP 582, Abidjan 22, Côte D'Ivoire
e-mail: couliba@yahoo.fr

Received: March 4, 2023; Accepted: May 1, 2023
2020 Mathematics Subject Classification: 35B35, 35P20, 93D15.
Keywords and phrases: Euler-Bernoulli beam, variable coefficients, Riesz basis, exponential stability.

How to cite this article: Bomisso Gossrin Jean-Marc, Koffi Claude Jean Joris, Touré Kidjégbo Augustin and Coulibaly Adama, Riesz basis property and exponential stability of a second order system in time with variable coefficients, Far East Journal of Dynamical Systems 36(1) (2023), 77-92. http://dx.doi.org/10.17654/0972111823003

Published Online: May 23, 2023

Abstract

This paper which is a variant [6] studies the Riesz basis property and the exponential stability of a flexible Euler-Bernoulli beam with variable coefficients, clamped at one end and submitted at its free end at two control forces. We begin by establishing the spectral properties of this dynamical system, which allows us to show that there exists a sequence of generalized eigenfunctions forming a Riesz basis for the energy space considered. Consequently, the exponential stability under some conditions is derived.

1. Introduction

Because of the many technological advances that the world is experiencing in recent years, mechanical problems with variable coefficients are nowadays very attractive but difficult to study. Developing an approach to solve this type of problems often involves the use of partial differential equations (PDEs), among which hyperbolic PDEs are included. In this work, we are interested in the motion of a flexible Euler-Bernoulli beam with variable coefficients clamped at one end and submitted at the other end to two force controls: the first one in rotational velocity and the second one in velocity. The motion of the so-called beam is in the framework of hyperbolic PDEs and the mathematical model describing the motion of such a beam is given by the following system of equations

$$m(x)y_{tt}(x, t) + (EI(x)y_{xx}(x, t))_{xx} = 0, \ \ t \geq 0, \ \ 0 < x < 1, \tag{1.1}$$

$$y(0, t) = y_x(0, t) = 0, \ \ t \geq 0, \tag{1.2}$$

$$EI(1)y_{xx}(1, t) = -\beta y_{xt}(1, t), \ \ t \geq 0, \tag{1.3}$$

$$(EI(\cdot)y_{xx}(\cdot, t))_x(1) = \alpha y_t(1, t), \ \ t \geq 0, \tag{1.4}$$

with the initial conditions

$$y(x, 0) = y_0(x), \quad y_t(x, 0) = y_1(x). \tag{1.5}$$

Let us specify in the system (1.1)-(1.4), $y(x, t)$ represents the transversal displacement of the beam at the position x and the time t; the variable coefficients $m(\cdot)$ and $EI(\cdot)$, respectively, the mass density and the flexural rigidity of the beam, with E the Young's modulus and I the second moment of inertia of the beam are non-negative functions and C^4 on $[0, 1]$. In addition, we also notice that α and β are given positive constants. It is important to note that several authors have had to work on Euler-Bernoulli beams, in particular [6] who dealt with the system (1.1)-(1.4) with constant coefficients that means $m(\cdot) = EI(\cdot) = 1$. It is proved that $-(1/\beta + \alpha)$ is a vertical asymptote in the complex plane for the spectrum. It is also possible to mention [5] which studied (1.1)-(1.4) with variable coefficients when $\beta = 0$. For our part, we devote this paper to the study of the Riesz basis property and the exponential stability of the system (1.1)-(1.4). To do this, we first establish in Section 2, the well-posedness of (1.1)-(1.4) in the sense of semi-groups. Then, after having written (1.1)-(1.4) in the form of a Sturn-Liouville problem, we take inspiration from the works of [3, 4, 7] and perform two essential transformations which allow us to obtain an eigenvalue problem in which the term of order 3 no longer appears. This will allow us to perform the spectral analysis of (1.1)-(1.4). As for Section 3, we establish by a similar approach to [5] the Riesz basis property of (1.1)-(1.4); this leads us to study its exponential stability.

2. Spectral Properties of Dynamical System (1.1)-(1.4)

This section is supposed to allow us to establish the spectral properties of the system (1.1)-(1.4). But first, we start by showing that the considered system is well posed according to the theory of the semi-group of contraction.

