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FAMILY OF RULED SURFACES WITH POINTWISE 1-TYPE GAUSS MAP IN A 3-DIMENSIONAL STRICT WALKER MANIFOLD

Papa Aly Cisse[1], Mamadou Eramane Bodian[1] and Ameth Ndiaye[2]

[1]Département de Mathématiques
UFR des Sciences et Technologies
UASZ, Ziguinchor, Senegal
e-mail: palycsse@gmail.com
eramane20era@yahoo.fr

[2]Département de Mathématiques
FASTEF, UCAD
Dakar, Senegal
e-mail: ameth1.ndiaye@ucad.edu.sn

Abstract

In this paper, we consider two special families of ruled surfaces in a three-dimensional Walker manifold which look like the ruled surfaces

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in a three-dimensional semi-Euclidean space. We obtain a necessary and sufficient condition for such a surface to be pointwise 1-type Gauss map. We obtain also, by the use of the concept of pointwise 1-type Gauss map, a characterization theorem for ruled surfaces of constant mean curvature.

1. Introduction

For a very long time, geometers have been interested in minimum surfaces. Specifically, the planes and helicoids are the only minimally ruled surfaces in a three-dimensional Euclidean space $\mathbb{E}^3$. In a three-dimensional Minkowski space $\mathbb{E}_1^3$, Kobayashi [16] classified a space-like Ruled minimal surface, which has been extended by de Woestijne to the Lorentz version [20].

Chen [8, 9] first proposed the idea of Euclidean immersions of finite type in the late 1970s. These are essentially submanifolds whose immersion into $\mathbb{E}$ is built using a finite number of eigen functions of its Laplacian that have $\mathbb{E}$ values. The book [9] contained the earliest findings on this topic; see [10] for a more current survey. Chen and Piccinni [11] conducted a general investigation on submanifolds of Euclidean space with finite type Gauss maps and categorized compact surfaces with 1-type Gauss maps within the context of the notion of finite type. A number of geometers also investigated pseudo-Euclidean spaces with finite type Gauss maps or submanifolds of Euclidean spaces (for further information, see [1-4]).

In [12], Choi studied the geometric properties of the Gauss map of ruled surfaces in a three-dimensional Minkowski space. In the paper [13], a characterization of the helicoid as ruled surfaces with pointwise 1-type Gauss map in 3-dimensional Euclidean space is obtained.

In [15], the authors studied ruled surfaces in a three-dimensional Minkowski space with pointwise 1-type Gauss map and obtain the complete classification theorems for those. They also obtain a new characterization of minimal ruled surfaces in a three-dimensional Minkowski space.

Motivated by the above work, we construct two special families of ruled surfaces in a strict Walker 3-manifold and we give a sufficient and necessary condition for these surfaces to be pointwise 1-type Gauss map.

In this paper, the ambient space is a three-dimensional Walker manifold. The strict Walker manifolds are described in terms of suitable coordinates (x, y, z) of the manifolds $\mathbb{R}^3$ and their metric depends on an arbitrary function of two variables $f = f(y, z)$ and their metric tensor is given by

$$g_f^{\varepsilon} = \varepsilon dy^2 + 2dxdz + fdz^2, \tag{1.1}$$

where $\varepsilon = \pm 1$. Curvature properties and a complete characterization of locally symmetric or locally conformally flat three-dimensional Walker manifolds have been studied in [7]. Also, in [5] the authors obtained a complete classification of parallel surfaces in a Lorentzian three-dimensional strict Walker manifold (i.e., admitting a parallel null vector field) as the ambient space. Some results on minimal graphs on three-dimensional Walker manifolds can be found in [14]. In [18], Niang et al. constructed two special families of ruled surfaces in a three-dimensional strict Walker manifold. They show that the local degeneracy (resp. non-degeneracy) to one of these families has a strong consequence on the geometry of the ambient Walker manifold. In [19], the same authors studied the geometry of minimal translation surfaces in a strict Walker 3-manifold. Based on the existence of two isometries, they classify minimal translation surfaces on this class of manifold.

