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EXISTENCE OF CLASSICAL WEAKENED SOLUTION ON THE AXIS OF THIRD CENTRALLY SYMMETRIC MIXED PROBLEM FOR SECOND ORDER HOMOGENEOUS THREE-DIMENSIONAL GENERAL HYPERBOLIC EQUATION

Damébé KOLANI[*], **P. Komivi SILIADIN, Lamedjogue LABITE and Kokou TCHARIE**

Department of Mathematics
University of Lomé (TOGO)
BP: 1515, Lomé, TOGO
e-mail: kolanida@gmail.com
psilia1705@gmail.com
lamedjogue1@gmail.com
tkokou09@yahoo.fr

Abstract

We prove the existence of the classical weakened solution on the axis

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[*] Corresponding author

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of the third mixed problem with central symmetry for the general homogeneous three-dimensional hyperbolic equation of the second order with minimal conditions on the initial data.

1. Introduction and Position of the Problem

In the cylinder $\overline{P} = \overline{G} \times [0, T]$ with $G = \{x \in \mathbb{R}^3 / |x| = r < R\}$, we consider the following mixed problem:

$$\frac{\partial^2 u}{\partial t^2} + \sum_{i=1}^{3} a_i(x, t) \frac{\partial^2 u}{\partial x_i \partial t} - \Delta u(x, t) + \sum_{i=1}^{3} b_i(x, t) \frac{\partial u}{\partial x_i}$$

$$+ c(x, t) \frac{\partial u}{\partial t} + q(x, t) u = 0, \tag{1.1}$$

with the initial conditions

$$\begin{cases} u(x, 0) = \varphi(|x|) \\ \dfrac{\partial u(x, 0)}{\partial t} = \psi(|x|), \end{cases} \tag{1.2}$$

and the third type boundary condition

$$\left(\frac{\partial u(x, t)}{\partial n} + \frac{1}{|x|} u(x, t) \right) \Bigg|_{\Gamma} = 0, \tag{1.3}$$

where φ and ψ are defined over the entire ball $\overline{G}$, $\frac{\partial}{\partial n}$ is the derivative at the point (x, t) along the external normal n of the lateral surface

$$\Gamma = \{(x, t) \in \overline{P} : |x| = R, 0 \le t \le T\}$$

of the closed cylinder $\overline{P}$.

Because of the central symmetry, the normal derivative is equal to the radial differential operator, i.e., $\frac{\partial}{\partial n} = \frac{\partial}{\partial r}$ with $|x| = r$.

It is assumed that the coefficients of the equation (1.1) are real and continuous in the closed cylinder $\overline{P}$ and that their first derivatives

$\frac{\partial a_i(x, t)}{\partial x}, \frac{\partial b_i(x, t)}{\partial x}, \frac{\partial c(x, t)}{\partial x}$ and $\frac{\partial q(x, t)}{\partial x}$ are bounded in $\overline{P}$. In other words,

$$\begin{cases} a_i, b_i, c, q \in C(\overline{P}), \\ \frac{\partial a_i(x, t)}{\partial x}, \frac{\partial b_i(x, t)}{\partial x}, \frac{\partial c(x, t)}{\partial x} \text{ and } \frac{\partial q(x, t)}{\partial x} \in L_\infty(\overline{P}). \end{cases} \tag{1.4}$$

The nature of the central symmetry manifests itself in the coefficients of the equation (1.1) as follows:

$$\begin{cases} a_i(x, t) = x_i a(r, t), \\ b_i(x, t) = x_i b(r, t), \\ c(x, t) = c(r, t), \\ q(x, t) = q(r, t). \end{cases} \tag{1.5}$$

The coefficients $a_i(x, t)$ and $b_i(x, t)$ obey the reconciliation condition at the origin of the axis of symmetry of the equation (1.1) and the boundary conditions of the study area (1.3). In other words,

$$\begin{cases} \sum_{i=1}^{3} a_i(x, t)|_\Gamma = \sum_{i=1}^{3} b_i(x, t)|_\Gamma = 0, & 0 \le t \le T \\ \sum_{i=1}^{3} a_i(x, t)|_{|x|=0} = \sum_{i=1}^{3} b_i(x, t)|_{|x|=0} = 0, & 0 \le t \le T \end{cases} \tag{1.6}$$

with the variables (r, t), the differential properties of the coefficients of the equation are expressed as follows:

$$\begin{cases} a(r, t), b(r, t), c(r, t), q(r, t) \in C(\overline{Q}) \\ \frac{\partial a(r, t)}{\partial r}, \frac{\partial b(r, t)}{\partial r}, \frac{\partial c(r, t)}{\partial r}, \frac{\partial q(r, t)}{\partial r} \in L_\infty(\overline{Q}) \text{ with } Q = (0, R) \times (0, T). \end{cases} \tag{1.7}$$

For the homogeneous general hyperbolic equation (1.1), let us pose the following problem: determine the function $u(x, t)$ belonging to the class $C^2_{\{r=0\}}(\overline{P})$ which transforms the equation (1.1) into an identity in $P \backslash \{0\}$

$\times [0, T]$ checking the initial conditions (1.2) and the boundary condition (1.3) on the side surface Γ.

Based on the requirement on the unknown function $u(x, t)$, lwe give the following definition.

Definition 1. We call *weakened classical solution* on the axis $r = 0$ of the mixed problem (1.1), (1.2), (1.3), function $u(x, t) \in C^2_{\{r=0\}}(\overline{P})$ transforming the equation (1.1) into an identity in the cylinder from which we exclude the axis $P\backslash\{0\} \times [0, T]$ and satisfying the conditions (1.2), (1.3) in the usual sense.