Let the functional space $H_E^2(0, 1)$ be defined by

$$H_E^2(0, 1) := \{u \in H^2(0, 1) : u(0) = u_x(0) = 0\}$$

and consider the Hilbert space $\mathcal{H} := H_E^2(0, 1) \times L^2(0, 1)$ with the inner product denoted by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and defined by

$$\langle u, v \rangle_{\mathcal{H}} := \int_0^1 (m(x) g_1(x) \overline{g_2(x)} + EI(x) f_1''(x) \overline{f_2''(x)}) dx \tag{2.1}$$

with $u = (f_1, g_1) \in \mathcal{H}$ and $v = (f_2, g_2) \in \mathcal{H}$. Denote by $\| \cdot \|_{\mathcal{H}}$ its corresponding norm. Now, consider the linear operator $\mathcal{A}$ defined on its domain

$$\mathcal{D}(\mathcal{A})(\subset \mathcal{H}) := \{(f, g)^T \in (H_E^2(0, 1) \cap H^4(0, 1)) \times H_E^2(0, 1) :$$
$$EI(1) f''(1) = -\beta g'(1), \ (EIf'')'(1) = \alpha g(1)\} \tag{2.2}$$

by

$$\mathcal{A}(f, g)^T := \left(g, -\frac{1}{m}(EIf'')'' \right)^T, \quad \forall (f, g)^T \in \mathcal{D}(\mathcal{A}). \tag{2.3}$$

Then, in the form of an evolutionary problem, (1.1)-(1.4) is written

$$\begin{cases} \dfrac{d}{dt} Y(t) = \mathcal{A} Y(t) \\ Y(0) = Y_0 \in \mathcal{H}, \end{cases} \tag{2.4}$$

with $Y(t) = (y(\cdot, t), y_t(\cdot, t))^T$ for all $t > 0$.

The problem (1.1)-(1.4) being equivalent to (2.4), we state the following fundamental result about well-posedness of (1.1)-(1.4).

Proposition 2.1. *Consider the linear operator* $\mathcal{A}$ *defined by* (2.2)-(2.3) *in a closed and dense domain. Then* $\mathcal{A}$ *is an infinitesimal generator of a* C_0*-semigroup of contraction on* $\mathcal{H}$ *denoted as* $\{T(t)\}_{t \geq 0}$ *and defined by* $T(t) = e^{t\mathcal{A}}$. *Furthermore,* $\mathcal{A}$ *has compact resolvent and* $0 \in \rho(\mathcal{A})$. *Therefore, the spectrum* $\sigma(\mathcal{A})$ *of the operator* $\mathcal{A}$ *is formed only by the isolated eigenvalues.*

Proof. We begin the proof by showing $\mathcal{A}$ is an infinitesimal generator of a C_0-semigroup of contraction on $\mathcal{H}$. To do this, let $(f, g)^T \in \mathcal{D}(\mathcal{A})$. Then, by a double integration by parts, we obtain

$$\langle \mathcal{A}(f, g)^T, (f, g)^T \rangle_{\mathcal{H}} = \int_0^1 EI(x)(\overline{f''(x)}g''(x) - f''(x)\overline{g''(x)})dx$$

$$- (\alpha| g(1)|^2 + \beta| g'(1)|^2).$$

Therefore, by taking the real part of the above inner product, we get

$$\mathrm{Re}(\langle \mathcal{A}(f, g)^T, (f, g)^T \rangle_{\mathcal{H}}) = -(\alpha| g(1)|^2 + \beta| g'(1)|^2) \leq 0.$$

This shows that the operator $\mathcal{A}$ is dissipative. Moreover, by using the Lax-Milgram Theorem, we prove the *m*-dissipativity of the operator $\mathcal{A}$. Thus, by the Lumer-Phillips Theorem [8, p. 14], we conclude that the operator $\mathcal{A}$ is an infinitesimal generator of a C_0-semigroup of contraction on $\mathcal{H}$. Now, we show that $\mathcal{A}$ has compact resolvent and $0 \in \rho(\mathcal{A})$. Clearly, we have to show that

$$\forall z = (f, g)^T \in \mathcal{H}, \quad \exists!\, y = (\phi, \psi)^T \in \mathcal{D}(\mathcal{A}) : z = \mathcal{A}y.$$

Indeed, with some computations we show that for all $z = (f, g)^T \in \mathcal{H}$, the relation $z = \mathcal{A}y$ implies

$$\begin{cases} \psi(x) = f(x), \\ \phi(x) = \int_0^x \int_0^s \left[\frac{\alpha(1-\zeta)\psi(1) + \beta\psi'(1)}{EI(\zeta)} + \frac{1}{EI(\zeta)} \int_\zeta^1 \int_\eta^1 m(t) g(t) dt d\eta \right] d\zeta ds. \end{cases}$$

Thus, $\mathcal{A}^{-1}$ exists and $0 \in \rho(\mathcal{A})$. Consequently, according to Sobolev Imbedding Theorem [1, pp. 85-86] the proof is complete. □

