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GENERALIZED SAMUEL NUMBER $\overline{w}_\varphi(\theta)$ AND AXE-FILTRATIONS ON A SEMIMODULE

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Abstract

This work extends the theory of Samuel numbers to semimodules by introducing a generalized number, $\overline{w}_\varphi(\theta)$, for two axe-filtrations. We establish its existence under regularity conditions, including the Approximable by Powers (AP) and weakly good properties. By

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adapting the concept of valuative reduction, we prove that this invariant is well-defined and robust, laying the foundation for a quantitative analysis of filtration structures on semimodules.

1. Introduction

The theory of filtrations, pioneered by Samuel and Rees for Noetherian local rings, is fundamental in commutative algebra for studying ideals and singularities through invariants like Samuel numbers [10, 11]. While this classical framework relies on an additive group structure, many algebraic systems, such as those in tropical geometry or language theory, are based on semirings, which only feature an additive monoid. This work extends the theory of Samuel numbers to this broader context of semimodules over semirings, a non-trivial generalization due to the absence of classical tools like exact sequences. We introduce a generalized Samuel number, denoted $\overline{w}_{\varphi}(\theta)$, for two axe-filtrations φ and θ on a B-semimodule M. Our main contribution is to establish that this number is well-defined under specific regularity conditions on the filtrations, notably the AP (Approximable by Powers) property and the weakly good property. Furthermore, we adapt the concept of valuative reduction from rings to semimodules and prove that our generalized Samuel number is invariant under this reduction, confirming its robustness. These results provide a solid foundation for a quantitative theory of axe-filtrations on semimodules.

1. Preliminaries

Definition 1.1. (1) A *monoid* is a set B equipped with an internal composition law $*$ that is associative, commutative, and has an identity element denoted by e_B.

(2) Let $(B, *)$ be a monoid and A be a subset of B.

We say that A equipped with the law $*$ from B is a *submonoid* of B if $(A, *)$ is a monoid such that $e_A = e_B$.

(3) A *pre-semiring* is a set B equipped with both an addition and a multiplication, and which is a monoid for each of these operations.

(4) A *semiring* is a set B equipped with an addition and a multiplication satisfying:

(i) $(B, +, .)$ is a pre-semiring;

(ii) Multiplication is distributive with respect to addition;

(iii) 0 is the annihilator of B.

(5) Let $(B, +, .)$ be a semiring and A be a subset of B. We say that A is a *subsemiring* of B if $(A, +, .)$ is a semiring such that $0_A = 0_B$ and $1_A = 1_B$.

Example 1.2. (1) Every commutative unitary ring is a semiring.

(2) The sets $(\mathbb{N}, +, .)$, $(\mathbb{R}_+, +, .)$, $(\mathbb{Q}_+, +, .)$ are semirings. Thus $\mathbb{N}$ is a subsemiring of $\mathbb{Q}_+$ and $\mathbb{Q}_+$ is a subsemiring of $\mathbb{R}_+$.

(3) $\sqrt{2}\mathbb{N}$ is a submonoid of $(\mathbb{R}_+, +)$.

Definition 1.3. Let B be a semiring.

(1) Let $(M, +)$ be a monoid.

We say that M is a B-semimodule if the map $(b; x) \mapsto bx$ from $B \times M$ to M satisfies the following conditions:

(i) $\forall b \in B$ and $\forall x_1, x_2 \in M$, $b(x_1 + x_2) = bx_1 + bx_2$;

(ii) $\forall b_1, b_2 \in B$ and $\forall x \in M$, $(b_1 + b_2)x = b_1x + b_2x$;

(iii) $\forall b_1, b_2 \in B$ and $\forall x \in M$, $b_1(b_2x) = (b_1b_2)x$;

(iv) $\forall x \in M$, $1_B \cdot x = x$;

(v) $\forall b \in B$, $\forall x \in M$, $0_B \cdot x = b \cdot 0_M = 0_M$.

(2) Let $(M, +, .)$ be a B-semimodule and N be a subset of M.

We say that N is a subsemimodule of M if $(N, +, .)$ is a B-semimodule with $0_N = 0_M$.

Example 1.4. (1) $\mathbb{Z}_-$, $\mathbb{Q}_-$ and $\mathbb{R}_-$ are $\mathbb{N}$-semimodules.

(2) $\sqrt{2}\mathbb{Z}_-$ is a subsemimodule of the $\mathbb{N}$-semimodule $\mathbb{R}_-$.

Definition 1.5. Let B be a semiring.

(1) We say that $g = (G_n)_{n\in\mathbb{N}\bigcup\{+\infty\}}$ is a quasi-filtration of B if:

(i) $G_0 = B$ is a subsemiring of B;

(ii) $G_\infty = (0)$;

(iii) $G_n G_m \subseteq G_{m+n}$, $\forall m, n \in \mathbb{N}$;

(iv) $G_{n+1} \subseteq G_n$, $\forall n \in \mathbb{N}^*$.

(2) A quasi-filtration $g = (G_n)_{n\in\mathbb{N}\bigcup\{+\infty\}}$ of the semiring B is said to be AP if there exists a sequence of positive integers $(k_n)_{n\in\mathbb{N}}$ such that:

(i) $G_{mk_n} \subseteq G_n^m$, $\forall m, n \in \mathbb{N}$;

(ii) $\lim_{n\to+\infty} \dfrac{k_n}{n} = 1$.