Such a definition makes it possible to formulate the posed problem for the equation (1.1) in a more laconic way: find the weakened classical solution on the axis $r = 0$ of the mixed problem (1.1), (1.2), (1.3).

This formulation had been adopted by Yashkin and Yurchuk in (see [13]) for a similar problem with a second-order hyperbolic equation not containing the mixed term $\sum_{i=1}^{3} a_i(x, t)\frac{\partial^2 u}{\partial x_i \partial t}$. In 2020, Siliadin et al. in the work [7], demonstrated the existence and uniqueness of the classical solution of the first mixed problem for the equation (1.1) containing this mixed term.

The objective of the present work is to extend the results of the article [7] on the third mixed problem for the equation (1.1), which is a more general equation than all those that have been studied previously on this topic. The novelty of our work lies in the fact that not only do we study the equation containing the mixed term $\sum_{i=1}^{3} a_i(x, t)\frac{\partial^2 u}{\partial x_i \partial t}$ but also and most importantly, we determine the necessary and sufficient conditions for the existence of the classical weakened solution on the axis for the third mixed problem of a three-dimensional hyperbolic equation of the second order with central symmetry. For this type of problem, this kind of question had never been asked before. Our work aims at extending and complementing the results of the work of Yashkin, Yurchuk, Siliadin, Tcharie and others.

To solve the problem thus posed, we must define the necessary and sufficient conditions to be imposed on the initial data φ and ψ so that the solution $u(x, t)$ of the third mixed problem (1.1), (1.2), (1.3) is classical everywhere but weakened on the axis of symmetry.

2. Necessary and Sufficient Conditions for the Existence of Weakened Classical Solution on the Axis of the Third Centrally Symmetric Mixed Problem for Second-order Three-dimensional Homogeneous General Hyperbolic Equation

We consider in the cylinder $\overline{P} = \overline{G} \times [0, T],$ the mixed problem (1.1), (1.2), (1.3). Passing to spherical coordinates, our problem reduces to the following mixed problem whose equation depends on a single space variable with Bessel operator in the main part:

$$\frac{\partial^2 u}{\partial t^2} + ra(r, t)\frac{\partial^2 u}{\partial r \partial t} - \left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right) + rb(r, t)\frac{\partial u}{\partial r}$$

$$+ c(r, t)\frac{\partial u}{\partial t} + q(r, t)u = 0, \ (r, t) \in Q, \tag{2.1}$$

with the initial conditions

$$\begin{cases} u(r, 0) = \varphi(r) \\ \dfrac{\partial u(r, 0)}{\partial t} = \psi(r), \quad 0 \le r \le R, \end{cases} \tag{2.2}$$

and the third type boundary condition

$$\left.\left(\frac{\partial u(r, t)}{\partial r} + \frac{1}{r}u(r, t)\right)\right|_{r=R} = 0, \quad 0 \le t \le T, \tag{2.3}$$

(2.1) is an equation of the one-dimensional hyperbolic type of the second order without right-hand side.

It is obvious that the problems (1.1), (1.2), (1.3) and (2.1), (2.2), (2.3) are equivalent everywhere in the defined space except on the axis $|x| = r = 0,$

because the functional definition of the change in spherical coordinates is equal to zero in the case of central symmetry, only for $r = 0$.

Remark 1. If we were looking for the solution of the problem (1.1), (1.2), (1.3) belonging to the class $C^2_{\{r=0\}}(\overline{P})$ in the sense of Definition 1, then it would be sufficient to consider one of the following conditions:

$$\lim_{|x|\to 0} |x|\Delta u(x, t) = 0, \tag{2.4}$$

$$\lim_{|x|\to 0} |x|\frac{\partial^2 u}{\partial t^2}(x, t) = 0. \tag{2.5}$$

For the following, we use the condition (2.4).

If the function $u(x, t)$ is a classical solution (weakened or usual) of the problem (1.1), (1.2) and

$$u(x, t)|_{\Gamma} = 0, \tag{2.6}$$

then from the equation (1.1), from the condition (2.6) as well as from the

$$\sum_{i=1}^{3} b_i(x, t)|_{\Gamma} = 0, \quad 0 \le t \le T, \tag{2.7}$$

by passing to the limit, we have the conciliation condition

$$\Delta u|_{\Gamma} = 0. \tag{2.8}$$

According to Definition 1, the solution of the singular problem (2.1), (2.2), (2.3) precisely the function $u(r, t)$ which transforms the equation (2.1) into an identity in Q and satisfying (2.2), (2.3) is sought in the class of functions $u(r, t) \in C^1(\overline{Q}) \cap C^2((0, R] \times [0, T])$, for which the following limiting equalities are true:

$$\lim_{r\to 0} r\left(\frac{\partial^2 u(r, t)}{\partial r^2} + \frac{2}{r}\frac{\partial u(r, t)}{\partial r}\right) = 0, \tag{2.9}$$

$$\lim_{r\to 0} r\frac{\partial^2 u}{\partial r\partial t}(r,\ t) = 0. \tag{2.10}$$

Moreover, the solution of the problem (2.1)-(2.3) must respect the condition (2.8) which for the variables *r* and *t* is of the form

$$\lim_{r\to R}\left(\frac{\partial^2 u(r,\ t)}{\partial r^2} + \frac{2}{r}\frac{\partial u(r,\ t)}{\partial r}\right) = 0, \tag{2.11}$$

$$\lim_{r\to R}\frac{\partial^2 u(r,\ t)}{\partial t^2} = 0. \tag{2.12}$$

The conditions (2.8), (2.11) and (2.12) are additional reconciliation conditions.