Since the problem (1.1)-(1.4) is well-posed in the sense of C_0-semigroups of contraction, we now establish its spectral properties. To do this, we start by considering $\lambda \in \sigma(\mathcal{A})$ and $\Psi = (\phi, \psi)^T = (\phi, \lambda\phi)^T$ the eigenfunction of $\mathcal{A}$ associated to λ. So, we obtain for $0 < x < 1$, $\phi \neq 0$ satisfies the following eigenvalue problem

$$\begin{cases} \phi^{(4)}(x) + \dfrac{2EI'(x)}{EI(x)}\phi'''(x) + \dfrac{EI''(x)}{EI(x)}\phi''(x) + \lambda^2 \dfrac{m(x)}{EI(x)}\phi(x) = 0, \\ \phi(0) = \phi'(0) = 0, \\ \phi''(1) = -\dfrac{\beta\lambda}{EI(1)}\phi'(1), \\ \phi'''(1) = -\dfrac{EI'(1)}{EI(1)}\phi''(1) + \dfrac{\alpha\lambda}{EI(1)}\phi(1). \end{cases} \tag{2.5}$$

Now, in order to simplify computations, we use two essential transformations like in [7]. Firstly, we apply a transformation in space by a scaling

$$f(z) := \phi(x) \text{ with } z := \frac{1}{h}\int_0^x \left(\frac{m(\zeta)}{EI(\zeta)}\right)^{\frac{1}{4}} d\zeta, \text{ where } h := \int_0^1 \left(\frac{m(\zeta)}{EI(\zeta)}\right)^{\frac{1}{4}} d\zeta. \tag{2.6}$$

Once this transformation done, we obtain the following equivalent problem:

$$\begin{cases} f^{(4)}(z) + a(z)f'''(z) + b(z)f''(z) + c(z)f'(z) + \lambda^2 h^4 f(z) = 0, \\ f(0) = f'(0) = 0, \\ f''(1) + a_{11}f'(1) = 0, \\ f'''(1) + a_{21}f''(1) + a_{22}f'(1) + a_{23}f(1) = 0, \end{cases} \tag{2.7}$$

where the coefficients $a(z)$, $b(z)$ and $c(z)$ are the same as in [10] with

$$a_{11} = \frac{\beta\lambda}{z_x(1)EI(1)} + \frac{z_{xx}(1)}{z_x^2(1)} \text{ and } a_{23} = -\frac{\alpha\lambda}{EI(1)z_x^3(1)}.$$

Then, we perform a second transformation as was done in [7] in order to remove the third order term $a(z)f'''(z)$. To achieve this, we introduce the invertible state transformation

$$f(z) = \exp\left(-\frac{1}{4}\int_0^z a(\zeta)d\zeta\right)g(z), \quad z \in [0, 1], \tag{2.8}$$

which gives the following equivalent eigenvalue problem to (2.7):

$$\begin{cases} g^{(4)}(z) + b_1(z)g''(z) + c_1(z)g'(z) + d_1(z)g(z) + \lambda^2 h^4 g(z) = 0, \\ g(0) = g'(0) = 0, \\ g''(1) + b_{11}g'(1) + b_{12}g(1) = 0, \\ g'''(1) + b_{21}g''(1) + b_{22}g(1) + b_{23}g(1) = 0, \end{cases} \tag{2.9}$$

where the smooth functions $b_1(z), c_1(z), d_1(z)$ can be found in [10] and the coefficients b_{ij}, $(i, j) \in \{1, 2\} \times \{1, 2, 3\}$ are obtained by some computations.

In order to solve the eigenvalue problem (2.9), we follow an idea developed in [3, 4, 7]. So, in the sector S_1 defined by

$$S_1 = \left\{z \in \mathbb{C} : \frac{\pi}{4} \le \arg z \le \frac{\pi}{2}\right\},$$

we consider $\omega_j = ((-1)^j + i)/\sqrt{2}$, $j = 1, 2$ with $\omega_3 = -\omega_2$ and $\omega_4 = -\omega_1$ the roots of the characteristic equation $\theta^4 + 1 = 0$ satisfying

$$\mathrm{Re}(\rho\omega_1) \le \mathrm{Re}(\rho\omega_2) \le \mathrm{Re}(\rho\omega_3) \le \mathrm{Re}(\rho\omega_4), \quad \forall \rho \in S_1. \tag{2.10}$$

Assuming that $\lambda := \rho^2/h^2$, we notice that

$$\begin{cases} \mathrm{Re}(\rho\omega_1) = -|\rho|\sin\left(\arg\rho + \frac{\pi}{4}\right) \le -\frac{\sqrt{2}}{2}|\rho| < 0 \\ \mathrm{Re}(\rho\omega_2) = |\rho|\cos\left(\arg\rho + \frac{\pi}{4}\right) \le 0, \end{cases} \tag{2.11}$$

which implies

$$|e^{\rho\omega_2}| \le 1 \text{ and } |e^{\rho\omega_1}| = \mathcal{O}\left(e^{-\frac{\sqrt{2}}{2}|\rho|}\right) \text{ when } |\rho| \to \infty. \tag{2.12}$$

Now, we recall Theorem 2.4 of [7] which we write down in the form of Lemma 2.2. It is useful for establishing the spectral properties of (2.9).

