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NORM INEQUALITY OF COMMUTATOR GENERATED BY THE FRACTIONAL INTEGRAL OPERATOR IN HARDY-AMALGAM SPACES WITH VARIABLE EXPONENTS $\mathcal{H}^{p(\cdot), q}(\mathbb{R})^d$

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Abstract

Let $0 < \gamma < d$ and I_γ be the fractional integral operator of order γ and we consider $\mathbb{R}^d$ the Euclidean space of dimension d. In this paper, we aim to prove some boundedness properties of commutator $[b, I_\gamma]$ generated by the fractional integral operator I_γ and a suitable function b on variable exponent spaces. Roughly speaking, we prove

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that the commutator $[b, I_\gamma]$ is extendable to a bounded linear operator from Hardy-amalgam spaces with variable exponents denoted $\mathcal{H}^{p(\cdot),q}(\mathbb{R}^d)$ to variable exponent amalgam spaces $(L^{p(\cdot)}, l^p)(\mathbb{R}^d)$ under appropriate conditions on the exponent $p(\cdot)$.

0. Introduction

Let d be a positive integer, $\mathbb{R}^d$ be the d-dimensional Euclidean space endowed with the Euclidean norm and the Lebesgue measure denoted by dx. Let us consider the fractional integral operator I_γ $(0 < \gamma < d)$ of order γ defined by

$$I_\gamma f(x) := \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\gamma}},$$

where $f \in L^q(\mathbb{R}^d)$ and $\max\{1; p^+\} < q < \frac{d}{\gamma}$. For a locally integrable function b, the commutator generated by I_γ is defined by

$$[b, I_\gamma]f(x) := b(x)I_\gamma f(x) - I_\gamma(bf)(x).$$

We indicate that commutators of classical operators of harmonic analysis play a basic role in various topics of analysis and PDE; see for instance, [27] and [28], and the references therein. More especially, commutators are very useful when studying problems in relation with regularity of solutions of elliptic partial differential equations of second order, e.g., [4] and [5].

In this paper, we deal with norm inequality of the commutators. Let us make a historical remark of the boundedness of the commutators for the convenience of the reader. We point out that norm inequalities of the commutators generated by the fractional integral operator also known as Riesz potential operator have been extensively studied in the setting of Lebesgue spaces L^p $(p \in (1; \infty))$. Since then, these studies have undergone several directions and developments. Thus, in [3], Chanillo introduced the commutator $[b, I_\gamma]$ and proved that it is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ with $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n}$ and $1 < p < \frac{\gamma}{n}$ if $b \in BMO$.

In the same vein, Bui in [2] obtained the boundedness of $[b, I_\gamma]$ from $L^q(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ with $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n}$ and $1 < p < \frac{n}{\gamma}$ when $b \in BMO_\theta(\rho)$. Paluszynski gave the same proof in [26] but considering that b belongs to the Campanato space Λ_β, $0 < \beta < 1$, where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha + \beta}{n}$ and $1 < p < \frac{n}{\alpha + \beta}$. Furthermore, in the framework of these developments, Lu et al. in [23] considered the boundedness of $[b, I_\alpha]$ on the classical Hardy spaces when $b \in \Lambda_\beta$ $(0 < \beta \leq 1)$. In their research, they proved that if $\frac{n}{n + \beta} < p \leq 1$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha + \beta}{n}$, then $[b, I_\alpha]$ maps $H^p(\mathbb{R}^n)$ continuously into $L^p(\mathbb{R}^n)$. Moreover, at the endpoint $p = \frac{n}{n + \beta}$, they also showed that $[b, I_\alpha]$ maps $H^p(\mathbb{R}^n)$ continuously into weak $L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$. Moving in the same direction, recently, Di Fazio and Ragusa [10] proved that if b is in $BMO(\mathbb{R}^n)$, then the commutator $[b, I_\alpha]$ is bounded from the classical Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$, and conversely, under some restricted condition on α, if the commutator $[b, I_\alpha]$ is bounded from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$, then $b \in BMO(\mathbb{R}^n)$. Subsequently, studies concerning the boundedness of commutators associated with the Riesz potential were extended to Orlicz spaces by Fu et al. [19] and Guliyev et al. [20]. Thus, in the framework of these extensions to other spaces, Wang et al. proved in [35] that the commutator $[b, I_\sigma]$ is bounded from $L^{q_1(\cdot)}(\mathbb{R}^n)$

