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ON NICHOLS BICHARACTER ALGEBRAS

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Abstract

In this paper, we define two Lie operations, and with that we define the bicharacter algebras, Nichols bicharacter algebras, etc. We obtain explicit bases for $\mathfrak{L}(V)_R$ and $\mathfrak{L}(V)_L$ over a connected braided vector V of diagonal type with $\dim V = 2$ and $p_{1,1} = p_{2,2} = -1$. We give the sufficient and necessary conditions for $\mathfrak{L}(V)_R = \mathfrak{L}(V)$, $\mathfrak{L}(V)_L = \mathfrak{L}(V)$, $\mathfrak{B}(V) = F \oplus \mathfrak{L}(V)_R$ and $\mathfrak{B}(V) = F \oplus \mathfrak{L}(V)_L$, respectively. We show that if $\mathfrak{B}(V)$ is a connected Nichols algebra of diagonal type with $\dim V > 1$, then $\mathfrak{B}(V)$ is finite-dimensional if and only if $\mathfrak{L}(V)_L$ is finite-dimensional if and only if $\mathfrak{L}(V)_R$ is finite-dimensional.

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1. Introduction

Nichols algebras play a central role in the theory of pointed Hopf algebras, and the classification of (braided) Lie algebras has been an international hotspot and cutting-edge problem in the field of algebras. A great deal of attention has been paid to the question of finite-dimensionality of Nichols algebras (see [6, 7, 9, 10, 11, 21]). The interest in this problem arose from the work of Andruskiewitsch and Schneider [3] on classification of finite dimensional (Gelfand-Kirillov) pointed Hopf algebras which are generalizations of quantized enveloping algebras of semi-simple Lie algebras. On the other hand, Lie algebra arising from a Nichols algebra was studied in [1, 2]. The theory of Lie superalgebras has been developed systematically, which includes the representation theory and classifications of simple Lie superalgebras and their varieties [13]. A sophisticated multilinear version of the Lie bracket was considered in [14, 18, 16]. Various generalized Lie algebras have already appeared under different names, e.g., Lie color algebras, ε Lie algebras [19], quantum and braided Lie algebras [17], braided m-Lie algebras [23] and generalized Lie algebras [8].

In this work, we define two new Lie operations, this is one of the main results in this paper, which enables us to define the bicharacter algebras (all linear Lie algebras and Kac-Moody algebras are bicharacter algebras), Nichols bicharacter algebras, etc. Braided Lie algebra (with braided Lie operation $[u, v] = vu - p_{v,u}uv$) seems to be 'located' between two Lie algebras (the usual ones with Lie operation $[u, v]^{-} = uv - vu$ and bicharacter algebras with Lie operation $[u, v]_L = p_{v,u}uv - p_{u,v}vu$ or $[u, v]_R = p_{u,v}uv - p_{v,u}vu$), which is of great significance for revealing the relationship between braided Lie algebra and Lie algebra. In the rest of the paper, there is focus on Nichols bicharacter algebras, which is a cross research target in Lie algebras and Nichols algebras. It was proven that a Nichols algebra is finite-dimensional if and only if the corresponding Nichols bicharacter algebras are finite-dimensional. This provides a new method for determining when a Nichols algebra is finite dimensional.

This paper is organized as follows. In Section 2, we recall some results on Lie algebras and Nichols algebras, and define bicharacter algebras and Nichols bicharacter algebras. Section 3 presents explicit bases for $\mathfrak{L}(V)_L$ and $\mathfrak{L}(V)_R$, where V is a connected braided vector space of diagonal type with $\dim V = 2$ and $p_{1,1} = p_{2,2} = -1$. In Section 4, we present the sufficient and necessary conditions for $\mathfrak{L}(V)_R = \mathfrak{L}(V)$, $\mathfrak{L}(V)_L = \mathfrak{L}(V)$, $\mathfrak{B}(V) = F \oplus \mathfrak{L}(V)_R$ and $\mathfrak{B}(V) = F \oplus \mathfrak{L}(V)_L$, respectively. In Section 5, we prove that $\mathfrak{B}(V)$ is finite-dimensional if and only if $\mathfrak{L}(V)_L$ is finite-dimensional, or equivalently $\mathfrak{L}(V)_R$ is finite-dimensional, when $\mathfrak{B}(V)$ is a connected Nichols algebra of diagonal type with $\dim V > 1$.

$\mathbb{Z}$ denotes the set of integers and $\mathbb{N}$ that of positive integers. $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. F denotes a field of characteristic 0 and $F^* = F - \{0\}$.

2. Preliminaries

In this section, we recall some results on Lie algebras and Nichols algebras, and define bicharacter algebras and Nichols bicharacter algebras.