Example 1.6. (1) All filtrations of a semiring B are quasi-filtrations of B.

(2) Let $B = \mathbb{R}_+$ and $A = \mathbb{N}$.

A is a subsemiring of the semiring B. If p is a prime number, then by taking $F = p\mathbb{N} + \sqrt{p}\mathbb{N}$, F is a submonoid of $(B, +)$ such that $F^{n+1} \subseteq F^n$, $\forall n \in \mathbb{N}^*$.

Thus $g = (G_n = F^n)_{n\in\mathbb{N}\bigcup\{+\infty\}}$, where $G_\infty = (0)$ and $F^0 = A$, is a quasi-filtration of the semiring B which is not a filtration of B.

2. Study of Axe-filtrations on a Semimodule

Definition 2.1. Let B be a semiring and M be a B-semimodule.

(1) Let $\varphi = (M_n)_{n\in\mathbb{N}\bigcup\{+\infty\}}$ be a sequence of submonoids of $(M, +)$.

We say that φ is an axe-filtration of M if there exists a subsemiring A of B such that $M_{n+1} \subseteq AM_n$, $\forall n \in \mathbb{N}$ and $M_{\infty} = (0)$.

(2) Let A be a subsemiring of B. $AF_A(M)$ is the set of axe-filtrations of the B-semimodule M relative to A, that is

$$\varphi = (M_n) \in AF_A(M) \Rightarrow M_{n+1} \subseteq AM_n,\ \forall n \in \mathbb{N}.$$

The set $AF_A(M)$ is ordered by the relation:

$$\varphi = (M_n) \leq \psi = (F_n),\ \text{if } M_n \subseteq F_n \text{ for all } n \geq 0.$$

(3) Let $f = (F_n)_{n \in \mathbb{N} \cup \{+\infty\}}$ be a quasi-filtration of the semiring B such that $F_0 = A$ and $\varphi = (M_n)_{n \in \mathbb{N} \cup \{+\infty\}}$ is an axe-filtration of M.

(a) We say that φ is axe-compatible with f if $F_m M_n \subseteq AM_{n+m}$, $\forall m, n \in \mathbb{N}$.

(b) The axe-filtration φ of M is said to be *weakly f-good* if:

(i) φ is axe-compatible with f.

(ii) $\exists m \geq 1$ such that:

$$AM_n = \sum_{p=0}^{m} F_{n-p} M_p,\ \forall n \geq m.$$

(4) For any real number $\lambda > 0$, $\{\lambda\}$ is the smallest natural number greater than or equal to λ. If $\varphi = (M_n)_{n \in \mathbb{N} \cup \{+\infty\}}$ is an axe-filtration of the B-semimodule M, then $\varphi^{(\lambda)} = (M_{\{n\lambda\}})_{n \in \mathbb{N} \cup \{+\infty\}}$ is an axe-filtration of M.

(5) If $\varphi = (M_n)_{n \in \mathbb{N} \cup \{+\infty\}}$ is an axe-filtration of M and $a \in \mathbb{N}$, we define $t_a\varphi = (M'_n)_{n \in \mathbb{N} \cup \{+\infty\}}$, where $M'_0 = M_0$ and $M'_n = M_{a+n}$ if $n \geq 1$.

Then $t_a\varphi$ is an axe-filtration of M called the *truncated axe-filtration* of φ.

Example 2.2. (1) All filtrations of a semimodule M are axe-filtrations of M.

(2) Let $B = \mathbb{R}_+$ and $M = \mathbb{R}_-$; $A = \mathbb{N} + \sqrt{2}\mathbb{N}$, $\varphi = (M_n)_{n \in \mathbb{N} \bigcup \{+\infty\}}$, where $M_n = \sqrt{2}^n \mathbb{Z}_-$ and $M_\infty = (0)$. It can be easily shown that φ is an axe-filtration of the B-semimodule M. It should be noted that φ is not a filtration of M.

Proposition 2.3. *Let B be a semiring, M be a B-semimodule, and $f = (F_n)_{n \in \mathbb{N} \bigcup \{+\infty\}}$ be a quasi-filtration of B such that $F_0 = B$. Then the sequence $g = (F_n M)_{n \in \mathbb{N} \bigcup \{+\infty\}}$ is an axe-filtration of M, called the axe-filtration of the B-semimodule M relative to the quasi-filtration f of B, denoted by f_M or fM.*

Proof. (i) Since M is a B-semimodule and F_n is a submonoid of B for all $n \geq 0$, $F_n M$ is a submonoid of $(M, +)$ for all $n \geq 0$.

(ii) Since $F_0 = A$ is a subsemiring of B and $F_{n+1}M \subseteq BF_n M$, $\forall n \in \mathbb{N}$, f_M is an axe-filtration of M. □

3. Study of $\overline{w}_{fM}(gM)$

In [9], the Samuel number $\overline{w}_f(g)$ was defined for two quasi-filtrations f and g of a semiring B. In this section, we extend this notion to the Samuel number $\overline{w}_\varphi(\theta)$, where $\varphi = (M_n)_{n \in \mathbb{N}}$ and $\theta = (F_n)_{n \in \mathbb{N}}$ are axe-filtrations of the B-semimodule M by setting: $\overline{w}_\varphi(\theta) = \lim\limits_{n \to +\infty} \frac{w_\varphi(F_n)}{n}$ if this limit exists in $\overline{\mathbb{R}}_+$.