to $L^{q_2(\cdot)}(\mathbb{R}^n)$ with $\frac{1}{q_1(x)} - \frac{1}{q_2(x)} = \frac{\alpha + \beta}{n}$ and $q_1(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ with $q_1^+ < \frac{n}{\alpha + \beta}$. Beside, Wang proved in [34] the continuity of the commutator $[b, I_\sigma]$ on Herz-type Hardy spaces with variable exponent. Thus, many researches undertaken in classical harmonic analysis have been intended to norm inequalities involving certain integral operators in weighted amalgam spaces. See [17, 18, 29]. We can therefore see that there is a great deal of literature on norm inequalities of commutators associated with some classical operators such as Riesz potential, maximal operator and Calderon-Zygmund operators, etc. We would like to mention that this research is the continuation of our two previous works in which we characterized the Hardy-amalgam spaces with variable exponents $\mathcal{H}^{p(\cdot),q}$ by the mean of their atomic decomposition. Then, in the second work, that is the second paper, we proved the stability of the fractional integral operator I_γ on these spaces. Furthermore, we specify that the goal of this research is onefold. To this end, our task here is to show that the commutator of I_γ, $[b, I_\gamma]$ which is known to be bounded on some generalized spaces such as weighted Herz spaces with variable exponent [21], weighted amalgam spaces [33] etc., is also bounded on Hardy-amalgam spaces with variable exponents under suitable conditions. In other words, we show that if $b \in BMO(\mathbb{R}^d)$, then the commutator $[b, I_\gamma]$ has the same boundedness as the fractional integral operator obtained in [32]. Next, it is worth mentioning that to investigate the boundedness of linear operators in Hardy type spaces on $\mathbb{R}^d$, one usually appeals to atomic or molecular characterizations of these spaces. The characterizations suppose that a function or a distribution in Hardy type spaces can be written as a linear combination of functions of atoms or molecules. So working similarly, we indicate that to establish the boundedness of the commutators associated with the Riesz potential operator, the authors either used results deduced from the behaviour on atoms or molecules. However, in this manuscript, we are going to study the continuous mapping properties of the commutator $[b, I_\gamma]$ via a direct proof based on the molecular decomposition of the Hardy-amalgam spaces with variable exponents which is previously obtained in [30]. Now, we give some conventions. To this end, let d be a positive integer and $\mathbb{R}^d$, the d-dimensional Euclidean space. Next, let us consider $\varphi \in \mathcal{S}(\mathbb{R}^d)$ with support on $B(0, 1)$ such that $\int_{\mathbb{R}^d} \varphi dx = 1$, where $B(0, 1)$ is the unit open ball centered at 0, $\mathcal{S}(\mathbb{R}^d)$ is the space of complex-valued functions indefinitely differentiable and rapidly decreasing on $\mathbb{R}^d$ and $\mathcal{S}'(\mathbb{R}^d)$ stands for the space of tempered distributions on $\mathbb{R}^d$ and is by definition the topological dual of the space $\mathcal{S}(\mathbb{R}^d)$ equipped with the pointwise convergence topology. For all $t > 0$, φ_t represents the dilated function and is defined as

$$\varphi_t(x) = t^{-d}\varphi\left(\frac{x}{t}\right),$$

with x in $\mathbb{R}^d$. We point out that throughout the entire manuscript, the letter C is used for a positive constant that is independent of the main parameters but whose value can change from one occurrence to another. We specify that when a constant for example C depends on some basic parameters $\alpha, \gamma, ...,$ we will denote it by the expression $C(\alpha, \gamma, ...)$. Moreover, $|E|$ stands for the Lebesgue measure and χ_E denotes the characteristic function for a measure set $E \subset \mathbb{R}^d$. We mean a cube whose edges are parallel to the coordinate axes:

$$Q \equiv \prod_{j=1}^{d} [x_j; x_{j+r}],$$

with $r > 0$. Furthermore, we consider that all cubes used in this research are closed unless it is specified otherwise and $\mathcal{Q}$ represents the set of all cubes belonging to $\mathbb{R}^d$. Equivalently, we define the Hardy-amalgam spaces with variable exponents with balls instead of cubes. For a function $f : \mathbb{R}^d \to \mathbb{C}$, the symbol $supp(f)$ means the support of *f*, and for nonnegative *f* and *g*, $f \approx g$ means $f \lesssim g \lesssim f$.

1. Function Spaces with Variable Exponents

1.1. Some background on variable Lebesgue spaces

Variable exponent Lebesgue spaces are a generalization of the classical $L^p(\mathbb{R}^n)$ spaces, in which the constant exponent *p* is replaced by an exponent function $p(\cdot) : \mathbb{R}^n \to (0, \infty)$, namely, they consist of all functions *f* such that

$$\int_{\mathbb{R}^n} | f(x) |^{p(x)} dx < \infty.$$

The space $L^{p(\cdot)}(\mathbb{R}^n)$ is the set of all measurable functions *f* on $\mathbb{R}^n$ for which the quasi-norm $\| f \|_{L^{p(\cdot)}(\mathbb{R}^n)}$ is finite. These spaces were introduced by Birnbaum-Orlicz [1] and Orlicz [25], and were widely used in the study of harmonic analysis as well as partial differential equations; see for example [6, 7, 12, 13, 15]. But, for a systematic research about the variable exponent Lebesgue spaces, we refer the reader to [6, 14]. Recently, Nakai and Sawano [24] extended the theory of variable Lebesgue spaces via the study of the Hardy spaces with variable exponents on $\mathbb{R}^n$, and Sawano in [29] further gave more applications of these variable exponent Hardy spaces. We would like to indicate that before establishing our result, we need to state some basic definitions and conventions as we are working in variable exponent analysis. Thus, for a measurable subset $\Omega \subset \mathbb{R}^d$, we set as usual

$$p^-(\Omega) := \inf_{x \in \Omega} p(x)$$

and

$$p^-(\Omega) : \sup_{x \in \Omega} p(x).$$

Moreover, let $p(\cdot):\mathbb{R}^d \to (0, \infty)$ be a measurable function with $0 < p^- \leq p^+ < \infty$ and $\mathcal{P}$ be the set of all measurable functions $p(\cdot):\mathbb{R}^d \to [1, \infty)$ such that $1 < p^- \leq p^+ < \infty$. Let $\mathcal{B}$ stand for the set of $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ such that the Hardy-Littlewood maximal operator $\mathfrak{M}$ is bounded on $L^{p(\cdot)}$.

An important subset of $\mathcal{B}$ is the class of globally log-Hölder continuous functions $p \in LH$, with $1 < p^- \leq p^+ < \infty$. Let us recall that the measurable function $p(\cdot)$ is said to belong to $LH(\mathbb{R}^d)$, if $p(\cdot)$ fulfills the following conditions:

$$|p(x) - p(y)| \leq \frac{-C}{\ln(|x-y|)} \text{ when } |x-y| \leq \frac{1}{2}$$

and

$$|p(x) - p(y)| \leq \frac{C}{\ln(|x|+e)} \text{ when } |y| \geq |x|.$$

On the other hand, the variable exponent Hardy spaces theory was introduced independently by Nakai and Sawano in [24] and by Cruz-Uribe and Wang in [8]. After that, they characterized $\mathcal{H}^{p(\cdot)}(\mathbb{R})^n$ spaces via the atomic decomposition. Furthermore, as an application, they used the so-called atomic decomposition to establish the boundedness of some classical linear operators such as: the singular integral operators, Riesz potential operators, etc.