For the third mixed problem (1.1), (1.2), (1.3), we formulate and prove the existence theorem of the classical solution in the sense of Definition 1 with the minimal conditions on the initial functions $\varphi(|x|)$ and $\psi(|x|)$.

Theorem. *For the classical weakened solution* $u(x,\ t)$ *on the axis* $r = 0$ *of the third mixed problem* (1.1), (1.2), (1.3) *to exist, there is necessary and sufficient that the initial functions* φ *and* ψ *satisfy the conditions*

$$\varphi(r) \in C^1[0,\ R] \cap C^2(0,\ R],\ \lim_{r\to 0} r\Delta\varphi(r) = 0,\ \lim_{r\to R}\left(\frac{d\varphi(r)}{dr} + \frac{1}{r}\varphi(r)\right) = 0, \tag{2.13}$$

$$\psi(r) \in C[0,\ R] \cap C^1(0,\ R],\ \lim_{r\to 0} r\frac{d\psi(r)}{dr} = 0,\ \lim_{r\to R}\left(\frac{d\psi(r)}{dr} + \frac{1}{r}\psi(r)\right) = 0, \tag{2.14}$$

when the coefficients of the equation (1.1) *satisfy the conditions* (1.5) *and* (1.6).

Demonstration

- **Need**

Suppose there is a weakened classical solution $u(x,\ t)$ on the axis $r = 0$

of the problem (1.1), (1.2), (1.3). Then according to (see [13, p. 23]) the function $u(r, t) \in C^1(\overline{Q}) \cap C^2((0, R] \times [0, T])$ checks the problem (2.1), (2.2), (2.3), the weakened conditions (2.9), (2.10) as well as the conditions of reconciliation (2.11), (2.12).

Thus, as $u(r, t) \in C^1(\overline{Q}) \cap C^2((0, R] \times [0, T])$, is solution of mixed problem (2.1), (2.2), (2.3), it follows immediately from the relations (2.2) that the initial functions φ and ψ have the following differential properties:

$$\varphi(r) \in C^1[0, R] \cap C^2(0, R], \tag{2.15}$$

$$\psi(r) \in C[0, R] \cap C^1(0, R]. \tag{2.16}$$

From equality (2.3), we have

$$0 = \lim_{r \to R} \lim_{t \to 0} \left[\frac{\partial u}{\partial r}(r, t) + \frac{1}{r} u(r, t) \right] = \lim_{r \to R} \left(\frac{d\varphi}{dr}(r) + \frac{1}{r} \varphi(r) \right).$$

From (2.11), we can write

$$\begin{aligned} 0 &= \lim_{r \to R} \left(\frac{\partial^2 u(r, t)}{\partial r^2} + \frac{2}{r} \frac{\partial u(r, t)}{\partial r} \right) \\ &= \lim_{r \to R} \lim_{t \to T} \left(\frac{\partial^2 u(r, t)}{\partial r^2} + \frac{2}{r} \frac{\partial u(r, t)}{\partial r} \right) \\ &= \lim_{r \to R} \left(\frac{d^2 \varphi(r)}{dr^2} + \frac{2}{r} \frac{d\varphi(r)}{dr} \right) \\ &= \lim_{r \to R} \Delta\varphi(r) \\ &= \Delta\varphi(R). \end{aligned}$$

The results $\lim_{r \to 0} r \Delta\varphi(r) = 0$ and $\lim_{r \to 0} r \frac{d\psi(r)}{dr} = 0$ are respectively obtained by rewriting the weakened conditions (2.9) and (2.10).

It follows that the initial functions φ and ψ are respectively the elements of the class of functions $C^1[0, R] \cap C^2(0, R]$ and $C[0, R] \cap C^1(0, R]$ and we have the reconciliation balance conditions on the axis $r = 0$,

$$\lim_{r\to 0} r\left(\frac{d^2\varphi(r)}{dr^2} + \frac{2}{r}\frac{d\varphi(r)}{dr}\right) = 0, \ \lim_{r\to 0} r\frac{d\psi(r)}{dr} = 0.$$

Passing to the limit when $t \to 0$, of the boundary condition (2.3), we obtain

$$0 = \lim_{r\to R}\lim_{t\to 0}\left(\frac{\partial u(r, t)}{\partial r} + \frac{1}{r}u(r, t)\right)$$

$$= \lim_{r\to R}\left(\frac{d\varphi(r)}{dr} + \frac{1}{r}\varphi(r)\right).$$

Like $u(r, t) \in C^1(\overline{Q}) \cap C^2((0, R] \times [0, T])$, the functions $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial r\partial t}, \frac{\partial^2 u}{\partial t\partial r}$ are continuous in the rectangle $(0, R] \times [0, T]$. Therefore, the mixed derivatives are equal on the segment $r = R, 0 \le t \le T$, i.e., $\frac{\partial^2 u}{\partial r\partial t} = \frac{\partial^2 u}{\partial t\partial r}$. Then the limit when $t \to 0$ taken in the differentiation according to t of the boundary condition (2.3) gives

$$0 = \lim_{r\to R}\lim_{t\to 0}\left[\frac{\partial^2 u}{\partial t\partial r}(r, t) + \frac{1}{r}\frac{\partial u}{\partial t}(r, t)\right] = \lim_{r\to R}\left(\frac{d\psi}{dr}(r) + \frac{1}{r}\psi(r)\right).$$