Lemma 2.2. *For* $|\rho|$ *large enough*, *there are four linearly independent solutions* $g_k(z)$, $k = 1, 2, 3, 4$ *of the equation*

$$g^{(4)}(z) + b_1(z)g''(z) + c_1(z)g'(z) + d_1(z)g(z) + \lambda^2 h^4 g(z) = 0$$

such that for $l = 0, 1, 2, 3$,

$$g_k^{(l)}(z) = (\rho\omega_k)^l e^{\rho\omega_k z}[1], \tag{2.13}$$

where $[1] = 1 + \mathcal{O}(\rho^{-1})$.

Therefore, the asymptotic expressions for the boundary conditions of the eigenvalue problem (2.9) for large enough $|\rho|$ are the following for $k = 1, 2, 3, 4$:

$$U_4(g_k, \rho) = g_k(0, \rho) = [1], \tag{2.14}$$

$$U_3(g_k, \rho) = g_k'(0, \rho) = (\rho\omega_k)[1], \tag{2.15}$$

$$\begin{aligned} U_2(g_k, \rho) &= g_k''(1, \rho) + b_{11}g_k'(1, \rho) + b_{12}g_k(1, \rho) \\ &= \kappa\rho^3\omega_k e^{\rho\omega_k}[1], \end{aligned} \tag{2.16}$$

$$\begin{aligned} U_1(g_k, \rho) &= g_k'''(1, \rho) + b_{21}g_k''(1, \rho) + b_{22}g_k'(1, \rho) + b_{23}g_k(1, \rho) \\ &= (\rho\omega_k)^3 e^{\rho\omega_k}[1], \end{aligned} \tag{2.17}$$

where $\kappa = \dfrac{\beta}{z_x(1)EI(1)h^2}$.

Since the characteristic determinant of (2.9) is given by

$$\Delta(\rho) = (U_i(g_k, \rho))_{i=4,3,2,1,\, k=1,2,3,4}, \tag{2.18}$$

by substituting (2.14)-(2.17) into (2.18) and some computations made give

$$e^{2\rho\omega_2} = \frac{\omega_2 - \omega_1}{\omega_2 + \omega_1} + \mathcal{O}(\rho^{-1}) = -i + \mathcal{O}(\rho^{-1}). \tag{2.19}$$

Ignoring the higher order terms in (2.19), we obtain the following equation:

$$e^{2\rho\omega_2} = -i,$$

whose solutions $\widetilde{\rho}_k$ are given by

$$\widetilde{\rho}_k = \frac{\sqrt{2}}{2}\left(k - \frac{1}{4}\right)\pi(1 + i), \quad k = 1, 2, \ldots. \tag{2.20}$$

Let ρ_k be the solutions of (2.19). Applying Rouché's Theorem as in [7, p. 70], we get

$$\rho_k = \frac{\sqrt{2}}{2}\left(k - \frac{1}{4}\right)\pi(1 + i) + \mathcal{O}(k^{-1}), \quad k = N, N + 1, \ldots,$$

where N is a sufficiently large positive integer.

Furthermore, by applying Lemma 3.1 from [7], we can write $g_k^{(n)}(z)$, $n = 0, 1, 2$ as follows:

$$g_k^{(n)}(z) = \rho_k^n \det\begin{pmatrix} [1] & [1] & e^{\rho_k\omega_2}[1] & e^{\rho_k\omega_1}[1] \\ \omega_1^n e^{\rho_k\omega_1 z}[1] & \omega_2^n e^{\rho_k\omega_2 z}[1] & \omega_3^n e^{\rho_k\omega_2(1-z)}[1] & \omega_4^n e^{\rho_k\omega_1(1-z)}[1] \\ \omega_1 e^{\rho_k\omega_1}[1] & \omega_2 e^{\rho_k\omega_2}[1] & -\omega_2[1] & -\omega_1[1] \\ \omega_1^3 e^{\rho_k\omega_1}[1] & \omega_2^3 e^{\rho_k\omega_2}[1] & -\omega_2^3[1] & -\omega_1^3[1] \end{pmatrix}.$$