1.2. On Hardy-amalgam spaces with variable exponents

Here, for the convenience of the reader, we recall the definition of the Hardy-amalgam spaces with variable exponents.

Definition 1. Let $p(\cdot):\mathbb{R}^d \to (0, \infty)$ be a measurable function with $0 < p^- \leq p^+ < \infty$ and $1 < q \leq \infty$. Moreover, we assume that $p(\cdot)$ fulfills the local log-Hölder continuity and decay conditions. Thus, following the maximal function approach, the *Hardy-amalgam space* with variable exponent $\mathcal{H}^{p(\cdot),q}$ is defined on $\mathbb{R}^d$ as the set of all tempered distributions $f \in \mathcal{S}'(\mathbb{R})^d$ for which the quasi-norm is given by

$$\| f \|_{\mathcal{H}^{p(\cdot),q}(\mathbb{R}^d)} \equiv \| \mathcal{M}_\varphi(f) \|_{L^{p(\cdot),q}(\mathbb{R}^d)}$$

is finite, where $\mathcal{M}_\varphi$ is the maximal function and is defined as

$$\mathcal{M}_\varphi(f) := \sup_{t>0} | \varphi_t * f |.$$

Now, let us introduce the concept of an atom for $\mathcal{H}^{p(\cdot),q}$ spaces.

Definition 2. Consider a measurable function $p(\cdot)$ satisfying the *globally log-Hölder continuous conditions* and let us have a fixed positive integer *s* such that

$$s \geq s_{p(\cdot)} := \{ s \in \mathbb{Z}^+ / p^-(d + s + 1) > d \}.$$

Thus, a measurable function ***a*** is called $(p(\cdot), q, s)$-*atom* on $\mathbb{R}^d$ for $\mathcal{H}^{p(\cdot),q}$ if there exists a cube *Q* fulfilling the following conditions:

(i) $supp(\boldsymbol{a}) \subset Q$ for some $Q \subset \mathbb{R}^d$.

(ii) $\| \boldsymbol{a} \|_q \leq \dfrac{| Q |^{\frac{1}{q}}}{\| \chi_Q \|_{p(\cdot)}}$ (boundedness condition).

(iii) $\int_{\mathbb{R}^d} \boldsymbol{a}(x) x^\beta dx = 0$ for all multi-indexes $\beta \in \mathbb{Z}_+^d$ with $| \beta | \leq s$ (cancellation condition).

Now, as we are handling variable exponent spaces, there is a very useful result which must be quoted. This important result concerns the boundedness of the Hardy-Littlewood maximal operator $\mathfrak{M}$. Let us recall that given a locally integrable function *f*, the centered Hardy-Littlewood maximal function of *f*, $\mathfrak{M}(f)$ is defined as follows:

$$\mathfrak{M}f(x) := \sup_{\substack{B\subset\mathbb{R}^d \\ B\ni x}} \frac{1}{|B(x, r)|}\int_{B(x, r)} |f(y)|dy,$$

where $f \in L^{1, loc}(\mathbb{R}^d)$ and the supremum is taken over all balls $B(x, r)$ and $|B(x, r)|$ denotes the volume of $B(x, r)$.

Diarra proved the following result in [11, Proposition 5.2].

Proposition 1. *Let* $p(\cdot) \in LH$ *and* $1 < q < \infty$. *Then for all* $f \in L^{1, loc}(\mathbb{R}^d)$,

$$\| \mathfrak{M}(f) \|_{p(\cdot), q} \leq C\| f \|_{p(\cdot), q},$$

where C is a positive constant independent of f.

Another basic generalization of the above Proposition 1 and Fefferman-Stein's maximum inequality [16, Theorem 1] is the following lemma:

Lemma 1 [22, Proposition 11.8]. *Let* $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ *be a measurable function satisfying* $1 < p^- \leq p^+ < \infty$ *and* $1 < q \leq \infty$. *Moreover,* $p(\cdot)$ *satisfies the log-Hölder continuity conditions. Therefore, for* $1 < v \leq \infty$ *and for all sequences* $\{f_j\}_{j\geq 1}$ *of measurable functions,*

$$\left\| \left\{ \sum_{j=1}^{+\infty} (\mathfrak{M}(f_j))^v \right\}^{\frac{1}{v}} \right\|_{(L^{p(\cdot)}, l^q)(\mathbb{R}^d)} \approx \left\| \left\{ \sum_{j=1}^{+\infty} | f_j |^v \right\}^{\frac{1}{v}} \right\|_{(L^{p(\cdot)}, l^q)(\mathbb{R}^d)}$$

with the implicit constants independent of $\{f_j\}_{j\geq 1}$.

Next, we recall the atomic decomposition result obtained and proved in [31].