Thus, the initial functions $\varphi(r)$ and $\psi(r)$ satisfy all the requirements of (2.13) and (2.14). In other words, it has been shown that the conditions (2.13) and (2.14) are necessary conditions for the existence of the weakened classical solution on the axis $r = 0$ of the mixed problem (1.1), (1.2), (1.3). □

- **Sufficiency**

The change of variables

$$v(r, t) = ru(r, t) \tag{2.17}$$

of the unknown function u of the mixed problem (2.1), (2.2), (2.3) gives the auxiliary problem not containing the singularity with the boundary condition of the first type on the limit $r = 0$ and the boundary condition of the second type on the boundary $r = R$. This is the problem

$$\begin{aligned} \mathscr{L}v(r, t) \equiv \frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial r^2} + A(r, t)\frac{\partial^2 v}{\partial r \partial t} + B(r, t)\frac{\partial v}{\partial r} + C(r, t)\frac{\partial v}{\partial t} \\ + D(r, t)v = 0, \quad (r, t) \in Q = \,]0, R[\, \times \,]0, T[\end{aligned} \tag{2.18}$$

with the initial conditions

$$v(r, 0) = \Phi(r), \quad \frac{\partial v}{\partial t}(r, 0) = \Psi(r), \quad 0 \le r \le R, \tag{2.19}$$

and the boundary conditions of the first and second kind respectively

$$v(0, t) = 0, \quad \frac{\partial v}{\partial r}(R, t) = 0, \quad 0 \le t \le T. \tag{2.20}$$

In the problem (2.18)-(2.20), we designated the functions $A(r, t)$, $B(r, t)$, $C(r, t)$ and $D(r, t)$ by

$$\begin{cases} A(r, t) = ra(r, t), \\ B(r, t) = rb(r, t), \\ C(r, t) = c(r, t) - a(r, t), \\ D(r, t) = q(r, t) - b(r, t). \end{cases} \tag{2.21}$$

The initial functions Φ and Ψ are defined by

$$\begin{cases} \Phi(r) = r\varphi(r), \\ \Psi(r) = r\psi(r), \quad 0 \le r \le R. \end{cases} \tag{2.22}$$

The first equality of the boundary conditions (2.20) follows from the form of the change of variables (2.17) and from the fact that $u(r, t)$ is assumed to be bounded at the point $r = 0$. The second equality of the boundary conditions (2.20) follows from the form of the change (2.17) and from the boundary condition of the third type (2.3).

From the equalities (2.22), it follows from the conditions (2.13) and (2.14) that the initial functions Φ and Ψ according to the article [4] have the properties following:

$$\Phi \in C^2[0;\, R], \quad \Phi(0) = \frac{d^2\Phi(0)}{dr^2} = 0, \quad \frac{d\Phi(R)}{dr} = 0, \tag{2.23}$$

$$\Psi \in C^1[0;\, R], \quad \Psi(0) = \frac{d\Psi(R)}{dr} = 0. \tag{2.24}$$

Based on (1.4), we can also assume that the functions

$$A,\ B,\ C,\ D,\ \frac{\partial A}{\partial r},\ \frac{\partial A}{\partial t}$$

are continuous in $\overline{Q} = [0,\, R] \times [0,\, T]$, the functions

$$\frac{\partial^2 A}{\partial t^2},\ \frac{\partial B}{\partial r},\ \frac{\partial C}{\partial r},\ \frac{\partial D}{\partial r}$$

are bounded in $\overline{Q}$ and satisfy the conditions for reconciling the coefficients

$$\begin{cases} A(0,\, t) = \dfrac{\partial A(R,\, t)}{\partial r} = 0, \\ B(0,\, t) = \dfrac{\partial B(R,\, t)}{\partial r} = 0, \end{cases} \tag{2.25}$$

which follows from (1.6).

In the work [7], it was demonstrated that with the conditions verified and imposed on the coefficients of the equation (2.18), as well as on the initial functions Φ and Ψ, it exists a unique classical solution of the mixed problem (2.18), (2.19), (2.20) of the form

$$v(r, t) = \frac{\widetilde{\Phi}(h_2(g_2(r, t), 0)) + \widetilde{\Phi}(h_1(g_1(r, t), 0))}{2}$$

$$+ \frac{1}{2}\int_{h_1(g_1(r,t),0)}^{h_2(g_2(r,t),0)} \frac{\widetilde{\Psi}(\xi) + \frac{\widetilde{A}}{2}(\xi, 0)\phi'(\xi)}{\sqrt{\frac{\widetilde{A}^2}{4}(\xi, 0) + 1}} d\xi$$

$$+ \frac{1}{2}\int_0^t \int_{h_1(g_1(r,t),\xi)}^{h_2(g_2(r,t),\xi)} \frac{\widetilde{F}(\xi, \tau)}{\sqrt{\frac{\widetilde{A}^2}{4}(\xi, \tau) + 1}} d\xi d\tau \tag{2.26}$$

with

$$\widetilde{F} = \left[\left(\frac{\partial}{\partial t} + \left(\frac{\widetilde{A}}{2} - \sqrt{\frac{\widetilde{A}^2}{4} + 1}\right)\frac{\partial}{\partial r}\right)\left(\frac{\widetilde{A}}{2} + \sqrt{\frac{\widetilde{A}^2}{4} + 1}\right)\frac{\partial \widetilde{v}}{\partial r}\right]$$

$$+ \left[-\left(\left(\frac{\partial}{\partial t} - \frac{\widetilde{A}}{2}\frac{\partial}{\partial r}\right)\ln\sqrt{\frac{\widetilde{A}^2}{4} + 1}\right) + \frac{1}{2}\frac{\partial \widetilde{A}}{\partial r}\right]$$

$$\times \left[\frac{\partial \widetilde{v}}{\partial t} + \left(\frac{\widetilde{A}}{2} + \sqrt{\frac{\widetilde{A}^2}{4} + 1}\right)\frac{\partial \widetilde{v}}{\partial r}\right] - \widetilde{C}\frac{\partial \widetilde{v}}{\partial t} - \widetilde{B}\frac{\partial \widetilde{v}}{\partial r} - \widetilde{D}\widetilde{v}. \tag{2.27}$$