Notation 3.1. Let B be a semiring, M be a B-semimodule, $\varphi = (M_n)_{n \in \mathbb{N} \bigcup \{+\infty\}}$ be an axe-filtration of the B-semimodule M, and N be a submonoid of $(M, +)$. Then there exists a subsemiring A of B such that $M_{n+1} \subseteq AM_n, \ \forall n \geq 1$. We define

(i) $w_{\varphi}(N) = \inf\{r \in \mathbb{N}; M_r \subseteq N\}$, where $w_{\varphi}(N) = 0$ if $\{r \in \mathbb{N}; M_r \subseteq N\}$ is empty.

(ii) If $\theta = (F_n)_{n \in \mathbb{N} \bigcup \{+\infty\}}$ is an axe-filtration of M, then $\overline{v}_{\varphi}(\theta) = \lim_{n \to +\infty} \dfrac{1}{n} v_{\varphi}(F_n)$ if this limit exists.

(iii) If G is a submonoid of the semiring B, then we define

(a) $\overline{w}_{FM}(GM) = \overline{w}_{f_F M}(f_G M)$;

(b) $\overline{w}_{FM}(gM) = \overline{w}_{f_F M}(g_M)$

for any submonoid F of $(B, +)$ and any quasi-filtration g of B.

Remark 3.2. Let B be a semiring, and let $\varphi, \theta, \varphi'$ and θ' be filtrations of the B-semimodule M such that $\overline{w}_{\varphi}(\theta), \overline{w}_{\varphi}(\theta')$ and $\overline{w}_{\varphi'}(\theta)$ exist in $\mathbb{R}_+$. Then

(i) $\theta \leq \theta' \Rightarrow \overline{w}_{\varphi}(\theta) \geq \overline{w}_{\varphi}(\theta')$.

(ii) $\varphi \leq \varphi' \Rightarrow \overline{w}_{\varphi}(\theta) \geq \overline{w}_{\varphi'}(\theta)$.

Proposition 3.3. *Let B be a semiring, $\varphi = (M_n)_{n \in \mathbb{N} \bigcup \{+\infty\}}$ and $\theta = (F_n)_{n \in \mathbb{N} \bigcup \{+\infty\}}$ be axe-filtrations of the B-semimodule M, and let $\lambda > 0$ be a real number. Then the following assertions are equivalent*:

(i) $\overline{w}_{\varphi}(\theta)$ *exists in* $\overline{\mathbb{R}}_+$.

(ii) $\overline{w}_{\varphi^{(\lambda)}}(\theta)$ *exists in* $\overline{\mathbb{R}}_+$.

(iii) $\overline{w}_{\varphi}(\theta^{(\lambda)})$ *exists in* $\overline{\mathbb{R}}_+$.

If one of the equivalent assertions holds, *then*

$$\overline{w}_{\varphi}(\theta) = \lambda \overline{w}_{\varphi^{(\lambda)}}(\theta) = \frac{1}{\lambda}\overline{w}_{\varphi}(\theta^{(\lambda)}).$$

Proof. (i) $\Leftrightarrow$ (ii) First, suppose there exists an integer n_0 such that $w_{\varphi^{(\lambda)}}(F_{n_0}) = +\infty$. Then F_{n_0} contains no B-semimodule $M_{\{r\lambda\}}$ with $r \in \mathbb{N}$. Let m be an integer such that $m = 1$ if $\lambda \geq 1$ and $m\lambda \geq 1$ if $0 < \lambda < 1$. Then we have $\varphi^{(m\lambda)} \leq \varphi$ and $w_{\varphi^{(m\lambda)}}(F_{n_0}) = +\infty$. It follows that $w_{\varphi}(F_{n_0}) = +\infty$, and thus $w_{\varphi}(F_n) = +\infty$ for all $n \geq n_0$. Since $w_{\varphi^{(\lambda)}}(F_n) = +\infty$ for all $n \geq n_0$, we have $\overline{w}_{\varphi^{(\lambda)}}(\theta) = \overline{w}_{\varphi}(\theta) = +\infty = \dfrac{\overline{w}_{\varphi}(\theta)}{\lambda}$. Now suppose that $w_{\varphi^{\lambda}}(F_n) = s_n \in \mathbb{N}$, for all $n \in \mathbb{N}$. Then $F_n \supseteq M_{\{\lambda s_n\}}$. But $F_n \not\supseteq M_{\{\lambda(s_n - 1)\}}$.

So

$$\{\lambda(s_n - 1)\} < w_{\varphi}(F_n) \leq \{\lambda s_n\} \text{ and } \lambda(s_n - 1) < w_{\varphi}(F_n) \leq \lambda s_n + 1.$$

Therefore, we have

$$\frac{1}{n} w_{\varphi}(F_n) - \frac{\lambda}{n} < \frac{\lambda(s_n - 1)}{n} + \frac{1}{n} < \frac{1}{n}(w_{\varphi}(F_n) + 1) < \frac{\lambda s_n}{n} + \frac{2}{n}.$$

The equivalences (i) $\Leftrightarrow$ (ii) follow from the first case of the inequalities above.