The paper is organized as follows: in Section 2, we give some basic notions of the ambient space. And we recall the geometric properties of surfaces. In Section 3, we give the main results of the paper.

2. Preliminaries

2.1. Three-dimensional Walker manifold

Let M be the manifold $\mathbb{R}^3$ furnished with a metric g. Then we said that

(M, g) is *strict Walker* when it is Lorentzian and admits a parallel null vector field. Lorentzian three-dimensional manifolds admitting a parallel null vector field have been studied in [7]. These manifolds are described in terms of suitable local coordinates (x, y, z) and their metric depends on an arbitrary function of two variables $f = f(y, z)$. The metric g denoted by g_f^ε is given in these local coordinates by

$$g_f^\varepsilon = \varepsilon dy^2 + 2dxdz + fdz^2, \tag{2.1}$$

where $\varepsilon = \pm 1$. We denote this Lorentzian manifold by (M, g_f^ε).

Denote by ∇ the Levi-Civita connection of (M, g_f^ε). The curvature tensor R of (M, g_f^ε) is given by $R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$, with respect to the coordinate basis $(\partial_x, \partial_y, \partial_z)$, the only possibly nonvanishing components of ∇ is given by

$$\nabla_{\partial_y}\partial_z = \frac{1}{2} f_y \partial_x,$$

$$\nabla_{\partial_z}\partial_z = \frac{1}{2} f_z \partial_x - \frac{\varepsilon}{2} f_y \partial_y, \tag{2.2}$$

$$R(\partial_y, \partial_z)\partial_y = -\frac{1}{2} f_{yy} \partial_x,$$

$$R(\partial_y, \partial_z)\partial_z = \frac{\varepsilon}{2} f_{yy} \partial_y. \tag{2.3}$$

From (2.3) one can see that, if $f_{yy} = 0$, then (M, g_f^ε) is flat. And the Christoffel coefficients of the metric g_f^ε are given by

$$\Gamma_{23}^1 = \Gamma_{32}^1 = \frac{1}{2} f_y,$$

$$\Gamma_{33}^1 = \frac{1}{2} f_z,$$

$$\Gamma_{33}^2 = -\frac{\varepsilon}{2} f_y. \tag{2.4}$$

A surface $\Sigma \subset (M, g_f^{\varepsilon})$ is said to be *ruled* if every point of Σ is on (an open geodesic) segment of (M, g_f^{ε}). The details that a ruled surface is a 1-parameter family of differential geodesic can be found in [6]. Such a ruled surface is given by a smooth map φ from open set $U \subset \mathbb{R}^2$ into *M*, $(s, u) \mapsto \varphi(s, u)$ such that: for each *u*, the curve $s \mapsto \varphi(s, u)$ is a geodesic of *M*, and φ is an isometric immersion from $(U, \varphi^*(g_f^{\varepsilon}))$ into (M, g_f^{ε}). Since the vector field $X = \partial_x$ is parallel, the integral curve of *X* is geodesic by $\nabla_{\partial_x}\partial_x = 0$.

2.2. Recall of fundamental equation for surfaces

Let $\mathcal{D}$ be an open subset of the plane $\mathbb{R}^2$ satisfying this interval condition: horizontal or vertical lines intersect $\mathcal{D}$ in intervals (if at all). Then a *two-parameter map* is a smooth map $x : \mathcal{D} \to M$. Thus, *x* is composed of two interwoven families of *parameter curves*:

The *u*-parameter curve $v = v_0$ of *x* is $u \to x(u, v_0)$.