Let $E = \{e_1, e_2, ..., e_n\}$, $e_i =: (\overbrace{0, ..., 0, 1}^{i}, ..., 0) \in \mathbb{Z}^n$, $1 \le i \le n$. Let $A := \{x_1, x_2, ..., x_n\}$ be an alphabet and a basis of $V = \text{span}(A)$, likewise, associated with each letter x_i are an element $e_i = \deg(x_i)$ of $\mathbb{Z}^n$ and a character $\chi^{x_i} : \mathbb{Z}^n \to F^*$. Let the empty word be 1. Then $W := \{u \,|\, u$ is a word$\}$ is a basis for $T(V) = \text{span}(W)$. Let V^* denote the vector space dual to V and $\{y_1, y_2, ..., y_n\}$ denote the dual basis of V^*, where $\deg(y_i) = -e_i$, $\chi^{y_i} : \mathbb{Z}^n \to F^*$. Let $T(\overline{V})$ be free generated by $1 \cup \overline{V} := \{1, x_i, y_i \,|\, 1 \le i \le n\}$ and $\overline{W}$ be a basis of $T(\overline{V})$ with that each element in $\overline{W}$ is gotten from one in $1 \cup \overline{V}$ by multiplication elements of $1 \cup \overline{V}$. Then $\overline{W}$ is a monoid with 1, for all $u \in \overline{W}$, denote by $\deg(u)$ an element of the group $\mathbb{Z}^n$ which results from u by replacing each occurrence of the letter x_i (or y_i) with e_i (or $-e_i$). By χ^u we denote a character which results from u by replacing all x_i (or y_i) with χ^{x_i} (or χ^{y_i}). For a pair of words $u, v \in \overline{W}$, put $p_{u,v} = \chi^u(\deg v)$, where

$$p_{x_i,x_j} = \chi^{x_i}(\deg x_j), \quad p^{-1}_{x_i,x_j} = \chi^{x_i}(\deg y_j),$$

$$p^{-1}_{x_i,x_j} = \chi^{y_i}(\deg x_j), \quad p_{x_i,x_j} = \chi^{y_i}(\deg y_j).$$

Obviously, $p_{uu_1,v} = p_{u,v}p_{u_1,v}$, $p_{u,vv_1} = p_{u,v}p_{u,v_1}$, that is the operator $p_{\cdot,\cdot}$ is a bicharacter defined on $\overline{W}$. Moreover, we call $p_{\cdot,\cdot}$ is a symmetrical bicharacter if $p_{u,v} = p_{v,u}$ for all $u, v \in \overline{W}$. Define $u \circ_L v := p_{v,u}uv$, $u \circ_R v := p_{u,v}uv$. Clearly, $(T(\overline{V}), \circ_L)$ and $(T(\overline{V}), \circ_R)$ are associative algebras. Assume that $A(V)$ is in general a quotient algebra of some subalgebras of $T(\overline{V})$, then $(A(V), \circ_L)$ and $(A(V), \circ_R)$ are associative algebras. Especially, $(T(V), \circ_L)$ and $(T(V), \circ_R)$ are associative algebras. Let I be the (two sided) ideal in $T(V)$ generated by all $x_ix_j - x_jx_i$, $1 \le i, j \le n$ and call $S(V) = T(V)/I$ the symmetric algebra on V.

Lemma 2.1. *If* $(A, \circ)$ *is an associative algebra together with the bilinear operation* $[\cdot, \cdot]$ *defined by* $[u, v] = u \circ v - v \circ u$ *for any homogeneous elements* $u, v \in A$, *then* $(A, [\cdot, \cdot])$ *is a Lie algebra.*

If we define the bilinear operations $[u, v]_L = u \circ_L v - v \circ_L u$ and $[u, v]_R = u \circ_R v - v \circ_R u$ for all $u, v \in W$, then $(A(V), [\cdot, \cdot]_L)$ and $(A(V), [\cdot, \cdot]_R)$ are Lie algebras, which are called *L-bicharacter algebra* and *R-bicharacter algebra*, written as $A(V)_L$ and $A(V)_R$, respectively. Both L-bicharacter algebra and R-bicharacter algebra are called *bicharacter algebras*.

Remark 2.2. (i) $(S(V), \circ_L)$ (resp. $(S(V), \circ_R)$) is a commutative associative algebra if and only if $p_{\cdot,\cdot}$ is a symmetrical bicharacter.

(ii) Let $S(V)_L$ and $S(V)_R$ denote the Lie algebras generated by V in $S(V)$ under Lie operations $[u, v]_L = p_{v,u}uv - p_{u,v}vu$, $[u, v]_R = p_{u,v}uv - p_{v,u}vu$, respectively, for any homogeneous elements $u, v \in S(V)$. Then $[u, v]_L = (p_{v,u} - p_{u,v})uv = -[u, v]_R$ and $S(V)_L = S(V)_R$.

(iii) Linear Lie algebras and Kac-Moody algebras are bicharacter algebras, where $p_{v,u} = p_{u,v} = 1$.

Let $h_i =: x_i y_i - y_i x_i$, $\langle \alpha_j, \alpha_i \rangle$ be Cartan integers, $1 \le i, j \le n$. Then $(ad\ u)_L(v) =: [u, v]_L$, $(ad\ u)_R(v) =: [u, v]_R$, for any $u, v \in T(\overline{V})$.