Theorem 1. *Let* $p(\cdot):\mathbb{R}^d \to (0,\infty)$ *such that* $p(\cdot)\in LH$ *and* $0 < p^- \leq p^+ \leq \infty$ *and* $1 < q < \infty$. *Assume that* $f \in \mathcal{H}^{p(\cdot),q}(\mathbb{R}^d)$, $L \in \mathbb{Z}^+$ *and* $L^q_{comp}(\mathbb{R}^d)$ *be the set of all* $L^q(\mathbb{R}^d)$ *functions with compact support. Then there exist a family of* $\{(\boldsymbol{a}_j, Q_j)\}_{j\geq 1}$ *in* $\mathcal{A}(p(\cdot),\infty,s)$ *and a sequence of nonnegative scalars* $\{\lambda_j\}_{j=1}^{\infty}\subset\mathbb{R}^+$ *such that*

$$\left\| \left\{ \sum\nolimits_{j=1}^{\infty} \left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{p(\cdot)}} \right)^{\alpha} \chi_{Q_j} \right\}^{\frac{1}{\alpha}} \right\|_{\frac{p(\cdot)}{\alpha},\frac{q}{\alpha}} < \infty$$

and $f := \sum_{j=1}^{\infty}\lambda_j\boldsymbol{a}_j$ *in* $\mathcal{H}^{p(\cdot),q}(\mathbb{R}^d)$ *and in the topology of* $\mathcal{S}'(\mathbb{R}^d)$. *Moreover, the series* $f := \sum_{j=1}^{\infty}\lambda_j\boldsymbol{a}_j$ *converges to f in* $\mathcal{H}^{p(\cdot),q}(\mathbb{R}^d)$ *and satisfies*

$$\left\| \left\{ \sum\nolimits_{j=1}^{\infty} \left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{p(\cdot)}} \right)^{\alpha} \chi_{Q_j} \right\}^{\frac{1}{\alpha}} \right\|_{\frac{p(\cdot)}{\alpha},\frac{q}{\alpha}} \leq C\|f\|_{\mathcal{H}^{p(\cdot),q}}$$

for some $\alpha \in (0; \min(1, p^-))$ *and* $C > 0$ *independent of f.*

Remark. Furthermore, we say that a tempered distribution $f \in \mathcal{H}^{p(\cdot),q}$ if and only if there exists a family of measurable functions (atoms) $\{\boldsymbol{a}_j\}_{j\geq 0}$ each of which is supported in a cube Q_j and a family of scalars $\{\lambda_j\}_{j\geq 0}$ such that $f := \sum_{j\geq 0}\lambda_j\boldsymbol{a}_j$ in the sense of distributions in the topology of $\mathcal{S}'(\mathbb{R}^d)$ and

$$\left\| \left\{ \sum_{j=1}^{\infty} \left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^d)}} \right)^{\alpha} \chi_{Q_j} \right\}^{\frac{1}{\alpha}} \right\|_{\frac{p(\cdot)}{\alpha}, \frac{q}{\alpha}} < \infty$$

for any fixed $0 < \alpha < q$ depending on the family of atoms.

Moreover,

$$\| f \|_{\mathcal{H}^{p(\cdot), q}} \approx \left\| \left\{ \sum\nolimits_{j=1}^{\infty} \left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^d)}} \right)^{\alpha} \chi_{Q_j} \right\}^{\frac{1}{\alpha}} \right\|_{\frac{p(\cdot)}{\alpha}, \frac{q}{\alpha}}$$

and in particular,

$$\| f \|_{\mathcal{H}^{p(\cdot), q}}$$

$$\approx \inf \left\{ \left\| \left\{ \sum\nolimits_{j=1}^{\infty} \left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^d)}} \right)^{\alpha} \chi_{Q_j} \right\}^{\frac{1}{\alpha}} \right\|_{\frac{p(\cdot)}{\alpha}, \frac{q}{\alpha}} : f = \sum\nolimits_{j \geq 0} \lambda_j \boldsymbol{a}_j \right\},$$

where the infimum is taken over all possible representations of *f*.

Corollary. *Given* $1 < q < \infty,$ *let us consider* $p(\cdot)$ *which satisfies log-Hölder continuous conditions and a positive integer s such that*

$$s \geq s_{p(\cdot)} := \{ s \in \mathbb{Z}^{+} / p^{-}(d + s + 1) > d \}.$$

Next, let us also consider $\mathcal{H}_{fin}^{p(\cdot), q, s}$ the subspace of all the finite linear combinations of $(p(\cdot), q, s)$-atoms such that $\mathcal{H}_{fin}^{p(\cdot), q, s} \subseteq \mathcal{H}^{p(\cdot), q}$. Then

(i) For every $\mathcal{H}_{fin}^{p(\cdot),q,s}$, we set

$$\| f \|_{\mathcal{H}_{fin}^{p(\cdot),q,s}}$$

$$\equiv \inf\left\{ \left\| \left\{ \sum_{j=1}^{\infty} \left(\frac{|\lambda_j|}{\| \chi_{Q_j} \|_{L^{p(\cdot)}(\mathbb{R}^d)}} \right)^{\alpha} \chi_{Q_j} \right\}^{\frac{1}{\alpha}} \right\|_{\frac{p(\cdot)}{\alpha}, \frac{q}{\alpha}} : f = \sum_{j \geq 0} \lambda_j \boldsymbol{a}_j \right\}$$

for some $\alpha \in (0; \min(1, p^{-}))$, where the infimum is taken over all the possible representations of *f*. Thus, $\| \cdot \|_{\mathcal{H}_{fin}^{p(\cdot),q,s}}$ defines a quasi-norm on $\mathcal{H}_{fin}^{p(\cdot),q,s}$ spaces.

(ii) Besides, the subspace $\mathcal{H}_{fin}^{p(\cdot),q,s}$ is dense in $\mathcal{H}^{p(\cdot),q}$ spaces for the quasi-norm $\| \cdot \|_{\mathcal{H}^{p(\cdot),q}}$.

Proof. We give the proof of item (ii) of the corollary because this point helps us to establish the result of our work.

(ii) Given $1 < q < \infty$, it is obvious to see that $\mathcal{H}_{fin}^{p(\cdot),q,s} \subseteq \mathcal{H}^{p(\cdot),q}$. Moreover, taking $q \gg 1$, i.e., picking *q* sufficiently very large, we see that $\mathcal{H}_{fin}^{p(\cdot),q,s}$ is dense in $\mathcal{H}^{p(\cdot),q}$ for the quasi-norm $\| \cdot \|_{\mathcal{H}^{p(\cdot),q}}$ by the above remark.