The *v*-parameter curve $u = u_0$ of x is $v \to x(u_0, v)$. The partial velocities $x_u = dx(\partial_u)$, $x_v = dx(\partial_v)$ are vector fields on *x*. Evidently $x_u(u_0, v_0)$ is the velocity vector at u_0 of the *u*-parameter curve $v = v_0$, and symmetrically for $x_v(u_0, v_0)$. If *x* lies in the domain of a coordinate system $x^1, ..., x^n$, then its coordinate functions $x_i \circ x\,(1 \leq i \leq n)$ are real-valued functions on $\mathcal{D}$, and

$$x_u = \sum \frac{\partial x^i}{\partial u}\partial_i, \quad x_v = \sum \frac{\partial x^i}{\partial v}\partial_i.$$

So far *M* could be a smooth manifold; now suppose it is semi-Riemannian. If *Z* is a smooth vector field on *x*, then its *partial covariant derivatives* are:

$Z_u = \dfrac{DZ}{\partial u}$, the covariant derivative of *Z* along *u*-parameter curves, and

$Z_v = \dfrac{DZ}{\partial v}$, the covariant derivative of *Z* along *v*-parameter curves.

Explicitly, $Z_u(u_0, v_0)$ is the covariant derivative at u_0 of the vector field $u \to Z(u, v_0)$ on the curve $u \to x(u, v_0)$.

In terms of coordinates, $Z = \sum Z^i \partial_i$, where each $Z^i = Z(x^i)$ is a real valued function on $\mathcal{D}$. Then

$$Z_u = \sum_k \left\{ \frac{\partial Z^k}{\partial u} + \sum_{i, j} \Gamma_{ij}^k Z^i \frac{\partial x^j}{\partial u} \right\} \partial_k. \tag{2.5}$$

In the special case $Z = x_u$ the derivative $Z_u = x_{uu}$ gives the accelerations of u-parameter curves, while x_{vv} gives v-parameter accelerations. With coordinate notation as above we have

$$x_{uv} = \sum_k \left\{ \frac{\partial^2 x^k}{\partial v \partial u} + \sum_{i, j} \Gamma_{ij}^k \frac{\partial x^i}{\partial u} \frac{\partial x^j}{\partial v} \right\} \partial_k. \tag{2.6}$$

This formula is symmetric in u and v, since Γ_{ij}^k is symmetric in i and j.

Let ϕ be an isometric immersion of a subset $U \subset \mathbb{R}^2$ in (M_f^2, g_f). Then the first fundamental form of the immersion is given by

$$\begin{cases} E = g_f^{\varepsilon}(\phi_*(\partial_u), \phi_*(\partial_u)) \\ F = g_f^{\varepsilon}(\phi_*(\partial_u), \phi_*(\partial_v)) \\ G = g_f^{\varepsilon}(\phi_*(\partial_v), \phi_*(\partial_v)). \end{cases} \tag{2.7}$$

The canonical connection of the map ϕ which is also denoted by $\overline{\nabla}$ is defined as follow: let $X, Y \in \Gamma(U)$ be vector fields of U and $m = (u, v) \in U$. Let $\overline{Y} \in \Gamma(M_f^3)$ be a local extension of Y in some neighborhood of m such that $\overline{Y}_{\varphi(m)} = Y_m$. Then

$$\overline{\nabla}_X \phi_*(Y)_m = (\overline{\nabla}_{d_m \phi(X)} \overline{Y})_{\phi(m)}. \tag{2.8}$$

For detail see Jacques Lafontaine Riemannian Geometry [17, p. 114].

For all vector fields $X, Y \in \chi(U)$ the fundamental Gauss form is

$$\overline{\nabla}_X \phi_*(Y) = \phi_*(\nabla_X Y) + \Pi(X, Y), \tag{2.9}$$

where ∇ is the Levi-Civita connection of the metric g_f^{ε} and $\Pi(X, Y)$ is the second fundamental form of the immersion ϕ. In particular, for the surface immersed in M_f^3, the second fundamental form Π is given by

$$\Pi(X, Y) = h(X, Y)N, \tag{2.10}$$

where h is also called the *second fundamental form* of the ϕ and η is a unitary normal along the map ϕ. Thus, equation (2.9) becomes

$$\overline{\nabla}_X \phi_*(Y) = \phi_*(\nabla_X Y) + h(X, Y)N. \tag{2.11}$$

Definition 2.1. The surface ϕ is said to be a *pointwise* 1-*type Gauss map* if

$$\Delta N = \lambda N, \tag{2.12}$$

where λ is a non zero constant and Δ is the Laplacian of the induced metric given by

$$\Delta N = \frac{1}{\sqrt{\det[g_{ij}]}} \sum_{i,j}^{3} \frac{\partial}{\partial x^i}\left(\sqrt{\det[g_{ij}]}\, g^{ij} \frac{\partial \eta}{\partial x^j}\right). \tag{2.13}$$