Proposition 2.3 (See [12, Proposition 18.1, Theorem 18.3]). (1) *With the above notation*, $(A_V)_L$ *is generated by* $\{x_i, y_i \mid 1 \le i \le n\}$ *and* $p_{\cdot,\cdot}$ *is a symmetrical bicharacter*, *then* $(A_V)_L$ *is a finite dimensional semisimple Lie algebra if and only if these generators satisfy the following relations*:

(i) $[h_i, h_j]_L = 0, 1 \le i, j \le n.$

(ii) $[x_i, y_i]_L = p^{-1}_{x_i, x_i} h_i, [x_i, y_j]_L = 0 \text{ if } i \ne j.$

(iii) $[h_i, x_j]_L = \langle \alpha_j, \alpha_i \rangle x_j, [h_i, y_j]_L = -\langle \alpha_j, \alpha_i \rangle y_j, 1 \le i, j \le n.$

(iv) $(ad\ x_i)_L^{-\langle \alpha_j, \alpha_i \rangle + 1}(x_j) = 0 \text{ if } i \ne j.$

(v) $(ad\ y_i)_L^{-\langle \alpha_j, \alpha_i \rangle + 1}(y_j) = 0 \text{ if } i \ne j.$

(2) *With the above notation*, $(A_V)_R$ *is generated by* $\{x_i, y_i \mid 1 \le i \le n\}$ *and* $p_{\cdot,\cdot}$ *is a symmetrical bicharacter*, *then* $(A_V)_R$ *is a finite dimensional semisimple Lie algebra if and only if these generators satisfy the following relations*:

(i) $[h_i, h_j]_R = 0, 1 \leq i, j \leq n.$

(ii) $[x_i, y_i]_R = p^{-1}_{x_i, x_i} h_i, [x_i, y_j]_R = 0$ *if* $i \neq j$.

(iii) $[h_i, x_j]_R = \langle \alpha_j, \alpha_i \rangle x_j, [h_i, y_j]_R = -\langle \alpha_j, \alpha_i \rangle y_j, 1 \leq i, j \leq n.$

(iv) $(ad\ x_i)_R^{-\langle \alpha_j, \alpha_i \rangle + 1}(x_j) = 0$ *if* $i \neq j$.

(v) $(ad\ y_i)_R^{-\langle \alpha_j, \alpha_i \rangle + 1}(y_j) = 0$ *if* $i \neq j$.

Example 2.4. When $\dim V = 1$, $(A_V)_R$ is a Lie algebra with generators x_1, y_1 and relations $[x_1, y_1]_R = p^{-1}_{x_1, x_1} h_1$, $[h_1, x_1]_R = 2x_1$, $[h_1, y_1]_R = -2y_1$. Moreover, $(A_V)_R = (A_V)_L$.

From now on let $V = \{x_1, ..., x_n\}$ be a basis of vector space V and $C(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$ with $q_{ij} = p_{x_i, x_j}$. Then V is called a *braided vector space* of diagonal type, $\{x_1, ..., x_n\}$ is called *canonical basis* and $(q_{ij})_{n \times n}$ is called *braided matrix*. In this case, $x_i \circ_L x_j = (g_j . x_i) x_j$, $x_i \circ_R x_j = x_i (g_i . x_j)$. Throughout this paper all of braided vector spaces are connected and of diagonal type without special announcement.

Let $S_m \in End_k(V^{\otimes m})$ and $S_{1,j} \in End_k(V^{\otimes j+1})$ denote the maps

$$S_m = \prod_{j=1}^{m-1} (id^{\otimes m-j-1} \otimes S_{1,j}),$$

$$S_{1,j} = id + C_{12}^{-1} + C_{12}^{-1} C_{23}^{-1} + \cdots + C_{12}^{-1} C_{23}^{-1} \cdots C_{j,j+1}^{-1}$$

(in leg notation) for $m \geq 2$ and $j \in \mathbb{N}$. Then the subspace $S = \bigoplus_{m=2}^{\infty} \ker S_m$ of the tensor algebra $T(V)$ is a two-sided ideal, and algebra $\mathfrak{B}(V) = T(V)/S$ is termed the Nichols algebra associated to (V, C) (see [9, Definition 1.2.2]).

Let $R_m := \{\alpha \mid \alpha$ is a primitive mth root of 1$\}$. Set $R_3 := \{1, \omega, \omega^2\}$. Write i for x_i in short.

Lemma 2.5. *Assume that* $i \neq j$. *Then*

(i) $[x_i, x_j]_L = 0$ *if and only if* $p_{i,j} = p_{j,i} = 1$.

(ii) $[x_i, x_j]_R = 0$ *if and only if* $p_{i,j} = p_{j,i} = 1$ *or* $p_{i,j} = \omega$, $p_{j,i} = \omega^2$ *or* $p_{i,j} = \omega^2$, $p_{j,i} = \omega$.

Proof. (i) $\langle y_i, [x_i, x_j]_L\rangle = (p_{j,i} - 1)x_j$, $\langle y_j, [x_i, x_j]_L\rangle = (1 - p_{i,j})x_i$.

(ii) $\langle y_i, [x_i, x_j]_R\rangle = p_{i,j}^{-1}(p_{i,j}^2 - p_{j,i})x_j$,

$\langle y_j, [x_i, x_j]_R\rangle = p_{j,i}^{-1}(p_{i,j} - p_{j,i}^2)x_i$. □

Let $\mathfrak{L}^-(V)$, $\mathfrak{L}(V)_L$ and $\mathfrak{L}(V)_R$ denote the Lie algebras generated by V in $\mathfrak{B}(V)$ under Lie operations $[u, v]^- = uv - vu$, $[u, v]_L = p_{v,u}uv - p_{u,v}vu$, $[u, v]_R = p_{u,v}uv - p_{v,u}vu$, respectively, for any homogeneous elements $u, v \in \mathfrak{B}(V)$. Then $(\mathfrak{L}^-(V), [\cdot, \cdot]^-)$, $(\mathfrak{L}(V)_L, [\cdot, \cdot]_L)$ and $(\mathfrak{L}(V)_R, [\cdot, \cdot]_R)$ are called *Nichols Lie algebra*, *Nichols L-bicharacter algebra* and *Nichols R-bicharacter algebra* of V, respectively. Let $\mathfrak{L}(V)$ denote the braided Lie algebras generated by V in $\mathfrak{B}(V)$ under Lie operations $[u, v] = vu - p_{u,v}vu$, for any homogeneous elements $u, v \in \mathfrak{B}(V)$. Then $(\mathfrak{L}(V), [\cdot, \cdot])$ is called *Nichols braided Lie algebra* of V.