Hence, the solution g_k associated to (2.9) admits the following asymptotic function

$$g_k(z) = 2(1 + i)\left(\cos\left(k - \frac{1}{4}\right)\pi z - \sin\left(k - \frac{1}{4}\right)\pi z - e^{-\left(k-\frac{1}{4}\right)\pi z}\right) + \mathcal{O}(k^{-1}), \tag{2.21}$$

$$\rho_k^{-2} g_k''(z) = 2(1-i)\left(\sin\left(k-\frac{1}{4}\right)\pi z - \cos\left(k-\frac{1}{4}\right)\pi z + e^{-\left(k-\frac{1}{4}\right)\pi z}\right) + \mathcal{O}(k^{-1}). \tag{2.22}$$

Moreover,

$$\rho_k^{-1} g_k'(z) = 2\sqrt{2}\left(\sin\left(k-\frac{1}{4}\right)\pi z - \cos\left(k-\frac{1}{4}\right)\pi z + e^{-\left(k-\frac{1}{4}\right)\pi z}\right) + \mathcal{O}(k^{-1}). \tag{2.23}$$

Consequently, we can notice that $\rho_k^{-2} g_k'(z) = \mathcal{O}(k^{-1})$. In addition, if we consider (2.6) and (2.12), we get the asymptotic expansion of (λ_k, ϕ_k). Finally, we can conclude this section by stating the following proposition which summarizes the spectral properties of the problem (1.1)-(1.4).

Proposition 2.3. *Let* $\mathcal{A}$ *be the operator defined by* (2.2)-(2.3). *Then*,

(i) *there is a family of eigenvalues* $(\lambda_k, \overline{\lambda_k})$ *of* $\mathcal{A}$ *which satisfies*

$$\begin{cases} \lambda_k = \dfrac{\rho_k^2}{h^2}, \ h = \displaystyle\int_0^1 \left(\frac{m(\zeta)}{EI(\zeta)}\right)^{1/4} d\zeta, \\ \rho_k = \dfrac{\sqrt{2}}{2}\left(k-\dfrac{1}{4}\right)\pi(1+i) + \mathcal{O}(k^{-1}), \end{cases} \tag{2.24}$$

where $k \to \infty$ *is a sufficiently large integer*. *Moreover*, *the eigenvalues* λ_k *are geometrically simple when k is a sufficiently large integer*;

(ii) *there is a solution* ϕ_k *to* (2.5) *corresponding to* λ_k *having the following asymptotic expansion*

$$e^{\frac{1}{4}\int_0^z a(\zeta)\,d\zeta} \phi_k(x) = 2(1+i)\left(\cos\left(k-\frac{1}{4}\right)\pi z - \sin\left(k-\frac{1}{4}\right)\pi z - e^{-\left(k-\frac{1}{4}\right)\pi z}\right) + \mathcal{O}(k^{-1}), \tag{2.25}$$

$$\lambda_k^{-1}\phi_k''(x) = 2(1-i)\theta(x)\left(\sin\left(k-\frac{1}{4}\right)\pi z - \cos\left(k-\frac{1}{4}\right)\pi z - e^{-\left(k-\frac{1}{4}\right)\pi z}\right) + \mathcal{O}(k^{-1}), \tag{2.26}$$

where $\theta(x) = \left(\frac{m(x)}{EI(x)}\right)^{\frac{1}{2}} e^{-\frac{1}{4}\int_0^z a(\zeta)d\zeta}.$

3. Riesz Basis Property and Exponential Stability of the Dynamical System (1.1)-(1.4)

In this last section, our aim is to establish the Riesz basis property and the exponential stability of the system (1.1)-(1.4). To achieve this, we will refer to Theorem 3.3 [5]. So, considering (1.1)-(1.4) with $\alpha = \beta = 0,$ we get the following uncontrolled system:

$$m(x)y_{tt}(x, t) + (EI(x)y_{xx}(x, t))_{xx} = 0, \ t > 0, \ 0 < x < 1, \tag{3.1}$$

$$y(0, t) = y_x(0, t) = y_{xx}(1, t) = (EIy_{xx})_x(1, t) = 0, \ t > 0, \tag{3.2}$$

whose associated operator $\mathcal{B} : \mathcal{D}(\mathcal{B})(\subset \mathcal{H}) \to \mathcal{H}$ is such that

$$\begin{cases} \mathcal{B}(f, g)^T = \left(g, -\frac{1}{m}(EIf'')''\right)^T, \ \forall (f, g)^T \in \mathcal{D}(\mathcal{B}), \\ \mathcal{D}(\mathcal{B}) = \{(f, g)^T \in (H_E^2(0, 1) \cap H^4(0, 1)) \times H_E^2(0, 1) : f''(1) = f'''(1) = 0\}. \end{cases} \tag{3.3}$$