Let now u, v be two elements in M_f^3. Denoted by (e_1, e_2, e_3) the canonical frame in $\mathbb{R}^3$. By using the canonical isomorphism C in the semi-Riemannian manifold M_f^3 given by

$$C : \Gamma(M_f^3) \to \Lambda(M_f^3),$$

$$X \mapsto X^*$$

we find that

$$C^{-1}\{w \mapsto \det(u, v, w)\} \tag{2.14}$$

is the vector product $u \times v$ of u and v.

If $u = (u_1, u_2, u_3) \in M_f^3$ and $v = (v_1, v_2, v_3) \in M_f^3$, then

$$u \times v = (u_1 v_2 - u_2 v_1 - f(u_2 v_3 - u_3 v_2), -\varepsilon(u_1 v_3 - u_3 v_1), u_2 v_3 - u_3 v_2). \tag{2.15}$$

3. Main Results

A curve $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$ is a geodesic of (M, g_f^ε) if the following relations are satisfied:

$$\begin{cases} \dfrac{d^2\gamma_1(t)}{dt^2} = f_y \dfrac{d\gamma_2}{dt}\dfrac{d\gamma_3}{dt} + \dfrac{1}{2} f_z \left(\dfrac{d\gamma_3}{dt}\right)^2 \\ \dfrac{d^2\gamma_2(t)}{dt^2} = -\dfrac{\varepsilon}{2} f_y \left(\dfrac{d\gamma_3}{dt}\right)^2 \\ \dfrac{d^2\gamma_3(t)}{dt^2} = 0. \end{cases} \tag{3.1}$$

These equations have the following trivial solutions: $\gamma_1(t) = a_1 t + b_1$, $\gamma_2(t) = a_2 t + b_2$, and $\gamma_3(t) = b_3$, where $a_1, a_2, b_1, b_2, b_3 \in \mathbb{R}$. From these solutions, one gets the following ruled surfaces made by affine straight lines.

3.1. Ruled surfaces of type 1

Let $r \in \mathbb{R}$ and $b : \mathbb{R} \to \mathbb{R}$ be a smooth function.

We denote by $\Sigma_1(r, b)$ the surface in M defined by the equation:

$$x + \varepsilon r y - \varepsilon r^2 z - b(z) = 0.$$

The surface $\Sigma_1(r, b)$ can be parametrized by the map:

$$\varphi : \mathbb{R} \times \mathbb{R} \to M,$$

$$(y, z) \mapsto y(-\varepsilon r, 1, 0) + (b(z), rz, z). \tag{3.2}$$

The Gauss map of the family σ_1 is given by

$$\eta = \frac{\varphi_u \times \varphi_v}{\| \varphi_u \times \varphi_v \|}. \tag{3.3}$$

Using (2.7), we have the induced metric of the surface $\Sigma(r, b)$ given by

$$g = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \varepsilon & 0 \\ 0 & 2b' + \varepsilon r^2 + f \end{pmatrix}.$$

The determinant is given by

$$| g | = \varepsilon(2b' + \varepsilon r^2 + f).$$

By an easy computation, we get

$$g^{ij} = \begin{pmatrix} \varepsilon & 0 \\ 0 & \dfrac{1}{2b' + \varepsilon r^2 + f} \end{pmatrix}.$$

Using the equation (3.3), we have

$$\eta = \frac{((-\varepsilon r^2 - b') - f, r, 1)}{\sqrt{| 2b' + \varepsilon r^2 + f |}}.$$

If we use the formula in (2.13), then we get

$$\Delta\eta = \frac{1}{\sqrt{D}} \partial_v \left(\frac{-b'' - f_v}{D} (1, 0, 0) - \frac{2b'' + f_v}{2D^2} (A, r, 1) \right), \tag{3.4}$$

where $D = | 2b' + \varepsilon r^2 + f |$ and $A = -\varepsilon r^2 - b' - f$.