Remark 2.6. Assume that $p_{\cdot,\cdot}$ is a symmetrical bicharacter. Then $\mathfrak{L}^-(V) = \mathfrak{L}(V)_L = \mathfrak{L}(V)_R$ since

$$[u, v]_R = [u, v]_L = p_{u,v}[u, v]^-$$

for any homogeneous elements $u, v \in \mathfrak{B}(V)$.

Let $l_u^0[v]_L := v$, $l_u^i[v]_L := [u, l_u^{i-1}[v]_L]_L$, $i \geq 1$. Similarly define $r_u^i[v]_L$. Let $l_u^0[v]_R := v$, $l_u^i[v]_R := [u, l_u^{i-1}[v]_R]_R$, $i \geq 1$. Similarly define $r_u^i[v]_R$. In fact, $l_u^i[v]_R := [u, [u, \ldots, [u, v]_R \cdots]_R]_R$.

3. Nichols Bicharacter Algebra with $\dim V = 2$ and $p_{1,1} = p_{2,2} = -1$

In this section, we obtain explicit bases for $\mathfrak{L}(V)_L$ and $\mathfrak{L}(V)_R$, where V is a connected braided vector space of diagonal type with $\dim V = 2$ and $p_{1,1} = p_{2,2} = -1$.

Lemma 3.1 (See [22, Lemma 4.1]). *Assume that* $\mathfrak{B}(V)$ *is a connected Nichols algebra of diagonal type with* $\dim V = 2$ *and* $p_{1,1} = p_{2,2} = -1$.

(i) *If u is a non-zero monomial, then there exists* $\alpha \in F^*$ *such that* $u = \alpha(x_1x_2)^k x_1$, *or* $\alpha x_2(x_1x_2)^k$, *or* $\alpha(x_1x_2)^{k+1}$, *or* $\alpha(x_2x_1)^{k+1}$, $k \geq 0$.

(ii) *Let*

$$P := \{x_2(x_1x_2)^k, (x_1x_2)^k x_1, (x_1x_2)^k, x_2(x_1x_2)^k x_1, 0 \leq k < \operatorname{ord}(p_{1,2}p_{2,1})\}.$$

Then P is a basis of $\mathfrak{B}(V)$.

3.1. Nichols R-bicharacter algebra

Lemma 3.2. *Assume that* $\mathfrak{L}(V)_R$ *is a connected Nichols R-bicharacter algebra of diagonal type with* $\dim V = 2$ *and* $p_{1,1} = p_{2,2} = -1$. *Then*

(i) $(r_{x_1}r_{x_2})^i[x_1]_R = (-2)^i(p_{1,2}p_{2,1})^{\frac{i(i+1)}{2}}(x_1x_2)^i x_1$ *and*

$(r_{x_2}r_{x_1})^i[x_2]_R = (-2)^i(p_{1,2}p_{2,1})^{\frac{i(i+1)}{2}} x_2(x_1x_2)^i$ *for* $i \geq 0$.

(ii) *For any method* σ *adding bracket* $[\cdot, \cdot]_R$ *and* $k > 0$, *there exist*

$\alpha_k, \beta_k \in F$ *such that* $\sigma(x_{i_1}, x_{i_2}, ..., x_{i_{2k+1}}) = \beta_k(x_1x_2)^k x_1$ *or* $\beta_k x_2(x_1x_2)^k$ *and* $\sigma(x_{i_1}, x_{i_2}, ..., x_{i_{2k}}) = \alpha_k(p_{1,2}^k(x_1x_2)^k - p_{2,1}^k(x_2x_1)^k)$.

(iii) $[[[[[x_1, x_2]_R, x_1]_R, x_2]_R, ..., x_1]_R]_R$

$$= 2^{k-1}(p_{1,2}p_{2,1})^{\frac{k(k-1)}{2}}(p_{1,2}^k(x_1x_2)^k - p_{2,1}^k(x_2x_1)^k),$$

where there exist $2k - 1$ *brackets in the left hand side.*

(iv)

$$\langle y_1, p_{1,2}^i(x_1x_2)^i - p_{2,1}^i(x_2x_1)^i\rangle = p_{1,2}^i(1 + (-p_{1,2}^{-2}p_{2,1})^i)x_2(x_1x_2)^{i-1},$$

$$\langle y_2, p_{1,2}^i(x_1x_2)^i - p_{2,1}^i(x_2x_1)^i\rangle = -p_{2,1}^i(1 + (-p_{1,2}p_{2,1}^{-2})^i)(x_1x_2)^{i-1}x_1.$$

Proof. (i) It can be proved by induction on i.