(i) $\Rightarrow$ (iii) This follows from the fact that

$$\frac{1}{n} w_{\varphi}(F_{\{\lambda n\}}) = \frac{\{\lambda n\}}{n} \frac{1}{\{\lambda n\}} w_{\varphi}(F_{\{\lambda n\}})$$

and that $\lim_{n \to +\infty} \dfrac{1}{n}\{\lambda n\} = \lambda.$

(iii) $\Rightarrow$ (i) For $n \in \mathbb{N}$, let q_n be the greatest integer less than or equal to $\frac{n}{\lambda}$. We have $\{\lambda q_n\} \leq \{\lambda(q_n + 1)\}$ and

$$\frac{q_n}{n}\frac{w_{\varphi}(F_{\{\lambda q_n\}})}{q_n} \leq \frac{w_{\varphi}(F_n)}{n} \leq \frac{(q_n + 1)}{n}\frac{w_{\varphi}(F_{\{\lambda(q_n+1)\}})}{q_n + 1}.$$

Then we obtain (iii) $\Rightarrow$ (i) by taking the limit as n tends to $+\infty$. □

Proposition 3.4. *Let B be a semiring, f and g be two quasi-filtrations of the semiring B, and M be a B-semimodule. Then the generalized Samuel number* $\overline{w}_{fM}(gM)$ *exists in* $\overline{\mathbb{R}}_+$ *for any AP quasi-filtration f.*

Proof. Let $f = (F_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ and $g = (G_n)_{n\in\mathbb{N}\cup\{+\infty\}}$. Suppose that $\underline{\lim}\frac{w_{fM}(G_nM)}{n} = a \in \overline{\mathbb{R}}_+$ and, since the sequence $(w_{fM}(G_nM))_{n\in\mathbb{N}}$ is increasing, $w_{fM}(G_nM) \to +\infty$ as $n \to +\infty$. Since f is an AP quasi-filtration, there exists a sequence of positive integers $(k_j)_{j\in\mathbb{N}}$ such that $F_{k_jn} \subseteq F_j^n$ for all j, n and $\lim_{j\to+\infty}\frac{k_j}{j} = 1$. If s, n are two integers greater than or equal to 1, then the chain of inclusions $F_{nk_{w_{fM}(G_sM)}}M \subset F^n_{w_{fM}(G_sM)}M \subset G_s^nM \subset G_{ns}M$ gives the inequality

$$w_{fM}(G_{ns}M) \leq nk_{w_{fM}}(G_sM). \qquad (*)$$

Take two integers $n > m \geq 1$ and let q_n be the smallest integer greater than or equal to $\frac{n}{m}$. From $(*)$, we have

$$w_{fM}(G_nM) \leq w_{fM}(G_{q_nm}M) \leq q_nk_{w_{fM}(G_mM)}.$$

Thus, we have

$$\frac{w_{fM}(G_nM)}{n} \leq \frac{q_nm}{n}\frac{k_{w_{fM}(G_mM)}}{m},$$

from which

$$\overline{\lim}\frac{w_{fM}(G_nM)}{n} \le \frac{k_{w_{fM}(G_mM)}}{m}.$$

Since

$$\frac{k_{w_{fM}(G_mM)}}{m} = \frac{w_{fM}(G_mM)}{m}\frac{k_{w_{fM}(G_mM)}}{w_{fM}(G_mM)},$$

we obtain

$$\overline{\lim}\frac{w_{fM}(G_mM)}{n} \le \inf_m \frac{k_{w_{fM}(G_mM)}}{m} \le \underline{\lim}\frac{w_{fM}(G_mM)}{m},$$

and the sequence $\left(\frac{w_{fM}(G_mM)}{m}\right)_{m\in\mathbb{N}}$ converges in $\overline{\mathbb{R}}_+$. □

Proposition 3.5. *Let B be a semiring, f and g be two quasi-filtrations of the semiring B such that f is an AP quasi-filtration, and let M be a B-semimodule. Then the sequences*

$$\left(\frac{\overline{w}_{fM}(G_nM)}{n}\right)_{n\in\mathbb{N}^*}$$

and $(n\overline{w}_{F_nM}(gM))_{n\in\mathbb{N}}$ *converge in* $\overline{\mathbb{R}}_+$, *and*

(i) $\overline{w}_{fM}(gM) = \lim_{n\to+\infty}\frac{\overline{w}_{fM}(G_nM)}{n} = \inf_n \frac{\overline{w}_{fM}(G_nM)}{n}$;

(ii) $\overline{w}_{fM}(gM) = \lim\limits_{n\to+\infty} n\overline{w}_{F_nM}(gM)$.

Proof. Let f be an AP quasi-filtration and $(k_n)_{n\in\mathbb{N}}$ be a sequence of positive integers such that $F_{k_nm} \subset F_n^m$ for all n, m with $\lim\limits_{n\to+\infty}\frac{k_n}{n} = 1$.

(i) We know from Proposition 3.4 that $\overline{w}_{fM}(G_nM)$ exists in $\overline{\mathbb{R}}_+$ for all integers $n \geq 0$. Furthermore, since the sequence $(\overline{w}_{fM}(G_nM))_{n\in\mathbb{N}}$ is increasing, we can assume that $\overline{w}_{fM}(G_nM) < +\infty$ for all integers n. Since $\overline{w}_{fM}(GM) \leq k_{w_{fM}(gM)}$ for any submonoid G of B, we obtain $\overline{\lim}\frac{\overline{w}_{fM}(G_nM)}{n} \leq \overline{w}_{fM}(gM)$. On the other hand, for any integer n and any integer m, we have $G_n^m M \subset G_{nm}M$, hence $\frac{w_{fM}(G_n^m M)}{nm} \geq \frac{w_{fM}(G_{nm}M)}{nm}$. As $m \to +\infty$, we get $\frac{\overline{w}_{fM}(G_nM)}{n} \geq \overline{w}_{fM}(gM)$ for all n. It follows that the sequence $\left(\frac{\overline{w}_{fM}(G_nM)}{n}\right)_{n\in\mathbb{N}^*}$ converges to $\overline{w}_{fM}(gM)$. Moreover, $\overline{w}_{fM}(gM) = \inf_n \frac{\overline{w}_{fM}(G_nM)}{n}$.