We have the following theorem.

Theorem 3.1. *Let* $\Sigma(r, b)$ *be the family of ruled surfaces given in* (3.2). *Then* $\Sigma(r, b)$ *is a pointwise* 1-*type Gauss map iff*:

(1) $b''(v) + f_v(u, v) = 0$,

(2) *The expressions with* $(-\varepsilon r^2 - b' - fr, 1)$ *in* $\Delta\eta$ *are proportional to* η.

In particular, if $b(v)$ *is linear and* $f(u, v)$ *is independent to* v, *then* $\Sigma(r, b)$ *is pointwise* 1-*type Gauss map with* $\lambda = 0$.

Proof. Suppose that the family of surfaces is pointwise 1-type Gauss map. Then by an easy computation we get

$$\partial_v\left(\frac{-b'' - f_v}{D}\right) = \frac{(-b''' - f_{vv})D + (b'' + f_v)(2b'' + f_v)}{D^2}.$$

Let us put $F = b'' + f_v$. Then we have

$$\partial_v\left(\frac{-b'' - f_v}{D}\right) = \frac{-F'D + (F)(F + b'')}{D^2}.$$

On the other hand, using the derivative computation we get

$$\partial_v\left(\frac{2b'' + f_v}{2D^2}(A, r, 1)\right) = \partial_v\left(\frac{2b'' + f_v}{2D^2}\right)(A, r, 1) + \frac{2b'' + f_v}{2D^2}\partial_v(A, r, 1).$$

That is

$$\partial_v\left(\frac{2b'' + f_v}{2D^2}(A, r, 1)\right) = \frac{(F' + b''')D - 2(F + b'')^2}{2D^{5/2}}\eta + \frac{F + b''}{2D^2}(-F, 0, 0),$$

where

$$\eta = \frac{1}{\sqrt{D}}(A, r, 1).$$

Thus, we get

$$\Delta\eta = \frac{-F'D + (F)(F + b'')}{D^2} - \frac{F(F + b'')}{2D^2}(1, 0, 0) - \frac{(F' + b''')D - 2(F + b'')^2}{2D^{5/2}}\eta.$$

The proportionality of $\Delta\eta$ and η, implies that the terms in $(1, 0, 0)$ must vanish, and then we have

$$F = b'' + f_v = 0.$$

The terms in $(A, r, 1)$ on $\Delta\eta$ must be proportional to η. Then we have

$$\frac{b'''D - 2b''^2}{2D^{5/2}}\eta = \lambda\eta.$$

By simplification we have

$$\frac{b'''D - 2b''^2}{2D^{5/2}} = \lambda.$$

Thus, the terms on $(A, r, 1)$ are proportional to η if and only if this relation is satisfied.

Conversely, suppose that we have

(1) $b'' + f_v = 0,$

(2) The terms on $(A, r, 1)$ in $\Delta\eta$ are proportional to η.

The condition $b'' + f_v = 0$ implies that the terms on $(1, 0, 0)$ in $\partial_v\eta$ vanish. Thus, $\partial_v\eta$ is given by

$$\partial_v\eta = \frac{b'''D - 2b''^2}{2D^3}(A, r, 1).$$

The proportionality of the terms on $(A, r, 1)$ in $\Delta\eta$ to η implies that

$$\Delta\eta = \lambda\eta.$$

Then the family of the surfaces $\Sigma(r, b)$ is pointwise 1-type Gauss map. □

Corollary 3.2. *If* $b(v)$ *is linear and* $f = f(u)$, *then we have*

$$\Delta\eta = 0.$$

Thus the family of surfaces $\Sigma(r, b)$ *is pointwise* 1-*type Gauss map with* $\lambda = 0$.

Now, we show the relation between the mean curvature H and the Laplacian $\Delta\eta$. We have the following theorem.