(ii) We show this by induction on k. It is clear for $k = 1$. Now assume $k > 1$. Then we have

$$\sigma(x_{i_1}, x_{i_2}, ..., x_{i_{2k}}) = [\sigma_1(x_{i_1}, x_{i_2}, ..., x_{i_{2s}}), \sigma_2(x_{i_{2s+1}}, x_{i_2}, ..., x_{i_{2k}})]_R \text{ or}$$

$$\sigma(x_{i_1}, x_{i_2}, ..., x_{i_{2k}}) = [\sigma_1(x_{i_1}, x_{i_2}, ..., x_{i_{2s+1}}), \sigma_2(x_{i_{2s+2}}, x_{i_2}, ..., x_{i_{2k}})]_R.$$

In the first case, we have

$$\begin{aligned}
&\sigma(x_{i_1}, x_{i_2}, ..., x_{i_{2k}})\\
&= [\sigma_1(x_{i_1}, x_{i_2}, ..., x_{i_{2s}}), \sigma_2(x_{i_{2s+1}}, x_{i_2}, ..., x_{i_{2k}})]_R\\
&= [\alpha_s(p_{1,2}^s(x_1x_2)^s - p_{2,1}^s(x_2x_1)^s), \alpha_t(p_{1,2}^t(x_1x_2)^t - p_{2,1}^t(x_2x_1)^t)]_R\\
&\qquad\qquad\qquad \text{(by inductive assumption, where } s + t = k)\\
&= \alpha_k(p_{1,2}^k(x_1x_2)^k - p_{2,1}^k(x_2x_1)^k) \text{ (where } \alpha_k = 0).
\end{aligned}$$

In second case, we have

$$\sigma(x_{i_1}, x_{i_2}, \ldots, x_{i_{2k}})$$

$$= [\sigma_1(x_{i_1}, x_{i_2}, \ldots, x_{i_{2s+1}}), \sigma_2(x_{i_{2s+2}}, x_{i_2}, \ldots, x_{i_{2k}})]_R$$

$$= [\beta_s(x_{j_1}x_{j_2})^s x_{j_1}, \beta_t(x_{k_1}x_{k_2})^t x_{k_1})]_R$$

(by inductive assumption, where $s + t + 1 = k,\ j_1 \neq j_2$ and $k_1 \neq k_2$)

$$= \begin{cases} 0, & \text{when } k_1 = j_1 \\ \beta_s\beta_t(-1)^{s+t}(p_{j_1, j_2}p_{j_2, j_1})^{st}(p_{j_1, j_2}^k(x_{j_1}x_{j_2})^k & \\ \qquad - p_{j_2, j_1}^k(x_{j_2}x_{j_1})^k) & \text{when } k_1 \neq j_1 \end{cases}$$

$$= \alpha_k(p_{1,2}^k(x_1x_2)^k - p_{2,1}^k(x_2x_1)^k),$$

$$\sigma(x_{i_1}, x_{i_2}, \ldots, x_{i_{2k+1}})$$

$$= b[\sigma_1(x_{i_1}, x_{i_2}, \ldots, x_{i_{2s}}), \sigma_2(x_{i_{2s+1}}, x_{i_{2s+2}}, \ldots, x_{i_{2k+1}})]_R$$

(where $b = 1$ or -1)

$$= [\alpha_s(p_{1,2}^s(x_1x_2)^s - p_{2,1}^s(x_2x_1)^s), \beta_t(x_{k_1}x_{k_2})^t x_{k_1})]_R$$

(by inductive assumption, where $s + t = k$ and $k_1 \neq k_2$)

$$= \begin{cases} 2\alpha_s\beta_t(-1)^s(p_{1,2}p_{2,1})^{st+s}(x_1x_2)^k x_1, & \text{when } k_1 = 1 \\ 2\alpha_s\beta_t(-1)^{s+1}(p_{1,2}p_{2,1})^{st+s}(x_2x_1)^k x_2, & \text{when } k_1 = 2 \end{cases}$$

$$= \beta_k(x_{k_1}x_{k_2})^k x_{k_1}.$$

Consequently, (ii) holds.

(iii) We show this by induction on k,

$$[[[[[x_1, x_2]_R, x_1]_R, x_2]_R, \ldots, x_1]_R]_R$$

$$= 2^{k-2}(p_{1,2}p_{2,1})^{\frac{(k-1)(k-2)}{2}}[[(p_{1,2}^{k-1}(x_1x_2)^{k-1} - p_{2,1}^{k-1}(x_2x_1)^{k-1}), x_1]_R, x_2]_R$$

(by inductive assumption)

$$= 2^{k-2}(p_{1,2}p_{2,1})^{\frac{(k-1)(k-2)}{2}}[2(-1)^{k-1}(p_{1,2}p_{2,1})^{k-1}(x_1x_2)^{k-1}x_1, x_2]_R$$

$$= (-1)^{k-1}2^{k-1}(p_{1,2}p_{2,1})^{\frac{k(k-1)}{2}}((-1)^{k-1}p_{1,2}^{k}(x_1x_2)^{k} - (-1)^{k-1}p_{2,1}^{k}(x_2x_1)^{k})$$

$$= 2^{k-1}(p_{1,2}p_{2,1})^{\frac{k(k-1)}{2}}(p_{1,2}^{k}(x_1x_2)^{k} - p_{2,1}^{k}(x_2x_1)^{k}). \qquad \square$$

