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ON THE PROPERTIES OF A KIND OF RANDOM MATRICES

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Abstract

Starting from matrices whose columns generate an isotropic subspace and the work done by Qing-You in [8], important properties of a kind of random symplectic matrix are presented. We show that:

(1) it can be transformed into Jordan canonical form by a similar orthogonal transformation,

(2) it has a particular Schur canonical form, and

(3) its condition number is a constant and is the same as that of the matrix studied in [8], numerical examples are given to confirm our theoretical results.

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1. Introduction

Let J be a skew-symmetric and invertible matrix. Then a matrix $W \in \mathbb{R}^{2N\times 2N}$ is said to be *J-symplectic* if $W^T JW = J$. Symplectic matrices play an important role in many fields of science such as classical mechanics and mathematics [1, 2]. In mathematics, for example, they are very often found in the linear control theory for discrete-time systems, perturbation theory and Hamiltonian dynamical systems, etc. In perturbation theory and Hamiltonian dynamical systems, these types of matrices are used to study stability (resp. strong stability) and can be written as:

$$\widetilde{W} = (I + UU^T J)W, \tag{1.1}$$

where W is J-symplectic and $U \in \mathbb{R}^{2N\times 2N}$ is a random matrix whose columns generate an isotropic subspace $(\text{i.e., } \forall U \in \mathcal{L}, \text{ we have } U^T JU = 0$ and $\dim(\mathcal{L}) \leq N)$ [3, 4]. In this paper, we only study some properties of the random matrix $S = I - UU^T J$.

This allows us to propose a generalization of the work of Qing-You (see [8]) and obtain the following results:

(1) It can be transformed into a special canonical form of Schur.

(2) Its condition number has nothing to do with the orthogonal matrix $U \in \mathbb{R}^{2N\times k}$ and is the same as the matrix studied in [8].

Throughout this article, the identity (resp. null) matrix of order $2N$ is denoted by I_{2N} (resp. 0_{2N}) or just I (resp. 0) if the order is clear from the context. The transpose of a matrix (or vector) U is denoted by U^T. The direct sum $\bigoplus_{i=1}^{N} A_i$ of matrices A_i, denotes a diagonal matrix whose diagonal elements are the matrices A_i, $i = 1, ..., N$. Finally, the symbol $A(:, 1 : j)$

(resp. $A(:, j)$) denotes the first j columns of the matrix A (resp. the jth column of the matrix A).

2. Main Results

Let $k = 1, ..., N$ and $U \in \mathbb{R}^{2N\times k}$ be a random matrix whose columns belong to an isotropic subspace. Consider the following matrix S defined by:

$$S = I - UU^T J. \tag{2.1}$$

Then we have the following results:

Theorem 2.1. *S is J-symplectic and has inverse $S^{-1} = I + UU^T J$. Moreover, $1 \in \sigma(S)$ and $\det(S) = 1$.*

Proof. (1) Since

$$\begin{aligned} S^T JS &= (I - UU^T J)^T J(I - UU^T J) \\ &= (J + JUU^T J)(I - UU^T J) \\ &= J + JUU^T J - JUU^T J - JU\underbrace{U^T JU}_{=0}U^T J \\ &= J, \end{aligned}$$

S is J-symplectic. Noting that

$$S(I + UU^T J) = I - UU^T J + UU^T J - U\underbrace{U^T JU}_{=0}U^T J = I,$$

we get $S^{-1} = I + UU^T J$.

(2) We have $SU = U - U\underbrace{U^T JU}_{=0} = U.$ So, $1 \in \sigma(S)$.

On the other hand, $\det(S) = 1,$ because:

$$\det(S) = \det\left(\prod_{j=1}^{k}(I - u_j u_j^T J)\right) = \prod_{j=1}^{k}(\det(I - u_j u_j^T J)) = (1)^k = 1. \qquad \square$$

Theorem 2.2. *Let* $V \in \mathbb{R}^{2N\times k}$. *Then*

(i) $V \in \ker(S - I)$ *if and only if* $U^T JV = 0.$

(ii) $\dim(\ker(S - I)) = 2N - k,$ *where* $k = 1, ..., N.$

(iii) $(S - I)^2 V = 0.$

(iv) *For* $Y = JV,$ $SY = UU^T V + Y,$ $(S - I)Y \neq 0$ *and* $(S - I)^2 Y = 0.$

(v) *All eigenvalues of S are equal to* 1.