(ii) First, let us show that $\overline{w}_{fM}(gM) = \underline{\lim}\, n\overline{w}_{F_nM}(gM)$. We can assume that $k_n \geq n$ for all $n \geq 1$. Then if $g = (G_n)_{n\in\mathbb{N}\cup\{+\infty\}}$, the chain of inequalities $f_{F_{k_n}} \leq f^{(k_n)} \leq f_{F_n}$ gives us

$$\overline{w}_{F_{k_n}M}(gM) \leq \overline{w}_{f^{(k_n)}M}(gM) = \frac{1}{k_n}\overline{w}_{fM}(gM) \leq \overline{w}_{F_nM}(gM). \quad (*)$$

By multiplying the terms of $(*)$ by k_n and taking the lower limit, we obtain $\underline{\lim}\, k_n\overline{w}_{F_nM}(gM) \leq \overline{w}_{fM}(gM) \leq \underline{\lim}\, n\overline{w}_{F_nM}(gM)$ and $\underline{\lim}\, n\overline{w}_{F_nM}(gM) \leq \underline{\lim}\, k_n\overline{w}_{F_{k_n}M}(gM)$ because $k_n \geq n$ for all n. We have thus shown that $\overline{w}_{fM}(gM) = \underline{\lim}\, n\overline{w}_{F_nM}(gM)$. Now let us show that $\overline{\lim}\, n\overline{w}_{F_nM}(gM) = \overline{w}_{fM}(gM)$. Suppose that $\overline{w}_{fM}(gM) \in \mathbb{R}_+$ and take two integers $p \geq 1$ and $q \geq 1$ such that $\overline{w}_{fM}(gM) < \frac{p}{q}$; then $\overline{w}_{fM}(g^{(q)}M) < p$ and there exists an integer $N \geq 1$ such that for all $m \geq N$, we have $\overline{w}_{fM}(G_{mq}M) \leq mp$.

Consequently, for all integers $m \geq N$, we have $F_{mp} \subset G_{mp}$. Let n be an integer greater than or equal to 1 and q_m be the integer part of $\frac{m}{n}$. We then have the chain of inclusions $F_n^{(q_m+1)p}M \subset F_{n(q_m+1)p}M \subset F_{mp}M \subset G_{mq}M$, which implies the inequality $\frac{w_{F_nM}(G_{mq})}{mq} \leq \frac{p(q_m+1)}{mq}$. By taking the limit as $m \to +\infty$, we get $\overline{w}_{F_nM}(gM) \leq \frac{1}{n}\frac{p}{q}$, which means

$$\overline{\lim}\, n\overline{w}_{F_nM}(gM) \leq \frac{p}{q},$$

hence $\overline{\lim}\, n\overline{w}_{F_nM}(gM) \leq \overline{w}_{fM}(gM)$. □

4. Valuative Reductions on a Semimodule and $\overline{w}_{\varphi}(\theta)$

Definition 4.1. Let B be a semiring, $\varphi = (M_n)_{n \in \mathbb{N} \cup \{+\infty\}}$ be an axe-filtration of the B-semimodule M and $a \in \mathbb{N}$. We define $t_a\varphi = (M'_n)_{n \in \mathbb{N}}$, where $M'_0 = M_0$ and $M'_n = M_{a+n}$ for $n \geq 1$. Then $t_a\varphi$ is an axe-filtration of M called the *truncated axe-filtration* of φ.

Definition 4.2. Let B be a semiring, $\varphi = (M_n)_{n \in \mathbb{N} \cup \{+\infty\}}$ and $\theta = (U_n)_{n \in \mathbb{N} \cup \{+\infty\}}$ be axe-filtrations of the B-semimodule M. For any $x \in M$, we define $v_{\varphi}(x) = \sup\{n \in \mathbb{N}, x \in M_n\}$. We say that the axe-filtration φ is a valuative reduction of the filtration θ if there exists a constant $a \in \mathbb{N}$ such that for all $x \in M$ we have: $0 \leq v_{\theta}(x) - v_{\varphi}(x) \leq a$ with $v_{\varphi}(x) = +\infty$ if and only if $v_{\theta}(x) = +\infty$. This notion of valuative reduction was introduced by Rees [10].

Lemma 4.3. *Let B be a semiring, $\varphi = (M_n)_{n \in \mathbb{N} \cup \{+\infty\}}$ be an axe-filtration of the B-semimodule M, and N be a submonoid of M. Then the* *following assertions are equivalent*:

(i) $v_{\varphi}(N) = +\infty$;

(ii) *for all* $a \in \mathbb{N}$, $v_{t_a\varphi}(N) = +\infty$;

(iii) *there exists an* $a \in \mathbb{N}$ *such that* $v_{t_a\varphi}(N) = +\infty$.