Theorem 3.3. *Let* $\Sigma(r, b)$ *be the family of surfaces given by* (3.2). *If the family of the surfaces is pointwise* 1-*type Gauss map*, *then the mean curvature H depends on the function* $b(v)$, $f(u, v)$ *and the metric induced on the surfaces*.

Proof. Suppose that the family of surfaces is pointwise 1-type Gauss map. Then by the previous computation we get

$$\Delta\eta = \frac{-F'D + (F)(F + b'')}{D^2} - \frac{F(F + b'')}{2D^2}(1, 0, 0)$$

$$- \frac{(F' + b''')D - 2(F + b'')^2}{2D^{5/2}}\eta.$$

By hypothesis the terms on $(1, 0, 0)$ must vanish and we have

$$F = b'' + f_v = 0.$$

The terms on $(A, r, 1)$ must be proportional to η. That gives

$$\frac{b'''D - 2b''^2}{2D^{5/2}}\eta = \lambda\eta.$$

By simplification we have

$$\frac{b'''D - 2b''^2}{2D^{5/2}}\eta = \lambda.$$

The mean curvature H is related to $\Delta\eta$ by the relation

$$\Delta\eta = -2H\eta.$$

By the relation in (2.15), we obtain

$$\lambda = -2H.$$

Thus using the expression of λ obtained above, we have

$$\frac{b'''D - 2b''^2}{2D^{5/2}} = -2H,$$

and we obtain H as

$$H = -\frac{b'''D - 2b''^2}{4D^{5/2}}. \qquad \square$$

Example 3.4. Consider the surface $\Sigma(r, b)$ defined by the following functions:

- $b(v) = v$ (a linear function),
- $f(u, v) = u^2$ (a function independent of v),
- $r = 1$ (an arbitrary constant).

The surface $\Sigma(r, b)$ can be parametrized by

$$\varphi(u, v) = (u, v, -u^2 - v).$$

The induced metric on the surface is given by

$$g = \begin{pmatrix} \varepsilon & 0 \\ 0 & 2b' + \varepsilon r^2 + f \end{pmatrix},$$

with $b'(v) = 1$ and $f(u, v) = u^2$, we obtain

$$g = \begin{pmatrix} \varepsilon & 0 \\ 0 & 2(1) + \varepsilon(1)^2 + u^2 \end{pmatrix} = \begin{pmatrix} \varepsilon & 0 \\ 0 & 2 + \varepsilon + u^2 \end{pmatrix}.$$

The Gauss map η is given by

$$\eta = \frac{(-\varepsilon r^2 - b' - f, r, 1)}{\sqrt{2b' + \varepsilon r^2 + f}}.$$

Substituting the values, we obtain

$$\eta = \frac{(-\varepsilon(1)^2 - 1 - u^2, 1, 1)}{\sqrt{2(1) + \varepsilon(1)^2 + u^2}} = \frac{(-\varepsilon - 1 - u^2, 1, 1)}{\sqrt{2 + \varepsilon + u^2}}.$$

Since $b''(v) = 0$ and $f_v = 0$, we have $F = b'' + f_v = 0$. Thus, $\Delta\eta = 0$, which confirms that the surface is a pointwise 1-type Gauss map with $\lambda = 0$.