We also need the definition of a molecule for $\mathcal{H}^{p(\cdot),q}$ spaces.

Definition 3. Assume that $p(\cdot) \in LH$ and $0 < p^{-} \leq p^{+} \leq 1 < q < \infty$ and consider a fixed positive integer *s* such that

$$s \geq s_{p(\cdot)} \equiv \{ s \in \mathbb{Z}^{+} / p^{-}(d + s + 1) > d \}.$$

Thus, a *measurable function* $\mathcal{M}$ is a $p(\cdot)$, q, s-molecule centered at a cube Q on $\mathbb{R}^d$ in $\mathcal{H}^{p(\cdot),q}$ spaces if and only if it satisfies the following conditions:

(i) $\| \mathcal{M} \|_q \leq | Q |^{\frac{1}{q}} \| \chi_Q \|_{p(\cdot)}^{-1}$ if $x \in 2\sqrt{d}Q$.

(ii) $| \mathcal{M}(x) | \leq \dfrac{1}{\| \chi_Q \|_{p(\cdot)}} \left(1 + \dfrac{| x - x_Q |}{l(Q)} \right)^{-2d-2s-3}$ for every $x \notin 2\sqrt{d}Q$,

where x_Q and $l(Q)$ denote the center and the side length of Q, respectively.

(iii) $\int_{\mathbb{R}^d} x^{\beta} \mathcal{M}(x) dx = 0$ for every multi-indexes β of a nonnegative integer whose length is less than or equal to s.

If we set $\mathcal{M}_o(p(\cdot), q, s)$ to be the set of all pairs $(\mathcal{M}, Q)$, therefore, the above definition implies that $\mathcal{A}(p(\cdot), q, s) \subset \mathcal{M}_o(p(\cdot), q, s)$.

Now, we come to the core of our research.

2. Norm Inequalities for Commutators Associated with the Fractional Integral Operator

Let $0 < \gamma < d$ and I_γ be the fractional integral operator also called the *Riesz potential* and b a suitable function on the variable exponent space. Then the commutator of I_γ and b is the linear operator $[I_\gamma, b]$ defined by

$$[b, I_\gamma](f)(x) := b(x) I_\gamma(f)(x) - I_\gamma(bf)(x)$$

for all $x \in \mathbb{R}^d$. This formula makes sense when b is a locally integrable function on $\mathbb{R}^d$. We recall for the convenience of the reader that in [25], Orlicz proved the boundedness of the commutator $[I_\gamma, b]$ from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ under conditions $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{d}$ and $1 < p < \frac{d}{\gamma}$ whenever $b \in BMO(\mathbb{R}^d)$. Next, let us give the definition of the space $BMO(\mathbb{R}^d)$.

Then, a locally integrable function b is said to belong to $BMO(\mathbb{R}^d)$, $d \geq 1$ if it satisfies the following:

$$\| b \|_{BMO} := \sup_{\substack{r>0 \\ B:=ball}} | B(x, r) |^{-1} \int_{B(x, r)} | b(y) - b_{B(x, r)} | dy < \infty,$$

where the supremum is taken over all balls in $\mathbb{R}^d$ and

$$b_{B(x, r)} = \frac{1}{| B(x, r) |} \int_{B(x, r)} b(y) dy.$$

Remark. The above definition holds if we replace the ball by a cube Q. We have the following result.

Theorem 2. *Let s be an integer such that*

$$s \geq s_{p(\cdot)} \text{ with } s_{p(\cdot)} \equiv \min\{s \in \mathbb{Z}^+ / p^-(d + s + 1) > d\}$$

and $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$. *Consider* $\mathcal{L}^{r,s}_{comp}(\mathbb{R}^d)$ *to stand for the subspace of* $\mathcal{L}^r$*-functions with compact support such that* $1 < r \leq +\infty$. *We assume that* $p(\cdot)$ *satisfies* $0 < p^- \leq p^+ < \frac{\gamma}{d}$ *as well as log-Hölder continuity and decay conditions. Thus, for* $f \in \mathcal{L}^{r,s}_{comp}(\mathbb{R}^d)$ *and* $b \in BMO(\mathbb{R}^d)$, *the commutator* $[b, I_\gamma]$ *generated by the fractional integral operator of order* γ *maps* $\mathcal{H}^{p(\cdot),q}(\mathbb{R}^d)$ *continuously into* $(L^{p(\cdot)}, l^p)(\mathbb{R}^d)$ *spaces. In other words, for every* $f \in \mathcal{L}^{r,s}_{comp}(\mathbb{R}^d)$,

$$\| [I_\gamma, b] \|_{(L^{p(\cdot)}, l^p)(\mathbb{R}^d)} \leq C \| f \|_{\mathcal{H}^{p(\cdot),q}(\mathbb{R}^d)},$$

where C is a positive constant independent of f.

Proof. Consider $p(\cdot)$ satisfying the globally log-Hölder continuous conditions mentioned above and fixed positive integer s such that

$$s \geq s_{p(\cdot)} := \{s \in \mathbb{Z}^+ / p^-(d + s + 1) > d\}.$$

Also, consider the subspace $\mathcal{H}_{fin}^{p(\cdot),q,s}$-atoms. It is well known that $\mathcal{H}_{fin}^{p(\cdot),q} \subset \mathcal{H}^{p(\cdot),q}$. We pick up $f \in \mathcal{H}_{fin}^{p(\cdot),q}$. Therefore, there exist a family of $(p(\cdot), q, s)$-atoms $\{\boldsymbol{a}_j\}_{j\geq 0}$ and a sequence of nonnegative scalars $\{\lambda_j\}_{j\geq 0} \subset \mathbb{R}^+$ such that $f := \sum_{j\geq 0} \lambda_j \boldsymbol{a}_j$. Now, we set $\widetilde{Q}_j = 2\sqrt{d}Q_j$ with x_j and l_j stand for the center and the side-length of Q_j, respectively. We indicate that the proof is partially similar to the proof of Theorem 3.3 (see [19]). So,

$$|[b, I_\gamma](f)(x)| \leq \sum_{j\geq 0} |\lambda_j| |[b, I_\gamma](\boldsymbol{a}_j)\chi_{\widetilde{Q}_j}(x)|$$

$$+ \sum_{j\geq 0} |\lambda_j| |[b, I_\gamma](\boldsymbol{a}_j)\chi_{\mathbb{R}^d \setminus \widetilde{Q}_j}(x)| \text{ for all } x \in \mathbb{R}^d. \quad (1)$$