Proof. (a) We show that (i) implies (ii). If $v_{\varphi}(N) = +\infty$, then $N \subseteq M_p$ for all $p \in \mathbb{N}$. Thus, for all $a \in \mathbb{N}$, $N \subseteq M_{a+q}$ for all $q \in \mathbb{N}$, hence $v_{t_a\varphi}(N) = +\infty$.

(b) (ii) implies (iii) is obvious.

(c) We show that (iii) implies (i). Let $a \in \mathbb{N}$ such that $v_{t_a\varphi}(N) = +\infty$. Then $N \subseteq M_{a+q}$ for all $q \in \mathbb{N}^*$, and we have $N \subseteq M_p$ for all $p \geq a$. We deduce that $N \subseteq M_p$ for all $p \in \mathbb{N}$, thus $v_{\varphi}(N) = +\infty$. □

Theorem 4.4. *Let* B *be a semiring*, $\varphi = (M_n)_{n \in \mathbb{N} \cup \{+\infty\}}$ *and* $\theta = (U_n)_{n \in \mathbb{N} \cup \{+\infty\}}$ *be axe-filtrations of the B-semimodule M. Then the following assertions are equivalent*:

(i) φ *is a valuative reduction of* θ;

(ii) *there exists an integer* $b \geq 0$ *such that* $t_b\theta \leq \varphi \leq \theta$.

Proof. (a) We show that (i) implies (ii). Assume (i) is true. Then there exists $a \in \mathbb{N}$ such that $0 \leq v_{\theta}(x) - v_{\varphi}(x) \leq a$ for all $x \in M$. We have $v_{\varphi}(x) \geq v_{\theta}(x) - a$. If $n \geq a$, then $x \in U_n$ implies $v_{\theta}(x) \geq n$, so $v_{\varphi}(x) \geq n - a$, hence $x \in M_{n-a}$. Since $v_{\varphi}(x) \geq v_{\theta}(x)$, $M_n \subseteq U_n$ for all $n \in \mathbb{N}$. For all $n \geq a$, we have $M_n \subseteq U_n \subseteq M_{n-a} \subseteq U_{n-a}$. Letting $p = n - a$, for $n \geq a$ we have $U_{p+a} \subseteq M_p \subseteq U_p$ for all $p \in \mathbb{N}$, hence $t_a\theta \leq \varphi \leq \theta$, and we deduce that (i) implies (ii).

(b) Next, we show that (ii) implies (i).

Assume (ii) is true. Then there exists $b \in \mathbb{N}$ such that $t_b\theta \leq \varphi \leq \theta$.

Hence we have: $v_{t_b\theta}(x) \leq v_\varphi(x) \leq v_\theta(x)$ for all $x \in M$. (1)

If $v_\varphi(x) = +\infty$, then $v_\theta(x) = +\infty$ (from (1)).

1) If $v_\theta(x) = +\infty$, then $v_{t_b\theta}(x) = +\infty$ (Lemma 4.3).

(1) then implies $v_\varphi(x) = +\infty$.

Consequently, we have: $v_\varphi(x) = +\infty$ if and only if $v_\theta(x) = +\infty$.

We then deduce that $v_\varphi(x) \in \mathbb{N}$ if and only if $v_\theta(x) \in \mathbb{N}$. (2)

If $v_\theta(x) \in \mathbb{N}$, let us show that $v_\theta(x) \leq v_{t_b\theta}(x) + 1 + b$.

Let $v_{t_b\theta}(x) = r \in \mathbb{N}$. If $r \neq 0$, then $x \in U_{b+r}$ and $x \notin U_{b+r+1}$, we have

$$v_{t_b\theta}(x) + b \leq v_\theta(x) \leq v_{t_b\theta}(x) + 1 + b. \qquad (*)$$

2) If $r = 0$, then $x \in U_0 = M$ and $x \notin U_{b+1}$, we have

$$0 = v_{t_b\theta}(x) \leq v_\theta(x) \leq v_{t_b\theta}(x) + 1 + b. \qquad (**)$$

$(*)$ and $(**)$ imply $v_\theta(x) < v_{t_b\theta}(x) + 1 + b$. (3)

(1) and (3) imply $0 \leq v_\theta(x) - v_\varphi(x) \leq v_\theta(x) - v_{t_b\theta}(x) < 1 + b$, so we have $0 \leq v_\theta(x) - v_\varphi(x) \leq v_\theta(x) - v_{t_b\theta}(x) \leq b$. (4)

(4) implies that φ is a valuative reduction of θ, and the result is proven. □

Lemma 4.5. *Let B be a semiring, $\varphi = (M_n)_{n \in \mathbb{N} \cup \{+\infty\}}$ be an axe-filtration of the B-semimodule M, and N be a submonoid of M. Then the* *following assertions are equivalent*:

(i) $w_{\varphi}(N) = +\infty$;

(ii) *for all* $a \in \mathbb{N}$, $w_{t_a\varphi}(N) = +\infty$;

(iii) *there exists an* $a \in \mathbb{N}$ *such that* $w_{t_a\varphi}(N) = +\infty$.

Proof. (a) We show that (i) implies (ii). If $w_{\varphi}(N) = +\infty$, then there is no $r \in \mathbb{N}$ such that $M_r \subseteq N$. Suppose $s = w_{t_a\varphi}(N) \neq +\infty$. Then we have $M_{s+a} \subset N$ with $s \in \mathbb{N}$, which implies $w_{\varphi}(N) \leq s + a$, a contradiction.

(b) (ii) implies (iii) is obvious.