Theorem 3.3. *Assume that* $\mathfrak{L}(V)_R$ *is a connected Nichols R-bicharacter algebra of diagonal type with* $\dim V = 2$ *and* $p_{1,1} = p_{2,2} = -1$. *Let*

$$P_1 := \begin{cases} \{p_{1,2}^{k+1}(x_1x_2)^{k+1} - p_{2,1}^{k+1}(x_2x_1)^{k+1},\ x_2(x_1x_2)^{k},\ (x_1x_2)^{k}x_1,\ 0 \le k < m\} \\ \qquad - \{p_{1,2}^{m}(x_1x_2)^{m} - p_{2,1}^{m}(x_2x_1)^{m}\} \\ \qquad\qquad \textit{when } (-p_{1,2}^{-2}p_{2,1})^{m} = (-p_{1,2}p_{2,1}^{-2})^{m} = -1 \\ \qquad\qquad \textit{with } \operatorname{ord}(p_{1,2}p_{2,1}) = m < \infty \\ \{p_{1,2}^{k+1}(x_1x_2)^{k+1} - p_{2,1}^{k+1}(x_2x_1)^{k+1},\ x_2(x_1x_2)^{k},\ (x_1x_2)^{k}x_1, \\ \qquad 0 \le k < \operatorname{ord}(p_{1,2}p_{2,1})\}, \qquad \textit{otherwise}. \end{cases}$$

Then P_1 *is a basis of* $\mathfrak{L}(V)_R$. *Furthermore, if* $\operatorname{ord}(p_{1,2}p_{2,1}) = m < \infty$, *then*

$$\dim \mathfrak{L}(V)_R = \begin{cases} 3m - 1, & \textit{when } (-p_{1,2}^{-2}p_{2,1})^{m} = (-p_{1,2}p_{2,1}^{-2})^{m} = -1, \\ 3m, & \textit{otherwise}. \end{cases}$$

Proof. By Lemma 3.1(ii) and Lemma 3.2(iv), P_1 is linearly independent. It follows from Lemma 3.2(i) (iii) that $P_1 \subseteq \mathfrak{L}(V)_R$. By Lemma 3.2(ii), P_1 is a basis of $\mathfrak{L}(V)_R$. $\square$

3.2. Nichols L-bicharacter algebra

Lemma 3.4. *Assume that* $\mathfrak{L}(V)_L$ *is a connected Nichols L-bicharacter algebra of diagonal type with* $\dim V = 2$ *and* $p_{1,1} = p_{2,2} = -1$. *Then*

(i) $(r_{x_1}r_{x_2})^i[x_1]_L = (-2)^i(p_{1,2}p_{2,1})^{\frac{i(i+1)}{2}}(x_1x_2)^i x_1$ *and*

$$(r_{x_2}r_{x_1})^i[x_2]_L = (-2)^i(p_{1,2}p_{2,1})^{\frac{i(i+1)}{2}}x_2(x_1x_2)^i \text{ for } i \geq 0.$$

(ii) *For any method* σ *adding bracket* $[\cdot, \cdot]_L$ *and* $k > 0$, *there exist* α_k, $\beta_k \in F$ *such that* $\sigma(x_{i_1}, x_{i_2}, ..., x_{i_{2k+1}}) = \beta_k(x_1x_2)^k x_1$ *or* $\beta_k x_2(x_1x_2)^k$ *and* $\sigma(x_{i_1}, x_{i_2}, ..., x_{i_{2k}}) = \alpha_k(p_{2,1}^k(x_1x_2)^k - p_{1,2}^k(x_2x_1)^k)$.

(iii) $[[[[[x_1, x_2]_L, x_1]_L, x_2]_L, ..., x_1]_L]_L$

$$= 2^{k-1}(p_{1,2}p_{2,1})^{\frac{k(k-1)}{2}}(p_{2,1}^k(x_1x_2)^k - p_{1,2}^k(x_2x_1)^k),$$

where there exist $2k-1$ *brackets in the left hand side.*

(iv)

$$\langle y_1, p_{2,1}^i(x_1x_2)^i - p_{1,2}^i(x_2x_1)^i\rangle = (-1)^i(1 + (-p_{2,1})^i)x_2(x_1x_2)^{i-1},$$

$$\langle y_2, p_{2,1}^i(x_1x_2)^i - p_{1,2}^i(x_2x_1)^i\rangle = (-1)^{i+1}(1 + (-p_{1,2})^i)(x_1x_2)^{i-1}x_1.$$

Proof. (i) It can be proved by induction on *i*.

(ii) We show this by induction on *k*. It is clear for $k = 1$. Now assume $k > 1$. Then we have

$\sigma(x_{i_1}, x_{i_2}, ..., x_{i_{2k}}) = [\sigma_1(x_{i_1}, x_{i_2}, ..., x_{i_{2s}}), \sigma_2(x_{i_1}, x_{i_2}, ..., x_{i_{2t}})]_L$ or

$\sigma(x_{i_1}, x_{i_2}, ..., x_{i_{2k}}) = [\sigma_1(x_{i_1}, x_{i_2}, ..., x_{i_{2s-1}}), \sigma_2(x_{i_1}, x_{i_2}, ..., x_{i_{2t-1}})]_L.$

In the first case, we have

$$\sigma(x_{i_1}, x_{i_2}, \ldots, x_{i_{2k}})$$

$$= [\sigma_1(x_{i_1}, x_{i_2}, \ldots, x_{i_{2s}}), \sigma_2(x_{i_{2s+1}}, x_{i_2}, \ldots, x_{i_{2k}})]_L$$

$$= [\alpha_s(p_{2,1}^s(x_1x_2)^s - p_{1,2}^s(x_2x_1)^s), \alpha_t(p_{2,1}^t(x_1x_2)^t - p_{1,2}^t(x_2x_1)^t)]_L$$