It follows that, for all $x \in \mathbb{R}^d$, we get

$$|[b, I_\gamma](f)(x)| \leq \sum_{j\geq 0} |\lambda_j| |[b, I_\gamma](\boldsymbol{a}_j)(x)|\chi_{\widetilde{Q}_j}(x)$$

$$+ \sum_{j\geq 0} |\lambda_j| |[b, I_\gamma](\boldsymbol{a}_j)(x)|\chi_{\mathbb{R}^d \setminus \widetilde{Q}_j}(x). \quad (2)$$

Next, by taking the $(L^{p(\cdot)}, l^p)$-quasi-norm of both the sides, we obtain

$$\|[I_\gamma, b](f)\|_{p(\cdot),q} \leq \left\| \sum_{j\geq 0} |\lambda_j| |[b, I_\gamma](\boldsymbol{a}_j)(x)|\chi_{\widetilde{Q}_j}(x) + \sum_{j\geq 0} |\lambda_j| |[b, I_\gamma](\boldsymbol{a}_j)(x)|\chi_{\mathbb{R}^d \setminus \widetilde{Q}_j}(x) \right\|_{p(\cdot),q}$$

$$\leq \left\| \sum_{j\geq 0} |\lambda_j| |[b, I_\gamma](\boldsymbol{a}_j)(x)|\chi_{\widetilde{Q}_j}(x) \right\|_{p(\cdot),q} + \left\| \sum_{j\geq 0} |\lambda_j| |[b, I_\gamma](\boldsymbol{a}_j)(x)|\chi_{\mathbb{R}^d \setminus \widetilde{Q}_j}(x) \right\|_{p(\cdot),q}.$$

Then, since $b \in BMO(\mathbb{R}^d)$ and $1 < q < +\infty$, we have for some $\alpha \in \min(1, p^-)$ and also for $x \in \widetilde{Q}_j$:

$$\| [([I_\gamma, b]\boldsymbol{a}_j)\chi_{\widetilde{Q}_j}]^\alpha \|_{\frac{q}{\alpha}} \le \| [I_\gamma, b]\boldsymbol{a}_j \|_q^\alpha \le C(b, q, \alpha)\| \boldsymbol{a}_j \|_q^\alpha$$

which leads to the following:

$$\left\| \sum_{j\ge 0} | \lambda_j || [b, I_\gamma](\boldsymbol{a}_j) | \chi_{\widetilde{Q}_j}(x) \right\|_{p(\cdot), q}$$

$$\le \left\| \sum_{j\ge 0} | \lambda_j |^\alpha (| [b, I_\gamma](\boldsymbol{a}_j) | \chi_{\widetilde{Q}_j})^\alpha \right\|_{\frac{p(\cdot)}{\alpha}, \frac{q}{\alpha}}^{\frac{1}{\alpha}}$$

$$\le C(b, q, \alpha) \left\| \sum_{j\ge 0} \left(\frac{| \lambda_j |}{\| \chi_{Q_j} \|_{p(\cdot)}} \right)^\alpha \chi_{Q_j} \right\|_{\frac{p(\cdot)}{\alpha}, \frac{q}{\alpha}}^{\frac{1}{\alpha}}$$

by the virtue of the proof of theorem (see [32, proof of Theorem 3.3]).

Now, it remains to prove the boundedness of the following expression, that is,

$$\sum_{j\ge 0} | \lambda_j || [b, I_\gamma](\boldsymbol{a}_j)(x) | \chi_{\mathbb{R}^d \backslash \widetilde{Q}_j}(x).$$

In order to reach this goal, let us set $\max\{1, p^+\} < q < \frac{d}{\gamma}$. Thus, the fractional integral operator is then defined as follows:

$$I_\gamma f(x) := \int_{(\mathbb{R}^d)} \frac{f(y)dy}{| x - y |^{d-\gamma}},$$

where $f \in L^q(\mathbb{R}^d)$. In addition, let us write this fractional integral operator as a linear operator with standard kernel $K(x, y)$ having certain regularity such that

$$I_\gamma f(x) \equiv \int_{(\mathbb{R}^d)} K(x, y) f(y) dy,$$

where $K(x, y) = \dfrac{1}{|x - y|^{d-\gamma}}$ for all $x \neq y$.

Next, as in the case of the Calderon-Zygmund singular operator, the standard kernel $K(x, y)$ is a continuous function defined on the set $\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \,|\, x \neq y\}$ which satisfies the following estimates:

(i) $|K(x, y)| \leq B|x - y|^{-d}$, $B > 0$ if $x \neq y$.

(ii) $|K(x, y) - K(x, z)| \leq B\dfrac{|y - z|^\sigma}{|x - y|^{d+\sigma}}$, $\sigma > 0$, $B > 0$ if $|x - y| > 2|y - z|$ and

(iii) $|K(x, y) - K(w, y)| \leq B\dfrac{|x - w|^\sigma}{|x - y|^{d+\sigma}}$, $\sigma > 0$, $B > 0$ if $|x - y| > 2|x - w|$.