(c) (iii) implies (i). Let $a \in \mathbb{N}$ be such that $w_{t_a\varphi}(N) = +\infty$. Suppose $r = w_{\varphi}(N) \neq +\infty$. Then $r \in \mathbb{N}$ and we have $M_{r+a} \subset M_r \subset N$. It follows that $w_{t_a\varphi}(N) < r + a \in \mathbb{N}$, and $w_{t_a\varphi}(N) \neq +\infty$, a contradiction. □

Lemma 4.6. *Let B be a semiring*, $\varphi = (M_n)_{n\in\mathbb{N}}$ *be an axe-filtration of the B-semimodule M*, *and N be a submonoid of M*. *Then the following assertions are equivalent*:

(i) $w_{\varphi}(N) = +\infty$,

(ii) *for all* $k \in \mathbb{N}^*$ *such that* $w_{\varphi^{(k)}}(N) = +\infty$,

(iii) *there exists a* $k \in \mathbb{N}^*$ *such that* $w_{\varphi^{(k)}}(N) = +\infty$.

Proof. (a) We show that (i) implies (ii). If $w_{\varphi}(N) = +\infty$, then there is no integer n such that M_n is included in N. Thus, $\forall k \in \mathbb{N}^*$ and $\forall n \in \mathbb{N}$, $M_{nk} \not\subset N$, hence $w_{\varphi^{(k)}}(N) = +\infty$.

(b) (ii) implies (iii) is obvious.

(c) We show that (iii) implies (i). If there exists a $k \in \mathbb{N}^*$ such that $w_{\varphi^{(k)}}(N) = +\infty$, then $M_{nk} \not\subset N$ for any integer n, and we deduce $M_n \not\subset N$, $\forall n \in \mathbb{N}$. Therefore, $w_\varphi(N) = +\infty$. □

Proposition 4.7. *Let* B *be a semiring,* $\varphi = (M_n)_{n\in\mathbb{N}\cup\{+\infty\}}$, $\theta = (F_n)_{n\in\mathbb{N}\cup\{+\infty\}}$, *and* $\eta = (E_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ *be axe-filtrations of the B-semimodule M. If* φ *is a valuative reduction of* θ, *then* $\overline{w}_\eta(\theta)$ *exists if and only if* $\overline{w}_\eta(\varphi)$ *exists. Therefore,* $\overline{w}_\eta(\theta) = \overline{w}_\eta(\varphi)$ *if the defined terms exist.*

Proof. Since the axe-filtration φ is a reduction of the axe-filtration θ, there exists $a \in \mathbb{N}$ such that $F_{n+a} \subseteq M_n \subseteq F_n$, $\forall n$ (Theorem 4.4). To obtain the result, it is sufficient to replace v_η with w_η in the corresponding proof for $\overline{v}$. □

Proposition 4.8. *Let* B *be a semiring,* $\varphi = (M_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ *and* $\theta = (F_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ *be axe-filtrations of the B-semimodule M. Then the following assertions are equivalent*:

(i) $\overline{w}_\varphi(\theta)$ *exists in* $\overline{\mathbb{R}}_+$.

(ii) *For all* $a \in \mathbb{N}$, $\overline{w}_{t_a\varphi}(\theta)$ *exists in* $\overline{\mathbb{R}}_+$.

(iii) *There exists an* $a \in \mathbb{N}$ *such that* $\overline{w}_{t_a\varphi}(\theta)$ *exists in* $\overline{\mathbb{R}}_+$.

If one of the following assertions holds, then for all $a \in \mathbb{N}$, $\overline{w}_\varphi(\theta) = \overline{w}_{t_a\varphi}(\theta)$.

Proof. (a) We show that (i) implies (ii). Suppose there exists $n \in \mathbb{N}$ such that $w_\varphi(F_n) = +\infty$. Then $w_\varphi(F_k) = +\infty$, $\forall k \geq n$, hence for $a \in \mathbb{N}$, $\forall k \geq n$, we have $w_{t_a\varphi}(F_n) = +\infty$ (Lemma 4.5), and we deduce (i) implies (ii).

Suppose that $w_{\varphi}(F_n) \in \mathbb{N}$, $\forall n \in \mathbb{N}$. Then for all $a \in \mathbb{N}$, $w_{t_a\varphi}(F_n) \in \mathbb{N}$, $\forall n \in \mathbb{N}$ (Lemma 4.5). Let $w_{t_a\varphi}(F_n) = r \in \mathbb{N}$.

(1) If $r \neq 0$, we have $M_{a+r} \subset F_n$ and $M_{a+r-1} \not\subseteq F_n$, so

$$w_{\varphi}(F_n) \leq a + w_{t_a\varphi}(F_n). \qquad (*)$$

(2) If $r = 0$, then $M_0 \subset F_n$, hence $w_{\varphi}(F_n) = 0 = w_{t_a\varphi}(F_n)$, so

$$0 = w_{t_a\varphi}(F_n) \leq w_{\varphi}(F_n) \leq w_{t_a\varphi}(F_n) + a. \qquad (**)$$

The inequalities $(*)$ and $(**)$ imply for all $r \in \mathbb{N}$, $\frac{w_{t_a\varphi}(F_n)}{n} \leq \frac{w_{\varphi}(F_n)}{n} \leq \frac{a + w_{t_a\varphi}(F_n)}{n}$. As $n \to +\infty$, we deduce the existence of $\overline{w}_{t_a\varphi}(\theta)$ and we have $\overline{w}_{\varphi}(\theta) = \overline{w}_{t_a\varphi}(\theta)$.