In (2.26), $\widetilde{\Phi}$, $\widetilde{\Psi}$ designate the respective extensions of the functions Φ and Ψ of the segment $[0, R]$ on real axis $\mathbb{R}$ in the following way: first, we do an odd extension of these functions from $[0, R]$ on $[-R, 0]$ and then periodically with period $2R$ on real axis $\mathbb{R}$.

In the formula (2.27), we have designated by $\widetilde{v}$, $\widetilde{A}$, the respective extensions with respect to r of the functions v, A of $\overline{Q}$ over $\mathbb{R} \times [0, T]$.

It follows from the conditions (2.23), (2.24) (see [13, p. 32]) that

$$\widetilde{\Phi} \in C^2(\mathbb{R}) \text{ and } \widetilde{\Psi} \in C^1(\mathbb{R}), \tag{2.28}$$

$$\widetilde{\Phi}(R) = \frac{d^2\widetilde{\Phi}(R)}{dr^2} = \widetilde{\Psi}(R). \tag{2.29}$$

We designate by $\widetilde{C}$ and $\widetilde{D}$, the even extensions following the argument r of $\overline{Q}$ on $R \times [0, T]$ respectively of the functions C and D first on $[-R, 0]$, then periodically after period $2R$ on $\mathbb{R}$. By $\widetilde{B}$, the extension of the coefficient B obtained in the following way: we first extend the coefficient $b(r, t)$ following r in an even way of $[0, R]$ on $[-R, 0]$, then periodically after period $2R$ following r of $[-R, R] \times [0, T]$ on $\mathbb{R} \times [0, T]$ and we note this extension $\widetilde{b}$. Similarly $\widetilde{r}$ is the extension of r. This function $\widetilde{B} = \widetilde{r}\widetilde{b}(r, t)$ is piecewise continuous on $\mathbb{R} \times [0, T]$ and admits a derivative along r continues bit by bit on $\mathbb{R} \times [0, T]$.

Also from (2.25), we have (see [13, p. 32]):

$$\widetilde{B}(r, t) = 0, \text{ for } r = R; \tag{2.30}$$

$$\widetilde{B}(r, t) = \frac{\partial \widetilde{B}(r, t)}{\partial r} = 0 \text{ for } r = R. \tag{2.31}$$

So it is clear that $\widetilde{A}, \widetilde{B}, \widetilde{C}$ and $\widetilde{D} \in C(\mathbb{R} \times [0, T])$ and $\frac{\partial \widetilde{A}}{\partial r}, \frac{\partial \widetilde{C}}{\partial r}, \frac{\partial \widetilde{D}}{\partial r}$ are bounded in $\mathbb{R} \times [0, T]$. Likewise, $\widetilde{\Phi} \in C^2(\mathbb{R}), \widetilde{\Psi} \in C^1(\mathbb{R})$. Thus, using the change of variables (2.17), we obtain that the mixed problem (2.1), (2.2), (2.3) admits a solution of the form

$$u(r, t) = \frac{\widetilde{\Phi}(H_2(r, t)) + \widetilde{\Phi}(H_1(r, t))}{2r} + \frac{1}{2r}\int_{H_1(r,t)}^{H_2(r,t)} \widetilde{\chi}(\xi) d\xi$$

$$+ \frac{1}{2r}\int_0^t \int_{H_1(r,t-\tau)}^{H_2(r,t-\tau)} \widetilde{\mathbb{F}}(\xi, \tau) d\xi d\tau \in C^2((0, R] \times [0, T]), \tag{2.32}$$

with

$$\begin{cases} H_1(r, t) = h_1(g_1(r, t), 0), \\ H_2(r, t) = h_2(g_2(r, t), 0), \\ \widetilde{\chi}(\xi) = \dfrac{\widetilde{\Psi}(\xi) + \dfrac{\widetilde{A}}{2}(\xi, 0)\widetilde{\Phi}'(\xi)}{\sqrt{\dfrac{\widetilde{A}^2}{4}(\xi, 0) + 1}}, \\ \widetilde{\mathbb{F}}(\xi, \tau) = \dfrac{\widetilde{F}(\xi, \tau)}{\sqrt{\dfrac{\widetilde{A}^2}{4}(\xi, \tau) + 1}} \end{cases} \tag{2.33}$$

and

$$\begin{cases} \dfrac{\partial H_1}{\partial r} = \dfrac{\partial H_2}{\partial r} = 1, \\ \dfrac{\partial H_2}{\partial t} = \dfrac{\widetilde{A}}{2} + \sqrt{\dfrac{\widetilde{A}^2}{4} + 1}, \\ \dfrac{\partial H_1}{\partial t} = \dfrac{\widetilde{A}}{2} - \sqrt{\dfrac{\widetilde{A}^2}{4} + 1}. \end{cases} \tag{2.34}$$