Thus, from the item (ii), for $\sigma > 0$, $B > 0$,

$$|x - y| > 2|y - z| \Rightarrow \frac{|x - y|}{|y - z|} > 2$$

$$\Rightarrow \frac{|x - x_j|}{|x_j - y|} > 2$$

$$\Rightarrow \frac{|x_j - y|}{|x - x_j|} < 2.$$

Therefore, if $\frac{|x_j - y|}{|x - x_j|} < 2,$ then we obtain

$$|K(x, x_j) - K(x, y)| \leq B2^{\sigma}\frac{|x_j - y|^{\sigma}}{|x - x_j|^{d+\sigma}}$$

$$\Leftrightarrow |K(x, x_j) - K(x, y)| \leq A\frac{|x_j - y|^{\sigma}}{|x - x_j|^{d+\sigma}} \text{ for } A > 0. \tag{3}$$

On the other hand, we have for $b \in BMO(\mathbb{R}^d)$:

$$b(x) - b(y) = b(x) - bQ_j + bQ_j - b(y)$$

$$\Leftrightarrow |b(x) - b(y)| = |b(x) - bQ_j + bQ_j - b(y)|$$

$$\Leftrightarrow |b(x) - b(y)| \leq |b(x) - bQ_j| + |bQ_j - b(y)|$$

$$= |b(x) - bQ_j| + |b(y) - bQ_j|.$$

Thus, for a fixed $x \notin \widetilde{Q}_j,$ we obtain

$$|[b, I_\gamma](\boldsymbol{a}_j)(x)| = |I_\gamma([b(x) - b]\boldsymbol{a}_j)(x)|$$

$$\leq A\int_{Q_j}|K(x, x_j) - K(x, y)||b(x) - b(y)||\boldsymbol{a}_j(y)|dy$$

$$\leq A\frac{l_j^{\sigma}}{|x - x_j|^{d+\sigma}}\int_{Q_j}|b(x) - b(y)||\boldsymbol{a}_j(y)|dy. \tag{3}$$

But, $|b(x) - b(y)| \leq |b(x) - bQ_j| + |b(y) - bQ_j|.$ So, it follows that

$$|[b, I_\gamma](\boldsymbol{a}_j)(x)|$$

$$\leq \frac{l_j^{\sigma}}{|x - x_j|^{d+\sigma}}\int_{Q_j}(|b(x) - bQ_j| + |b(y) - bQ_j|)|\boldsymbol{a}_j(y)|dy$$

and this implies that

$$\begin{aligned}
&|\,[b, I_\gamma\,](\boldsymbol{a}_j)(x)\,| \\
&\le A\frac{l_j^\sigma}{|\,x - x_j\,|^{d+\sigma}}\int_{Q_j}(|\,b(x) - bQ_j\,||\,\boldsymbol{a}_j(y)\,|)dy \\
&\quad + A\frac{l_j^\sigma}{|\,x - x_j\,|^{d+\sigma}}\int_{Q_j}(|\,b(y) - bQ_j\,||\,\boldsymbol{a}_j(y)\,|)dy \\
&\le A\frac{l_j^\sigma}{|\,x - x_j\,|^{d+\sigma}}(|\,b(x) - bQ_j\,|)\int_{Q_j}|\,\boldsymbol{a}_j(y)\,|dy \\
&\quad + A\frac{l_j^\sigma}{|\,x - x_j\,|^{d+\sigma}}\int_{Q_j}(|\,b(y) - bQ_j\,||\,\boldsymbol{a}_j(y)\,|)dy.
\end{aligned}$$

Notice that if b is a locally integrable function belonging to $BMO(\mathbb{R}^d)$ space, then there exists $0 < \xi < d$ such that

$$|\,b(x) - bQ_j\,| \le C\left(\frac{|\,x - x_{Q_j}\,|}{l_{Q_j}}\right)^\xi,$$

for all $x \notin \widetilde{Q}_j$, where C is a positive constant independent of x and Q_j, see [9, proof of Proposition 13]. It follows that

$$\begin{aligned}
|\,[b, I_\gamma\,](\boldsymbol{a}_j)(x)\,| &\le A_1\frac{l_j^{\sigma-\xi}}{|\,x - x_j\,|^{d+\sigma-\xi}}\int_{Q_j}|\,\boldsymbol{a}_j(y)\,|dy \\
&\quad + A_1\frac{l_j^\sigma}{|\,x - x_j\,|^{d+\sigma}}\int_{Q_j}(|\,b(y) - bQ_j\,||\,\boldsymbol{a}_j(y)\,|)dy,
\end{aligned}$$

with $A_1 > 0.$ We also have

$$\int_{Q_j}|\,\boldsymbol{a}_j(y)\,|dy \le \|\,\boldsymbol{a}_j\,\|_{q(\mathbb{R}^d)} \le |\,Q_j\,|^{\frac{1}{q}}\|\,\chi_{Q_j}\,\|_{p(\cdot)}^{-1},$$

by Definition 2 and for any $q \in (1, \infty)$. Hence

$$A_1 \frac{l_j^{\sigma}}{|x - x_j|^{d+\sigma}} \int_{Q_j} (|b(y) - bQ_j|| \boldsymbol{a}_j(y)|) dy$$

$$\leq A_1 \frac{l_j^{\sigma}}{|x - x_j|^{d+\sigma}} \| b \|_{BMO} | Q_j |^{\frac{1}{q}} \| \chi_{Q_j} \|_{p(\cdot)}^{-1}.$$

From now, we follow the method used in [31]. However, before moving with the rest of the proof, we obtain

$$|[b, I_\gamma](\boldsymbol{a}_j)(x)| \leq A_1 \frac{l_j^{\sigma-\xi}}{|x - x_j|^{d+\sigma-\xi}} \frac{|Q_j|^{\frac{1}{q}}}{\| \chi_{Q_j} \|_{p(\cdot)}}$$

$$+ A_1 \frac{l_j^{\sigma}}{|x - x_j|^{d+\sigma}} \| b \|_{BMO} \frac{|Q_j|^{\frac{1}{q}}}{\| \chi_{Q_j} \|_{p(\cdot)}}.$$