(b) (ii) implies (iii) is obvious.

(c) We show that (iii) implies (i). Let $a \in \mathbb{N}$. If $w_{t_a\varphi}(F_n) = +\infty$, then $w_{\varphi}(F_n) = +\infty$ (Lemma 4.5), and as in (a), we deduce that (iii) implies (i). Suppose that $w_{t_a\varphi}(F_n) \in \mathbb{N}$, $\forall n \in \mathbb{N}$; using relations $(*)$ and $(**)$ from (a), we have $\frac{w_{t_a\varphi}(F_n)}{n} \leq \frac{w_{\varphi}(F_n)}{n} \leq \frac{a + w_{t_a\varphi}(F_n)}{n}$. If $\overline{w}_{t_a\varphi}(\theta)$ exists in $\overline{\mathbb{R}}_+$ and as $n \to +\infty$, we deduce that $\overline{w}_{\varphi}(\theta)$ exists in $\overline{\mathbb{R}}_+$. □

Proposition 4.9. *Let B be a semiring, and let* φ, θ, *and* η *be axe-filtrations of the B-semimodule M. If* φ *is a valuative reduction of* θ, *then the following assertions are equivalent*:

(i) $\overline{w}_{\theta}(\eta)$ *exists in* $\overline{\mathbb{R}}_+$.

(ii) $\overline{w}_{\varphi}(\eta)$ *exists in* $\overline{\mathbb{R}}_+$.

If one of the equivalent assertions holds, then $\overline{w}_{\varphi}(\eta) = \overline{w}_{\theta}(\eta)$.

Proof. (a) We show that (i) implies (ii). Let $\eta = (E_n)_{n\in\mathbb{N}}$. Since φ is a valuative reduction of θ, there exists $a \in \mathbb{N}$ such that $t_a\theta \leq \varphi \leq \theta$, which implies $\frac{w_\theta(E_n)}{n} \leq \frac{w_\varphi(E_n)}{n} \leq \frac{w_{t_a\theta}(E_n)}{n}$. If $\overline{w}_\theta(\eta)$ exists and $n \to +\infty$, then we have $\overline{w}_\theta(\eta) = \lim_{n\to+\infty} \frac{w_{t_a\theta}(E_n)}{n} = \lim_{n\to+\infty} \frac{w_\varphi(E_n)}{n}$ (Proposition 4.8). We deduce that $\lim_{n\to+\infty} \frac{w_\varphi(E_n)}{n}$ exists and that $\overline{w}_\varphi(\eta) = \overline{w}_\theta(\eta)$.

(b) We show that (ii) implies (i). Since φ is a valuative reduction of θ, there exists $a \in \mathbb{N}$ such that $t_a\theta \leq \varphi \leq \theta$, which implies $t_a\varphi \leq t_a\theta \leq \varphi$. As in (a), we can show that if $\overline{w}_\varphi(\eta)$ exists, then $\overline{w}_\theta(\eta)$ exists (Proposition 4.8), and we have $\overline{w}_\varphi(\eta) = \overline{w}_\theta(\eta)$. □

Theorem 4.10. *Let B be a semiring*, φ *and* θ *be axe-filtrations of the B-semimodule M*, *and* $f = (F_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ *and* $g = (G_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ *be axe-filtrations of B*. *If* φ *is weakly f-good*, θ *is weakly g-good*, *and f is an AP quasi-filtration*, *then* $\overline{w}_\varphi(\theta)$ *exists in* $\mathbb{R}_+$ *and*

$$\begin{aligned}\overline{w}_\varphi(\theta) &= \overline{w}_{fM}(gM)\\ &= \lim_{n\to\infty} \frac{\overline{w}_{fM}(G_nM)}{n}\\ &= \inf_n \frac{\overline{w}_{fM}(G_nM)}{n}\\ &= \lim_{n\to+\infty} n\overline{w}_{F_nM}(gM).\end{aligned}$$

Proof. If $\varphi = (M_n)_{n\in\mathbb{N}\cup\{+\infty\}}$ is weakly *f*-good, then *fM* is a valuative reduction of φ. Since φ is weakly *f*-good, we have $F_pM_q \subseteq M_{p+q}$ for all p, q, and there exists an integer $m \geq 1$ such that $M_n = \sum_{p=0}^{m} F_{n-p}M_p$ for all $n > m$, and $M_n \subseteq F_{n-m}M$ for all $n > m$. Therefore, $M_{n+m} \subseteq F_nM \subset M_n$ for all $n \in \mathbb{N}$. $t_m\varphi \leq fM \leq \varphi$, and consequently, fM is a valuative reduction of φ (Theorem 4.4). Since θ is weakly g-good, gM is a valuative reduction of θ. If f is an AP quasi-filtration, then using Propositions 3.4, 4.7 and 4.9, we have

$$\begin{aligned}\overline{w}_{\varphi}(\theta) &= \overline{w}_{fM}(gM)\\ &= \lim_{n\to+\infty}\frac{\overline{w}_{fM}(G_nM)}{n}\\ &= \inf_{n}\lim_{n\to+\infty}\frac{\overline{w}_{fM}(G_nM)}{n}\\ &= \lim_{n\to+\infty} n\overline{w}_{F_nM}(gM).\end{aligned}$$

□

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