From the nature of the extensions of the coefficients of the equation (2.18), of the initial functions and the relations (2.33), we deduce that

$$\begin{cases} \widetilde{\mathbb{F}}(\xi, \tau) \in C(\mathbb{R} \times [0, T]), \\ \widetilde{\mathbb{F}}(R, t) = 0. \end{cases} \tag{2.35}$$

We show that the function $u(r, t)$ defined by the formula (2.32) is a weakened classical solution on the axis $r = 0$ of the problem (2.1), (2.2), (2.3), i.e.,

$$u(r, t) \in C^1(\overline{Q}) \cap C^2((0, R] \times [0, T])$$

transforming the equation (2.1) into an identity satisfying the initial conditions (2.2), the boundary conditions (2.3), the conciliation conditions (2.11) and (2.12) as well as the weakened conditions (2.9), (2.10).

In [7], it was shown that $v \in C^2(\overline{Q})$ and that $u = \dfrac{v}{r}$ according to (2.17), we have for $r \neq 0$, $u(r, t) \in C^2((0, R] \times [0, T])$.

We now study the regularity of the solution $u(r, t)$ defined by the formula (2.32) when $r \to 0$,

$$\lim_{r\to 0} u(r, t) = \lim_{r\to 0} \frac{\widetilde{\Phi}(H_2(r, t)) + \widetilde{\Phi}(H_1(r, t))}{2r}$$

$$+ \lim_{r\to 0} \frac{1}{2r} \int_{H_1(r,t)}^{H_2(r,t)} \widetilde{\chi}(\xi) d\xi$$

$$+ \lim_{r\to 0} \frac{1}{2r} \int_0^t \int_{H_1(r,t-\tau)}^{H_2(r,t-\tau)} \widetilde{\mathbb{F}}(\xi, \tau) d\xi d\tau.$$

By applying the Hospital theorem, while taking into account the procedure for extending the functions $\widetilde{\Phi}$ and $\widetilde{\Psi}$, we have

$$\lim_{r\to 0} u(r, t) = \lim_{r\to 0} \frac{1}{2} \left\{ \frac{d\widetilde{\Phi}(H_2(r, t))}{dH_2(r, t)} + \frac{d\widetilde{\Phi}(H_1(r, t))}{dH_1(r, t)} \right\}$$

$$+ \lim_{r\to 0} \frac{1}{2} \{\widetilde{\chi}(H_2(r, t)) - \widetilde{\chi}(H_1(r, t))\}$$

$$+ \lim_{r\to 0} \frac{1}{2} \int_0^t [\widetilde{\mathbb{F}}(H_2(r, t-\tau), \tau) - \widetilde{\mathbb{F}}(H_1(r, t-\tau), \tau)] d\tau$$

$$= \lim_{r\to 0} \frac{1}{2} \frac{d\widetilde{\Phi}(H_2(r, t))}{dH_2(r, t)} + \lim_{r\to 0} \frac{1}{2} \frac{d\widetilde{\Phi}(H_1(r, t))}{dH_1(r, t)}$$

$$+ \frac{1}{2} \{\widetilde{\chi}(H_2(0, t)) - \widetilde{\chi}(H_1(0, t))\}$$

$$+ \frac{1}{2} \int_0^t [\widetilde{\mathbb{F}}(H_2(0, t-\tau), \tau) - \widetilde{\mathbb{F}}(H_1(0, t-\tau), \tau)] d\tau \qquad (2.36)$$

since at the end of the extension procedure of the function $\widetilde{\Phi}$, it is odd according to the first argument *r* (see Chapter 2 Section 4.2; [12, pp. 32-33]).

As the first and the second terms of the right-hand side of the equality (2.36) belong to $C^1(\overline{Q})$ according to (2.23), the second term also belongs to

$C^1(\overline{Q})$ according to (2.23) and (2.24) and the third term belongs to $C^1(\overline{Q})$, we deduce that $u(r, t)$ belongs to $C^1(\overline{Q}) \cap C^2((0, R] \times [0, T])$. From the formula (2.32), we proceed to the calculations in turn of the terms $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial r}$, $\frac{\partial^2 u}{\partial t^2}$, $\frac{\partial^2 u}{\partial r \partial t}$ and $\frac{\partial^2 u}{\partial r^2}$ and we get

$$\begin{aligned}\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} = \frac{1}{2r^3}\Bigg[& r^2\left(\frac{d^2\widetilde{\Phi}(H_2(r, t))}{dH_2^2(r, t)} + \frac{d^2\widetilde{\Phi}(H_1(r, t))}{dH_1^2(r, t)}\right) \\ & + r^2\left(\frac{d\widetilde{\chi}(H_2(r, t))}{dH_2(r, t)} - \frac{d\widetilde{\chi}(H_1(r, t))}{dH_1(r, t)}\right) \\ & + r^2\frac{\partial}{\partial r}\int_0^t (\widetilde{\mathbb{F}}(H_2(r, t-\tau), \tau) - \widetilde{\mathbb{F}}(H_1(r, t-\tau), \tau))d\tau\Bigg]. \end{aligned} \tag{2.37}$$

Basing on the formula (2.25), we notice that in the neighborhoods $(r \to 0)$ and $(r \to R)$, the characteristic equations

$$\begin{cases} \frac{dr}{dt} = \frac{A}{2} + \sqrt{1 + \frac{A^2}{4}} \\ \frac{dr}{dt} = \frac{A}{2} - \sqrt{1 + \frac{A^2}{4}} \end{cases} \tag{2.38}$$

of the equation (2.18) take the following simplified form:

$$\begin{cases} \frac{dr}{dt} = 1 \\ \frac{dr}{dt} = -1. \end{cases} \tag{2.39}$$