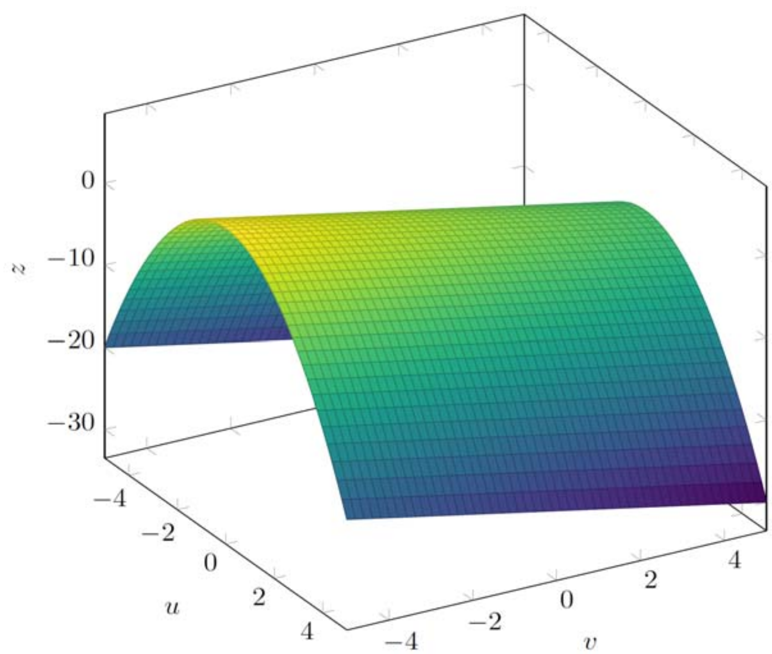
The mean curvature H is given by

$$H = -\frac{(F' + b''')D - 2(F + b'')^2}{4D^{5/2}},$$

with $F = 0$ and $b'' = 0$, we obtain $H = 0$, which confirms that the surface is minimal.

Here is the graph of the surface $\Sigma(r, b)$ defined by $z = -u^2 - v$:

Surface $\Sigma(r, b)$ with $b(v) = v$ and $f(u, v) = u^2$.



3.2. Ruled surfaces of type 2

Let $c : \mathbb{R} \to \mathbb{R}$ be a smooth function. Then we denote by $\Sigma_2(c)$ the family of surfaces of M defined by $y = c(z)$. These surfaces can be parametrized by

$$\psi : \mathbb{R} \times \mathbb{R} \to M, \tag{3.5}$$

$$(x, z) \mapsto x(1, 0, 0) + (0, c(z), z). \tag{3.6}$$

The matrix of the induced metric of ψ is given by

$$\begin{pmatrix} 0 & 1 \\ 1 & \varepsilon c'^2 + f \end{pmatrix}.$$

Then the unit normal vector field ξ of ψ is

$$\xi = c'\partial_x - \varepsilon\partial_y. \tag{3.7}$$

We have the following theorem.

Theorem 3.5. *Let* $\Sigma_2(c)$ *be the family of the surfaces given by* (3.5). *Then the normal vector* ξ *is harmonic. That is the family of the surfaces* $\Sigma_2(c)$ *is a pointwise* 1-*type Gauss map*.

Proof. The proof is based on the explicit calculation of $\Delta\xi$ using the Laplace operator.

The induced metric $\Sigma_2(c)$ is given by

$$g = \begin{pmatrix} 0 & 1 \\ 1 & G \end{pmatrix},$$

where $G = \varepsilon c'^2 + f$.

The partial derivatives of ξ are

$$\partial_x\xi = 0, \quad \partial_z\xi = c''\partial_x.$$

By an easy computation, we have

$$\Delta\xi = \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}g^{ij}\partial_j\xi).$$

Since $\partial_x\xi = 0$ and $g^{zz} = 0,$ this relation becomes

$$\Delta\xi = \frac{1}{\sqrt{|g|}}\partial_z(\sqrt{|g|}\cdot 0\cdot\partial_z\xi) = 0.$$

Thus, $\Delta\xi = 0,$ and the result follows. □

4. Conclusion

In this study, we examined special families of ruled surfaces within a three-dimensional strict Walker manifold and established conditions for these surfaces to possess a pointwise 1-type Gauss map. Our findings provide necessary and sufficient criteria for such surfaces and offer a characterization of ruled surfaces with constant mean curvature in this geometric setting.

Through explicit computations, we demonstrated that certain families of ruled surfaces satisfy the pointwise 1-type Gauss map condition under specific constraints on their defining functions. Additionally, we established a direct relationship between the mean curvature and the Laplacian of the Gauss map, reinforcing the interplay between curvature properties and geometric classifications.

These results contribute to the broader understanding of ruled surfaces in pseudo-Riemannian geometry and open avenues for further exploration of their applications in differential geometry and mathematical physics. Future research may extend these findings to higher-dimensional Walker manifolds or investigate other geometric structures with similar properties.

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