Proof. (i) As $V \in \ker(S - I) \Leftrightarrow UU^T JV = 0,$ by multiplying the two members of $UU^T JV = 0$ on the left by U^T and by the invertible matrix $(U^T U)^{-1},$ we obtain $U^T JV = 0.$

(ii) By the dimension theorem, we have

$$\begin{aligned}
&\dim(\mathbb{R}^{2N}) = \dim(\ker(S - I)) + rg(S - I)\\
&\Leftrightarrow 2N = \dim(\ker(S - I)) + rg(UU^T J)\\
&\qquad\quad = \dim(\ker(S - I)) + rg(U)\\
&\qquad\quad = \dim(\ker(S - I)) + k\\
&\Rightarrow \dim(\ker(S - I)) = 2N - k.
\end{aligned}$$

(iii) Since $(S - I)^2 = U\underbrace{U^T JU}_{=0}U^T J = 0,$ so $(S - I)^2 V = 0.V = 0.$

(iv) We have

$$SY = (I - UU^T J)JV = JV + UU^T V,$$

$$(S - I)Y = -UU^T JJV = UU^T V \neq 0,$$

and $(S - I)^2 Y = 0.Y = 0.$

(v) See in [8]. □

From property (iv) of Theorem 2.2, we can state the following remark.

Remark 2.1. If $V \in \mathbb{R}^{2N \times k}$ is such that $U^T V = I_k$, then for $Y = JV$, we have $SY = Y + U$.

Based on this remark, we can generalize Corollary 2 of [8] as follows:

Corollary 2.1. *If the matrix* $U \in \mathbb{R}^{2N \times k}$ *defined in* (2.1) *is such that* $UU^T = I_k$, *then there exists an invertible matrix* $X \in \mathbb{R}^{2N \times 2N}$ *such that*

$$X^{-1}SX = \begin{pmatrix} I_{2N-2k} & 0_{2k} \\ 0_{2k,\,2N-2k} & T_k \end{pmatrix}, \tag{2.2}$$

where

$$T_k = \bigoplus_{j=1}^{k} \mathcal{J}_j \text{ with } \mathcal{J}_j = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Proof. We know that $S = I - UU^T J$ can be written as:

$$S = \prod_{j=1}^{k} (I - u_j u_j^T J), \tag{2.3}$$

where u_j are the columns of matrix U.

(i) For $k = 1$, S becomes a rank-one perturbation of I. So, according to Corollary 2 of [8], S has the following Jordan form:

$$\begin{pmatrix} I_{2N-2} & 0 \\ 0 & T_1 \end{pmatrix} = \left(\bigoplus_{j=1}^{N-1} I_2 \right) \oplus T_1,$$

where

$$T_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(ii) For $k = 2$, $S = (I - u_2 u_2^T J)(I - u_1 u_1^T J)$ is a rank-one perturbation of $I - u_1 u_1^T J$. So, from Corollary 2 of [8], we get the following Jordan form:

$$\begin{pmatrix} I_{2N-2\times 2} & 0 \\ 0 & T_2 \end{pmatrix} = \left(\bigoplus_{j=1}^{N-1} I_2 \right) \oplus T_1,$$

where

$$T_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(iii) For $k \geq 3$, S becomes a rank-k perturbation of I. Applying Corollary 2 of [8] k-times, the Jordan form of S becomes:

$$\left(\bigoplus_{j=1}^{k} I_2 \right) \oplus T_k = \begin{pmatrix} I_{2N-2k} & 0_{2N-2k,\, 2k} \\ 0_{2k,\, 2N-2k} & T_k \end{pmatrix},$$

where

$$T_k = \bigoplus_{j=1}^{k} \mathcal{J}_j, \text{ with } \mathcal{J}_j = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \qquad \square$$

Now, we can state the following theorem:

Theorem 2.3. *Let* $U \in \mathbb{R}^{2N \times k}$ *be a matrix whose columns span an isotropic subspace. Let* $S = I - UU^T J$. *If* $UU^T = I$, *then*:

(1) *there exists an orthogonal matrix* $V \in \mathbb{R}^{2N \times 2N}$ *such that*

$$V^T S V = \begin{bmatrix} I_{2N-k} & V_1 \\ 0_{k,\, 2N-k} & I_k \end{bmatrix}, \tag{2.4}$$

where $V_1 \in \mathbb{R}^{(2N-k)\times k}$ *is an orthogonal matrix*,

(2) $cond_2(S) = \dfrac{3+\sqrt{5}}{2}.$

Proof. (1) Suppose $v_1, v_2, ..., v_{2N-k}$ to be an orthogonal basis of $\ker(S-I)$, $V_0 = JU$. Let

$$V = [v_1, v_2, ..., v_{2N-k}, V_0] \text{ with } V_0 = [v_{2N-k+1}, ..., v_{2N}].$$

Then we have

$$V = [v_1, v_2, ..., v_{2N-k}, v_{2N-k+1}, ..., v_{2N}].$$

Using Theorem 7 of [8], it is easy to verify that the columns of V are orthogonal in pairs. So, they form an orthogonal basis of $\mathbb{R}^{2N}$. Thus, V is orthogonal. Since $v_j \in \ker(S-I)$, $\exists k_{i,j} \in \mathbb{R}$ such that

$$\sum_{i=1}^{2N-k} k_{ij}^2 = 1$$

and

$$v_j = k_{1j}v_1 + k_{2j}v_2 + \cdots + k_{2N-k,j}v_{2N-k}, \ \forall j = 2N-k+1, ..., 2N.$$

Then we have

$$\begin{aligned} SV &= [Sv_1, Sv_2, ..., Sv_{2N-k}, SV_0] \\ &= [v_1, v_2, ..., v_{2N-k}, JU + U] \text{ (because } U^T U = I_k) \\ &= [v_1, v_2, ..., v_{2N-k}, v_{2N-k+1}, ..., v_{2N}] \begin{pmatrix} I_{2N-k} & V_1 \\ 0_{k,2N-k} & I_k \end{pmatrix} \end{aligned}$$

$$\Rightarrow V^T SV = \begin{pmatrix} I_{2N-k} & V_1 \\ 0_{k,2N-k} & I_k \end{pmatrix}.$$

(2) Since

$$S^T S = V\begin{bmatrix} I_{2N-k} & V_1 \\ 0_{k,2N-k} & I_k \end{bmatrix}^T \begin{bmatrix} I_{2N-k} & V_1 \\ 0_{k,2N-k} & I_k \end{bmatrix} V^T$$

$$= V\begin{bmatrix} I_{2N-k} & V_1 \\ V_1^T & V_1^T V_1 + I_k \end{bmatrix} V^T$$

$$= V\begin{bmatrix} I_{2N-k} & V_1 \\ V_1^T & 2I_k \end{bmatrix} V^T \text{ (because } V_1^T V_1 = I_k\text{)},$$

we have

$$\det(S^T S - \lambda I) = \begin{vmatrix} I_{2N-k} & V_1 \\ V_1^T & 2I_k \end{vmatrix}$$

$$= (1-\lambda)^{2N-k} \det(I_{2N-k}) \det\left((2-\lambda) I_k - \frac{1}{1-\lambda} V^T V\right)$$

$$= (1-\lambda)^{2N-k} \det\left(\left(2 - \lambda - \frac{1}{1-\lambda}\right) I_k\right)$$

$$= (1-\lambda)^{2N-k} \left(\frac{(2-\lambda)(1-\lambda)-1}{1-\lambda}\right)^k \det(I_k)$$

$$= (1-\lambda)^{2N-2k} (\lambda^2 - 3\lambda + 1)^k.$$

Hence,

$$\det(S^T S - \lambda I) = 0 \Leftrightarrow 1 - \lambda = 0 \text{ or } \lambda^2 - 3\lambda + 1 = 0$$

$$\Leftrightarrow \lambda = 1 \text{ or } \lambda = \frac{3+\sqrt{5}}{2} \text{ or } \lambda = \frac{3-\sqrt{5}}{2}.$$

This implies that

$$cond_2(\widetilde{S}) = \sqrt{\frac{\lambda_{\max}(S^T S)}{\lambda_{\min}(S^T S)}} = \sqrt{\frac{\frac{3+\sqrt{5}}{2}}{\frac{3-\sqrt{5}}{2}}} = \frac{3+\sqrt{5}}{2}. \qquad \square$$

Remark 2.2. The symplectic matrices S and $I - u_j u_j^T J$ have the same condition number (see [8]).