But

$$\frac{l_j^{\sigma-\xi}}{|x - x_j|^{d+\sigma-\xi}} \leq C_1 \{\mathfrak{M}(\chi_{Q_j})(x)\}^{\frac{d+\sigma-\xi}{d}}$$

and

$$\frac{l_j^{\sigma}}{|x - x_j|^{d+\sigma}} \leq C_2 \{\mathfrak{M}(\chi_{Q_j})(x)\}^{\frac{d+\sigma}{d}}.$$

Therefore,

$$|[b, I_\gamma](\boldsymbol{a}_j)(x)| \leq A_1 C_1 | Q_j |^{\frac{1}{q}} \frac{\{\mathfrak{M}(\chi_{Q_j})(x)\}^{\frac{d+\sigma-\xi}{d}}}{\| \chi_{Q_j} \|_{p(\cdot)}}$$

$$+ A_1 C_2 | Q_j |^{\frac{1}{q}} \frac{\{\mathfrak{M}(\chi_{Q_j})(x)\}^{\frac{d+\sigma}{d}}}{\| \chi_{Q_j} \|_{p(\cdot)}} \| b \|_{BMO}$$

$$\leq A_3 \frac{\{\mathfrak{M}(\chi_{Q_j})(x)\}^{\frac{d+\sigma-\xi}{d}}}{\| \chi_{Q_j} \|_{p(\cdot)}} + A_4 \frac{\{\mathfrak{M}(\chi_{Q_j})(x)\}^{\frac{d+\sigma}{d}}}{\| \chi_{Q_j} \|_{p(\cdot)}}$$

by considering $q \gg 1$. Next, by taking $p(\cdot)$, q-quasi-norm, we obtain

$$\left\| \sum_{j\ge 0} |\lambda_j| \, |[b, I_\gamma](\boldsymbol{a}_j)(x)| \chi_{\mathbb{R}^d \setminus \widetilde{Q}_j}(x) \right\|_{p(\cdot), q}$$

$$\le A_3 \left\| \sum_{j\ge 0} |\lambda_j| \frac{\{\mathfrak{M}(\chi_{Q_j})(x)\}^{\frac{d+\sigma-\xi}{d}}}{\|\chi_{Q_j}\|_{p(\cdot)}} \right\|_{p(\cdot), q}$$

$$+ A_4 \left\| \sum_{j\ge 0} |\lambda_j| \frac{\{\mathfrak{M}(\chi_{Q_j})(x)\}^{\frac{d+\sigma}{d}}}{\|\chi_{Q_j}\|_{p(\cdot)}} \right\|_{p(\cdot), q}$$

$$\le A_5 \left\| \sum_{j\ge 0} \left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{p(\cdot)}} \right)^{\theta} \chi_{Q_j} \right\|_{\frac{p(\cdot)}{\theta}, \frac{q}{\theta}}^{\frac{1}{\theta}}$$

for some $\theta \in (0; \min(1, p^-))$, by Lemma 1. Hence, the inequality (1) gives

$$|[b, I_\gamma](f)(x)| \le \sum_{j\ge 0} |\lambda_j| \, |[b, I_\gamma](\boldsymbol{a}_j)\chi_{\widetilde{Q}_j}(x)|$$

$$+ \sum_{j\ge 0} |\lambda_j| \, |[b, I_\gamma](\boldsymbol{a}_j)\chi_{\widetilde{Q}_j^c}(x)|,$$

for $x \in \mathbb{R}^d$. Taking the $p(\cdot)$, q-quasi-norm, we find

$$\left\| \sum_{j\ge 0} |\lambda_j| \, |[b, I_\gamma](\boldsymbol{a}_j)\chi_{\widetilde{Q}_j}| \right\|_{p(\cdot), q}$$

$$\le C(p(\cdot), q, b, \alpha) \left\| \sum_{j\ge 0} \left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{p(\cdot)}} \right)^{\alpha} \chi_{Q_j} \right\|_{\frac{p(\cdot)}{\alpha}, \frac{q}{\theta}}^{\frac{1}{\alpha}}$$

$$+ A_5(p(\cdot), q, b, \theta)\left\|\sum_{j\geq 0}\left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{p(\cdot)}}\right)^{\theta}\chi_{Q_j}\right\|_{\frac{p(\cdot)}{\theta}, \frac{q}{\theta}}^{\frac{1}{\theta}}$$

$$\leq C(p(\cdot), q, b, \tau)\left\|\sum_{j\geq 0}\left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{p(\cdot)}}\right)^{\tau}\chi_{Q_j}\right\|_{\frac{p(\cdot)}{\tau}, \frac{q}{\tau}}^{\frac{1}{\tau}}$$

by Lemma 1 and for some τ belonging to $(0; \min(1, p^{-}))$. Whence, we arrive at

$$\begin{aligned}\|[b, I_\gamma](f)\|_{p(\cdot), q} &\leq A_5\|f\|_{\mathcal{H}_{fin}^{p(\cdot), q}}\\ &\leq A_5\|f\|_{\mathcal{H}^{p(\cdot), q}}\end{aligned}$$

since the subspace $\mathcal{H}_{fin}^{p(\cdot), q}(\mathbb{R}^d)$ is dense in $\mathcal{H}^{p(\cdot), q}(\mathbb{R}^d)$ spaces, we finally obtain the result of the proof, that is,

$$\|[b, I_\gamma](f)\|_{(p(\cdot), q)(\mathbb{R}^d)} \leq C\|f\|_{\mathcal{H}^{p(\cdot), q}(\mathbb{R}^d)},$$

where A_5 and C are, respectively, positive constants independent of f.

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