(by inductive assumption, where $s + t = k$)

$$= \alpha_k(p_{2,1}^k(x_1x_2)^k - p_{1,2}^k(x_2x_1)^k) \text{ (where } \alpha_k = 0).$$

In second case, we have

$$\sigma(x_{i_1}, x_{i_2}, \ldots, x_{i_{2k}})$$

$$= [\sigma_1(x_{i_1}, x_{i_2}, \ldots, x_{i_{2s+1}}), \sigma_2(x_{i_{2s+2}}, x_{i_2}, \ldots, x_{i_{2k}})]_L$$

$$= [\beta_s(x_{j_1}x_{j_2})^s x_{j_1}, \beta_t(x_{k_1}x_{k_2})^t x_{k_1}]_L$$

(by inductive assumption, where $s + t + 1 = k$, $j_1 \neq j_2$ and $k_1 \neq k_2$)

$$= \begin{cases} 0, & \text{when } k_1 = j_1 \\ \beta_s\beta_t(-1)^{s+t}(p_{j_1,j_2}p_{j_2,j_1})^{st}(p_{j_2,j_1}^k(x_{j_1}x_{j_2})^k & \\ \qquad - p_{j_1,j_2}^k(x_{j_2}x_{j_1})^k) & \text{when } k_1 \neq j_1 \end{cases}$$

$$= \alpha_k(p_{2,1}^k(x_1x_2)^k - p_{1,2}^k(x_2x_1)^k),$$

$$\sigma(x_{i_1}, x_{i_2}, \ldots, x_{i_{2k+1}})$$

$$= b[\sigma_1(x_{i_1}, x_{i_2}, \ldots, x_{i_{2s}}), \sigma_2(x_{i_{2s+1}}, x_{i_{2s+2}}, \ldots, x_{i_{2k+1}})]_L$$

(where $b = 1$ or -1)

$$= [\alpha_s(p_{2,1}^s(x_1x_2)^s - p_{1,2}^s(x_2x_1)^s), \beta_t(x_{k_1}x_{k_2})^t x_{k_1}]_L$$

(by inductive assumption, where $s + t = k$ and $k_1 \neq k_2$)

$$= \begin{cases} 2\alpha_s\beta_t(-1)^s(p_{1,2}p_{2,1})^{st+s}(x_1x_2)^k x_1, & \text{when } k_1 = 1 \\ 2\alpha_s\beta_t(-1)^{s+1}(p_{1,2}p_{2,1})^{st+s}(x_2x_1)^k x_2, & \text{when } k_1 = 2 \end{cases}$$

$$= \beta_k(x_{k_1}x_{k_2})^k x_{k_1}.$$

Consequently, (ii) holds.

(iii) We show this by induction on k,

$$[[[[[x_1, x_2]_L, x_1]_L, x_2]_L, \dots, x_1]_L]_L$$

$$= 2^{k-2}(p_{1,2}p_{2,1})^{\frac{(k-1)(k-2)}{2}}[[(p_{2,1}^{k-1}(x_1x_2)^{k-1} - p_{1,2}^{k-1}(x_2x_1)^{k-1}), x_1]_L, x_2]_L$$

(by inductive assumption)

$$= 2^{k-2}(p_{1,2}p_{2,1})^{\frac{(k-1)(k-2)}{2}}[2(-1)^{k-1}(p_{1,2}p_{2,1})^{k-1}(x_1x_2)^{k-1}x_1, x_2]_L$$

$$= (-1)^{k-1}2^{k-1}(p_{1,2}p_{2,1})^{\frac{k(k-1)}{2}}((-1)^{k-1}p_{2,1}^k(x_1x_2)^k - (-1)^{k-1}p_{1,2}^k(x_2x_1)^k)$$

$$= 2^{k-1}(p_{1,2}p_{2,1})^{\frac{k(k-1)}{2}}(p_{2,1}^k(x_1x_2)^k - p_{1,2}^k(x_2x_1)^k). \qquad \square$$

Theorem 3.5. *Assume that* $\mathfrak{L}(V)_L$ *is a connected Nichols L-bicharacter algebra of diagonal type with* $\dim V = 2$ *and* $p_{1,1} = p_{2,2} = -1$. *Let*

$$P_2 := \begin{cases} \{p_{2,1}^{k+1}(x_1x_2)^{k+1} - p_{1,2}^{k+1}(x_2x_1)^{k+1},\ x_2(x_1x_2)^k,\ (x_1x_2)^k x_1,\ 0 \le k < m\} \\ \qquad - \{p_{2,1}^m(x_1x_2)^m - p_{1,2}^m(x_2x_1)^m\} \\ \qquad\qquad \textit{when } (-p_{2,1})^m = (-p_{1,2})^m = -1 \\ \qquad\qquad \textit{with } \operatorname{ord}(p_{1,2}p_{2,1}) = m < \infty \\ \{p_{2,1}^{k+1}(x_1x_2)^{k+1} - p_{1,2}^{k+1}(x_2x_1)^{k+1},\ x_2(x_1x_2)^k,\ (x_1x_2)^k x_1, \\ \qquad 0 \le k < \operatorname{ord}(p_{1,2}p_{2,1})\}, \qquad \textit{otherwise}. \end{cases}$$