The operator $\mathcal{B}$ is skew-adjoint and has compact resolvent on $\mathcal{H}$. Moreover, since Proposition 2.3 is still valid when $\alpha = \beta = 0,$ we have the following counterpart for the operator $\mathcal{B}.$

Lemma 3.1. *Let the operator* $\mathcal{B}$ *be defined by* (3.3) *and* $\nu_{k_0} \in \sigma(\mathcal{B})$ *of sufficiently large modulus. Then, each* ν_{k_0} *is geometrically simple and hence algebraically simple and the eigenvalues* $(\nu_{k_0}, \overline{\nu_{k_0}})$ *and the associated eigenfunctions* $\{(\nu_{k_0}^{-1}\Phi_{k_0}, \Phi_{k_0}) \cup \{$*their conjugates*$\}\}$ *of* $\mathcal{B}$ *have the following asymptotic expansion*

$$e^{\frac{1}{4}\int_0^z a(\zeta)\,d\zeta}\Phi_{k_0}(x) = 2(1+i)\left(\cos\left(k-\frac{1}{4}\right)\pi z - \sin\left(k-\frac{1}{4}\right)\pi z - e^{-\left(k-\frac{1}{4}\right)\pi z}\right) + \mathcal{O}(k^{-1}), \tag{3.4}$$

$$\nu_{k_0}^{-1}\Phi''_{k_0}(x) = 2(1-i)\theta(x)\left(\sin\left(k-\frac{1}{4}\right)\pi z - \cos\left(k-\frac{1}{4}\right)\pi z + e^{-\left(k-\frac{1}{4}\right)\pi z}\right) + \mathcal{O}(k^{-1}). \tag{3.5}$$

Now, note that according to a well-known result in functional analysis, since $\mathcal{B}$ is a skew-adjoint discrete operator in $\mathcal{H}$, the set of all ω-linearly independent eigenfunctions of $\mathcal{B}$ forms an orthogonal basis for $\mathcal{H}$. In addition since $(\Phi_{k_0}, \nu_{k_0}\Phi_{k_0})$ defined by (3.4)-(3.5) are approximately normalized, $\{(\Phi_{k_0}, \nu_{k_0}\Phi_{k_0})\} \cup \{\text{their conjugates}\}$ forms an orthogonal Riesz basis for $\mathcal{H}$. Therefore, considering (2.25), (2.26), (3.4) and (3.5), we deduce that there exists $N > 0$ such that

$$\sum_{k>N}^{\infty} \| (\lambda_k^{-1}\phi_k, \phi_k) - (\nu_{k_0}^{-1}\Phi_{k_0}, \Phi_{k_0}) \|_{\mathcal{H}}^2 = \sum_{k>N}^{\infty} \mathcal{O}(k^{-2}) < \infty;$$

the same result being obtained by reasoning with the conjugates. Thus, the hypotheses of Theorem 3.3 [5] are satisfied. Hence, the theorem below.

Theorem 3.2. *Consider the operator* $\mathcal{A}$ *defined by* (2.2)-(2.3). *Then*

(i) *There is a sequence of generalized eigenfunctions of* $\mathcal{A}$ *which forms a Riesz basis for the energy space* $\mathcal{H}$.

(ii) *The eigenvalues* $(\lambda_k, \overline{\lambda_k})$ *of* $\mathcal{A}$ *have the asymptotic expansion* (2.24).

(iii) *All* $\lambda \in \sigma(\mathcal{A})$ *of sufficiently large modulus is algebraically simple*. *Therefore*, $\mathcal{A}$ *generates a* C_0*-semi-group* $e^{\mathcal{A}t}$ *and* $\omega(\mathcal{A}) = S(\mathcal{A})$, *where*

$$\omega(\mathcal{A}) = \lim_{t\to+\infty} \frac{1}{\| e^{\mathcal{A}t} \|}$$

is the growth order of $e^{\mathcal{A}t}$ *and* $S(\mathcal{A}) = \sup\{\operatorname{Re}\lambda / \lambda \in \sigma(\mathcal{A})\}$ *is the spectral bound of* $\mathcal{A}$.