3. Numerical Example

In this section, we present numerical examples to confirm our theoretical results. All experiments are done with MATLAB.R2014a.

Example 3.1. (1) Consider the matrix

$$\widetilde{U} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

of rank 2, whose columns generate an isotropic subspace.

- For $U = \widetilde{U}(:, 1)$, S has the following Jordan form:

$$\mathcal{J} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Its condition number is about 2.6180 and all its eigenvalues are equal to 1.

- For $U = \widetilde{U}$, the Jordan canonical of S becomes:

$$\mathcal{J} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Its condition number is also about 2.6180 and all its eigenvalues are equal to 1.

(2) In this example, we consider the matrix

$$\widetilde{U} = \begin{bmatrix} -0.4191 & -0.1870 & 0.5768 \\ -0.5076 & -0.0260 & -0.4870 \\ -0.3469 & -0.3085 & 0.2541 \\ -0.0189 & -0.6515 & 0.1787 \\ -0.4492 & -0.2222 & -0.5033 \\ -0.4941 & 0.6288 & 0.2834 \end{bmatrix}$$

of rank 3 whose columns belong to an isotropic subspace.

- For $U = \widetilde{U}(:, 1)$, the Jordan canonical form of S is given by:

$$\mathcal{J} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

- With $U = \widetilde{U}(:, 1:2)$, the Jordan form of S becomes:

$$\mathcal{J} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

- Finally, for $U = \widetilde{U}$, we have the following Jordan form:

$$\mathcal{J} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

We also observe that all eigenvalues of S are equal to 1 and its condition number is about 2.6180 for any perturbation U.

4. Concluding Remarks

In this paper, we proposed a generalization of the result of Qing-You [8]. To realize this work, we used matrices whose columns generate an isotropic subspace and obtained the following results:

The matrix studied is J-symplectic, and its Jordan form behaved as a rank-k perturbation of the matrix studied in [8]. It also admits a particular Schur form and its condition number is identical to that introduced in [8] and is about 2.6180.

In our future work, we will see how to apply this theory to some problems in dynamical systems, in physics and economics, particularly, in the study of population dynamics.

References

[1] R. Abraham and J. Marsden, Foundations of Mechanics, Second ed., Addison-Wesley, Reading, 1978.

[2] V. I. Arnold, Mathematical Methods in Classical Mechanics, Springer-Verlag, Berlin, 1978.

[3] Traoré G. Y. Arouna, M. Dosso and J. C. Koua Brou, On a perturbation theory of Hamiltonian systems with periodic coefficients, International Journal of Numerical Methods and Applications 17(2) (2018), 47-89.

[4] M. Dosso, Traoré G. Y. Arouna and J.-C. Koua Brou, On rank one perturbation of Hamiltonian system with periodic coefficients, WSEAS Trans. Math. 15 (2016), 502-510.

[5] G. Freiling, V. Mehrmann and H. Xu, Existence, uniqueness and parametrization of Lagrangian invariant subspaces, SIAM J. Matrix Anal. Appl. 23(4) (2002), 1045-1069.

[6] F. Poloni and N. Strabić, Principal pivot transforms of quasidefinite matrix and semidefinite Lagrangian subspaces, Journal of Linear Algebra 31 (2016), 200-231.

[7] V. A. Yakubovich and V. M. Starzhinskii, Linear Differential Equations with Periodic Coefficients, Vols. 1 & 2, Wiley, New York, 1975.

[8] Yan Qing-You, The properties of a kind of random symplectic matrices, Appl. Math. Mech. 23(5) (2002), 590-596.