We can deduce from it that

$$\begin{cases} H_2(r, t) = r + t \\ H_1(r, t) = r - t \end{cases} \tag{2.40}$$

and when we find ourselves in the neighborhoods $(r \to 0)$ and $(r \to R)$, (2.37) becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} = \frac{1}{2r}\left[\left(\frac{d^2\widetilde{\Phi}(r+t)}{d(r+t)^2} + \frac{d^2\widetilde{\Phi}(r-t)}{d(r-t)^2}\right) + \left(\frac{d\widetilde{\chi}(r+t)}{d(r+t)} - \frac{d\widetilde{\chi}(r-t)}{d(r-t)}\right) + \frac{\partial}{\partial r}\int_0^t (\widetilde{\mathbb{F}}(r+(t-\tau), \tau) - \widetilde{\mathbb{F}}(r-(t-\tau), \tau))d\tau\right]. \quad (2.41)$$

According to the nature of the extensions of the functions $\widetilde{\Phi}$ and $\widetilde{\Psi}$ (see Chapter 2 Section 4.2; [12, pp. 32-33]), $\frac{d^2\widetilde{\Phi}(\xi)}{d\xi^2}$ is odd, $\frac{d\widetilde{\chi}(\xi)}{d\xi}$ is even and $\frac{\partial\widetilde{\mathbb{F}}}{\partial\xi}$ is even with respect to the variable ξ.

Passing to the limit when $r \to R$, we have

$$\lim_{r \to R} \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} = 0. \quad (2.42)$$

Which verifies the condition (2.11).

Also from equality (2.41), we have

$$r\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right) = \frac{1}{2}\left[\left(\frac{d^2\widetilde{\Phi}(r+t)}{d(r+t)^2} + \frac{d^2\widetilde{\Phi}(r-t)}{d(r-t)^2}\right) + \left(\frac{d\widetilde{\chi}(r+t)}{d(r+t)} - \frac{d\widetilde{\chi}(r-t)}{d(r-t)}\right) + \frac{\partial}{\partial r}\int_0^t (\widetilde{\mathbb{F}}(r+(t-\tau), \tau) - \widetilde{\mathbb{F}}(r-(t-\tau), \tau))d\tau\right]. \quad (2.43)$$

Passing to the limit when $r \to 0$ in (2.43), we get

$$\begin{aligned}\lim_{r\to 0} r\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right) &= \frac{1}{2}\left[\left(\frac{d^2\tilde{\Phi}(t)}{d(t)^2} + \frac{d^2\tilde{\Phi}(-t)}{d(-t)^2}\right) + \left(\frac{d\tilde{\chi}(+t)}{d(+t)} - \frac{d\tilde{\chi}(-t)}{d(-t)}\right)\right.\\ &\left.\quad + \int_0^t \left(\frac{\partial\tilde{\mathbb{F}}((t-\tau), \tau)}{\partial(t-\tau)} - \frac{\partial\tilde{\mathbb{F}}(-(t-\tau), \tau)}{\partial(-(t-\tau))}\right)d\tau\right] = 0. \quad (2.44)\end{aligned}$$

This proves that the condition (2.9) is verified.

Also basing on (2.40), in the neighborhoods $r \to 0$ and $r \to R$, we obtain the equality

$$\begin{aligned}\frac{\partial^2 u}{\partial r\partial t} = \frac{1}{2r^2}&\left[r\left(\frac{d^2\tilde{\Phi}(r+t)}{d(r+t)^2} - \frac{d^2\tilde{\Phi}(r-t)}{d(r-t)^2}\right) - \left(\frac{d\tilde{\Phi}(r+t)}{d(r+t)} + \frac{d\tilde{\Phi}(r-t)}{d(r-t)}\right)\right.\\ &+ r\left(\frac{d\tilde{\chi}(r+t)}{d(r+t)} + \frac{d\tilde{\chi}(r-t)}{d(r-t)}\right) - (\tilde{\chi}(r+t) - \tilde{\chi}(r-t))\\ &+ r\frac{\partial}{\partial r}\int_0^t (\tilde{\mathbb{F}}(r+(t-\tau), \tau) - \mathbb{F}(r-(t-\tau), \tau))d\tau\\ &\left. - \int_0^t (\tilde{\mathbb{F}}(r+(t-\tau), \tau) - \tilde{\mathbb{F}}(r-(t-\tau), \tau))d\tau\right]. \qquad (2.45)\end{aligned}$$

Using Taylor's formula with remainder in Peano form of functions that lie in terms of this last equality $\left(\text{i.e., } g(\pm rt) = g(\pm t) + \frac{dg(\pm t)}{d(\pm t)} r + o(r)\right)$, we have

$$\begin{aligned}\lim_{r\to 0} r\frac{\partial^2 u}{\partial r\partial t} = \lim_{r\to 0}\frac{1}{2r}&\left[r\left(\frac{d^2\tilde{\Phi}(r+t)}{d(r+t)^2} - \frac{d^2\tilde{\Phi}(r-t)}{d(r-t)^2}\right)\right.\\ &- \left(\frac{d\tilde{\Phi}(t)}{d(t)} + \frac{d^2\tilde{\Phi}(t)}{d(t)^2}r - \frac{d\tilde{\Phi}(-t)}{d(-t)} - \frac{d^2\tilde{\Phi}(-t)}{d(-t)^2}r + 0(r)\right)\\ &+ r\left(\frac{d\tilde{\chi}(r+t)}{d(r+t)} + \frac{d\tilde{\chi}(r-t)}{d(r-t)}\right)\end{aligned}$$