It remains to prove the exponential stability of the system (1.1)-(1.4) to close this paper. Referring to the third point of Theorem 3.2, we have $\omega(\mathcal{A}) = S(\mathcal{A})$; then it follows that the system (1.1)-(1.4) is exponentially stable if and only if there exists $\omega_0 > 0$ such that $\operatorname{Re}(\lambda) < -\omega_0$ for all $\lambda \in \sigma(\mathcal{A})$.

Lemma 3.3. *Let* λ_k *be defined by* (2.24). *Then, there exists* $\omega_0 > 0$ *such that*

$$\lim_{k\to\infty} \operatorname{Re}(\lambda_k) = -\omega_0 < 0,$$

where

$$\omega_0 = -\frac{\left(\alpha + \frac{m(1)EI(1)}{\beta}\right) e^{-\frac{1}{2}\int_0^1 a(\zeta)\,d\zeta}}{\int_0^1 m(x) e^{-\frac{1}{2}\int_0^z a(\zeta)\,d\zeta} dx}. \tag{3.6}$$

Proof. Let $(\lambda, \phi) = (\lambda_k, \phi_k) \neq 0$ satisfying for all $0 < x < 1$,

$$\lambda_k^2 m(x)\phi_k(x) + (EI(x)\phi_k''(x))'' = 0,$$

where ϕ_k is given by (2.25). Multiplying the above equation by $\overline{\phi_k}(x)$ and integrating over $(0, 1)$ with respect to x, we have

$$\lambda_k^2 \int_0^1 m(x)|\phi_k(x)|^2 dx + \int_0^1 (EI(x)\phi_k''(x))'' \overline{\phi_k}(x) dx = 0.$$

Then, performing a double integration by parts, we have

$$\lambda_k^2 \int_0^1 m(x)|\phi_k(x)|^2 dx + \int_0^1 EI(x)|\phi_k''(x)|^2 dx + \alpha\lambda_k|\phi_k(1)|^2 + \beta\lambda_k|\phi_k'(1)|^2 = 0.$$

Writing λ_k in algebraic form and considering the fact that $\text{Im}(\lambda_k) \neq 0$, the above relationship gives us

$$2\,\text{Re}(\lambda_k)\int_0^1 m(x)|\,\phi_k(x)\,|^2\,dx = -(\alpha|\,\phi_k(1)\,|^2 + \beta|\,\phi'_k(1)\,|^2).$$

Now, it is quite simple to show that

$$|\,\phi'_k(1)\,|^2 = (\beta^{-1}EI(1))^2\,|\lambda_k^{-1}\phi''_k(1)|^2.$$

So, we get

$$2\,\text{Re}(\lambda_k)\int_0^1 m(x)|\,\phi_k(x)\,|^2\,dx = -(\alpha|\,\phi_k(1)\,|^2 + \beta^{-1}EI^2(1)|\lambda_k^{-1}\phi''_k(1)|^2). \quad (3.7)$$

Now, using (2.25)-(2.26), we have

$$\lim_{k\to\infty}|\,\phi_k(1)\,|^2 = 16e^{-\frac{1}{2}\int_0^1 a(\zeta)\,d\zeta},$$

$$\lim_{k\to\infty}(EI^2(1)|\lambda_k^{-1}\phi''_k(1)|^2) = 16m(1)EI(1)e^{-\frac{1}{2}\int_0^1 a(\zeta)\,d\zeta},$$

and according to Riemann-Lebesgue Lemma

$$\lim_{k\to\infty}\int_0^1 m(x)|\,\phi_k(x)\,|^2\,dx = 8\int_0^1 m(x)e^{-\frac{1}{2}\int_0^z a(\zeta)\,d\zeta}dx.$$

Passing to the limit in (3.7), we obtain

$$\lim_{k\to\infty}\text{Re}(\lambda_k) = -\frac{\left(\alpha + \dfrac{m(1)EI(1)}{\beta}\right)e^{-\frac{1}{2}\int_0^1 a(\zeta)\,d\zeta}}{\displaystyle\int_0^1 m(x)e^{-\frac{1}{2}\int_0^z a(\zeta)\,d\zeta}dx} < 0. \qquad \square$$

The ultimate result of this paper is recorded in the following theorem.

Theorem 3.4. *Let the system* (1.1)-(1.4) *be exponentially stable for all* $\alpha \geq 0$ *and* $\beta > 0$. *Then, there exist strictly positive constants M and* ω *such that the energy* $E(t)$ *of the system* (1.1)-(1.4) *satisfies*

$$E(t) = \frac{1}{2}\int_0^1 \{m(x)y_t^2(x, t) + EI(x)y_{xx}^2(x, t)\}dx \leq Me^{-\omega t}E(0), \ \forall t \geq 0, \tag{3.8}$$

for any initial condition $(y(x, 0), y_t(x, 0))^T \in \mathcal{H}$.

Proof. By Lemma 3.3 and the fact that $\omega(\mathcal{A}) = S(\mathcal{A})$, it only remains to prove that $\text{Re}(\lambda) < 0$ for any $\lambda \in \sigma(\mathcal{A})$. Since the operator $\mathcal{A}$ is dissipative, $\text{Re}(\lambda) \leq 0$ for any $\lambda \in \sigma(\mathcal{A})$. So, we assume that $\text{Re}(\lambda) = 0$; in other words, take $\lambda := i\tau$, where $\tau \in \mathbb{R}^*$, an eigenvalue of the operator $\mathcal{A}$ on the imaginary axis and consider its corresponding eigenfunction $\Psi := (\phi, \psi)^T$. Then $\psi = \lambda\phi$. Therefore,

$$0 = \text{Re}(\langle \mathcal{A}\Psi, \Psi \rangle_{\mathcal{H}}) = -(\alpha|\psi(1)|^2 + \beta|\psi'(1)|^2),$$

$$0 = \|\Psi\|_{\mathcal{H}}^2 \text{Re}(\lambda) = \text{Re}(\langle \mathcal{A}\Psi, \Psi \rangle_{\mathcal{H}}) = -(\alpha|\psi(1)|^2 + \beta|\psi'(1)|^2).$$

Since $\alpha > 0$ and $\beta > 0$,

$$\psi'(1) = 0 \ \text{ and } \ \psi(1) = 0.$$

Thus $\phi(1) = 0$. So we end up with the fact that ϕ satisfies the following differential equation:

$$\begin{cases} \lambda^2 m(x)\phi(x) + (EI(x)\phi''(x))'' = 0, & 0 < x < 1, \\ \phi(0) = \phi'(0) = \phi(1) = \phi'(1) = \phi''(1) = \phi'''(1) = 0 \end{cases} \tag{3.9}$$

which we solve with the help of Rolle's Theorem. We show successively that zero is the unique solution of (3.9) [5]. Thus, there is no eigenvalue on the imaginary axis. Therefore, $\text{Re}(\lambda) < 0$. Referring to Theorem 3.2, since $\omega(\mathcal{A}) = S(\mathcal{A})$, we conclude that the system (1.1)-(1.4) is exponentially stable. □

Remark 3.5. We note that when the problem (1.1)-(1.4) is considered with constant coefficients, i.e., for $m(\cdot) = EI(\cdot) = 1$, then we effectively find the result of [6], that is,

$$\lim_{k\to\infty} \operatorname{Re}(\lambda_k) = -(\alpha + 1/\beta)$$

which makes $-(\alpha + 1/\beta)$ a vertical asymptote for the spectrum of the operator $\mathcal{A}$ in the complex plane.

References

[1] R. A. Adams, Sobolev spaces, Pure and Applied Mathematics, Academic Press, 65, New York-London, 1975.

[2] M. D. Aouragh and N. Yebari, Stabilisation exponentielle d'une quation des poutres d'Euler-Bernoulli à coefficients variables, Annales Mathématiques Blaise Pascal 16 (2009), 483-510.

[3] G. D. Birkhoff, On the Asymptotic Character of the Solutions of Certain Linear Differential Equations Containing a Parameter, Trans. Amer. Math. Soc. 9 (1908), 219-231.

[4] G. D. Birkhoff, Boundary Value and Expansion Problems of Ordinary Linear Differential Equations, Transactions of the American Mathematical Society 9 (1908), 373-395.

[5] B. Z. Guo, Riesz Basis Property and Exponential Stability of Controlled Euler-Bernoulli Beam Equations with Variable Coefficients, SIAM Journal on Control and Optimization 40 (2002), 1905-1923.

[6] Mensah E. Patrice, On the numerical approximation of the spectrum of a flexible Euler-Bernoulli beam with a force control in velocity and a moment control in rotating velocity, Far East Journal of Mathematical Sciences (FJMS) 126(1) (2020), 13-31.

[7] M. A. Naimark, Linear Differential Operators, Vol. I Ungar, New York, 1967.

[8] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York, 44 (1983).

[9] A. Shkalikov, Boundary Problem for Ordinary Differential Operators with Parameter in Boundary Conditions, J. Soviet Math. 33 (1986), 1311-1342.

[10] Koffi Claude Jean Joris, Bomisso Gossrin Jean-Marc, Touré Kidjégbo Augustin and Coulibaly Adama, Riesz basis property and exponential stability for a damped system and controlled dynamically, International Journal of Numerical Methods and Applications 22 (2022), 19-40.