$$- \left(\widetilde{\chi}(t) + \frac{d\widetilde{\chi}(t)}{dt} r - \widetilde{\chi}(-t) - \frac{d\widetilde{\chi}(-t)}{d(-t)} r + 0(r) \right)$$

$$+ r \int_0^t \left[\frac{\partial \widetilde{\mathbb{F}}(r + (t - \tau), \tau)}{\partial (r + (t - \tau))} - \frac{\partial \widetilde{\mathbb{F}}(r - (t - \tau), \tau)}{\partial (r - (t - \tau))} \right] d\tau$$

$$- \int_0^t (\widetilde{\mathbb{F}}((t - \tau), \tau) - \widetilde{\mathbb{F}}(-(t - \tau), \tau))$$

$$\left. + r \int_0^t \frac{\partial \widetilde{\mathbb{F}}((t - \tau), \tau)}{\partial ((t - \tau))} d\tau + r \int_0^t \frac{\partial \widetilde{\mathbb{F}}(-(t - \tau), \tau)}{\partial (-(t - \tau))} d\tau + 0(r) \right]. \quad (2.46)$$

From where

$$\lim_{r \to 0} r \frac{\partial^2 u}{\partial r \partial t} = \frac{0(r)}{2r} = 0.$$

Which proves the equality (2.10).

On the other hand, from the formula

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{2r} \left[\left(1 + \frac{\widetilde{A}^2}{4} \right) \left(\frac{d^2 \widetilde{\Phi}(H_2(r, t))}{dH_2^2(r, t)} + \frac{d^2 \widetilde{\Phi}(H_1(r, t))}{dH_1^2(r, t)} \right) \right.$$

$$+ \widetilde{A} \sqrt{1 + \frac{\widetilde{A}^2}{4}} \left(\frac{d^2 \widetilde{\Phi}(H_2(r, t))}{dH_2^2(r, t)} - \frac{d^2 \widetilde{\Phi}(H_1(r, t))}{dH_1^2(r, t)} \right)$$

$$+ \frac{\widetilde{A}}{2} \left(\frac{d\widetilde{\chi}(H_2(r, t))}{dH_2(r, t)} - \frac{d\widetilde{\chi}(H_1(r, t))}{dH_1(r, t)} \right)$$

$$+ \sqrt{1 + \frac{\widetilde{A}^2}{4}} \left(\frac{d\widetilde{\chi}(H_2(r, t))}{dH_2(r, t)} + \frac{d\widetilde{\chi}(H_1(r, t))}{dH_1(r, t)} \right)$$

$$\left. + \frac{\partial}{\partial t} \int_0^t [\widetilde{\mathbb{F}}(H_2(r, t - \tau), \tau) - \widetilde{\mathbb{F}}(H_1(r, t - \tau), \tau)] d\tau \right], \quad (2.47)$$

it follows that in the neighborhoods $(r \to 0)$ and $(r \to R)$, we obtain

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{2r}\left[\left(\frac{d^2\widetilde{\Phi}(r+t)}{d(r+t)^2} + \frac{d^2\widetilde{\Phi}(r-t)}{d(r-t)^2}\right) + \left(\frac{d\widetilde{\chi}(r+t)}{d(r+t)} + \frac{d\widetilde{\chi}(r-t)}{d(r-t)}\right)\right.$$

$$\left. + \frac{\partial}{\partial t}\int_0^t (\widetilde{\mathbb{F}}(r+(t-\tau), \tau) - \widetilde{\mathbb{F}}(r-(t-\tau), \tau))d\tau\right]. \tag{2.48}$$

From the formulas (2.41) and (2.48), we can deduce that in the neighborhoods $(r \to 0)$ and $(r \to R)$,

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right) = \frac{1}{2r}\left[\frac{\partial}{\partial t}\int_0^t (\widetilde{\mathbb{F}}(r+(t-\tau), \tau) - \widetilde{\mathbb{F}}(r-(t-\tau), \tau))d\tau\right.$$

$$\left. - \frac{\partial}{\partial r}\int_0^t (\widetilde{\mathbb{F}}(r+(t-\tau), \tau) - \widetilde{\mathbb{F}}(r-(t-\tau), \tau))d\tau\right].$$

Which then gives

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right) = \frac{\widetilde{\mathbb{F}}(r, t)}{r}. \tag{2.49}$$

Therefore, using the limit when $r \to R$, we have

$$\lim_{r\to R}\frac{\partial^2 u}{\partial t^2} - \lim_{r\to R}\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right) = \lim_{r\to R}\frac{\widetilde{\mathbb{F}}(r, t)}{r}.$$

Thus, according to the equality (2.42) on the one hand, the equalities (2.27) and (2.33) on the other hand, the formula (2.12) follows.

Similarly, passing to the limit when $r \to 0$, we have

$$\lim_{r\to 0} r\frac{\partial^2 u}{\partial t^2} - \lim_{r\to 0} r\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right) = \lim_{r\to 0}\widetilde{\mathbb{F}}(r, t).$$

From the equalities (2.27) and (2.33), we obtain the formula (2.5). □

Which proves the theorem.

In our future work, we will prove the existence of the classical weakened solution on the axis of the third centrally symmetric mixed problem for the second-order general hyperbolic equation with right-hand side.

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