Then P_2 *is a basis of* $\mathfrak{L}(V)_L$. *Furthermore, if* $\operatorname{ord}(p_{1,2}p_{2,1}) = m < \infty$, *then*

$$\dim \mathfrak{L}(V)_L = \begin{cases} 3m - 1, & \text{when } (-p_{2,1})^m = (-p_{1,2})^m = -1, \\ 3m, & \text{otherwise.} \end{cases}$$

Proof. By Lemma 3.1(ii) and Lemma 3.4(iv), P_2 is linearly independent. It follows from Lemma 3.4(i) (iii) that $P_2 \subseteq \mathfrak{L}(V)_L$. By Lemma 3.4(ii), P_2 is a basis of $\mathfrak{L}(V)_L$. □

4. Some Equivalent Characterizations

In this section, we give the sufficient and necessary conditions for $\mathfrak{L}(V)_R = \mathfrak{L}(V)$, $\mathfrak{L}(V)_L = \mathfrak{L}(V)$, $\mathfrak{B}(V) = F \oplus \mathfrak{L}(V)_R$ and $\mathfrak{B}(V) = F \oplus \mathfrak{L}(V)_L$, respectively.

Lemma 4.1. *Assume that* u_i *is a homogeneous element and* $u_iu_j = p_{u_i,u_j}u_ju_i$ *with* $p_{u_i,u_j} \in F^*$ *and* $p_{u_i,u_j}p_{u_j,u_i} = 1$ *for* $1 \le i, j \le k$ *and* $p_{u_i,u_j} = 1$ *when* $u_i = u_j$. *Then*

$$[u_1, \dots, u_m]_R = \prod_{j=1}^{m-1} p_{u_m,\dots,u_{j+1},u_j}(p^3_{u_j,u_m,\dots,u_{j+1}} - 1)u_m, \dots, u_2u_1,$$

where $[u_1, \dots, u_m]_R := [u_1, [u_2, \dots, [u_{m-1}, u_m]_R \cdots]_R]_R$.

Proof. We show lemma by induction on m. Obviously,

$$u_1[u_2, \dots, u_m]_R = p_{u_1,u_2,\dots,u_m}[u_2, \dots, u_m]_Ru_1.$$

Therefore,

$$\begin{aligned} [u_1, \dots, u_m]_R &= p_{u_1,u_2,\dots,u_m}u_1[u_2, \dots, u_m]_R - p_{u_2,\dots,u_m,u_1}[u_2, \dots, u_m]_Ru_1 \\ &= (p^2_{u_1,u_2,\dots,u_m} - p_{u_2,\dots,u_m,u_1})[u_2, \dots, u_m]_Ru_1 \\ &= \prod_{j=1}^{m-1} p_{u_m,\dots,u_{j+1},u_j}(p^3_{u_j,u_m,\dots,u_{j+1}} - 1)u_m, \dots, u_2u_1 \end{aligned}$$

(by inductive assumption). □

Lemma 4.2. *Let* $(L_0, [\cdot, \cdot]_R)$ *be a Lie algebra and* $u_1, u_2, ..., u_m \in L_0$. *If* σ *is a method of adding bracket* $[\cdot, \cdot]_R$ *on* $u_1, u_2, ..., u_m$, *then there exist some* $\tau_j \in \mathbb{S}_m$, $\xi_j = 1$ *or* -1 *such that*

$$(*)\ \ \sigma(u_1, u_2, ..., u_m) = \sum_{j=1}^{r} \xi_j [u_{\tau_j(1)}, ..., u_{\tau_j(m)}]_R.$$

Proof. We show $(*)$ by induction on m. Obviously, $(*)$ holds for $m = 2$. Assume $m > 2$. Let

$$\sigma(u_1, ..., u_m) = [\sigma_1(u_1, ..., u_s), \sigma_2(u_{s+1}, u_{s+2}, ..., u_m)]_R.$$

Now we show $(*)$ by induction on s. In case $s = 1$,

$$\begin{aligned} &\sigma(u_1, ..., u_m) \\ &= [u_1, \sigma_2(u_2, ..., u_m)]_R \\ &= \sum_{j=1}^{r} \xi_j [u_1, [u_{\tau_j(2)}, ..., u_{\tau_j(m)}]_R]_R \ \ (\text{where } \tau_j \in \mathbb{S}_{\{2,3,...,m\}} \text{ for } 1 \le j \le r) \\ &= \sum_{j=1}^{r} \xi_j [u_{\tau_j(1)}, u_{\tau_j(2)}, ..., u_{\tau_j(m)}]_R. \end{aligned}$$

Therefore, $(*)$ holds.

Assume $s > 1$ and

$$\sigma_1(u_1, ..., u_s) = [\sigma_3(u_1, ..., u_k), \sigma_4(u_{k+1}, u_{k+2}, ..., u_s)]_R.$$

We have

$$\begin{aligned} &\sigma(u_1, ..., u_m) \\ &= [[\sigma_3(u_1, ..., u_k), \sigma_4(u_{k+1}, u_{k+2}, ..., u_s)]_R, \sigma_2(u_{s+1}, u_{s+2}, ..., u_m)]_R \\ &= [[\sigma_3(u_1, ..., u_k), \sigma_2(u_{s+1}, u_{s+2}, ..., u_m)]_R, \sigma_4(u_{k+1}, u_{k+2}, ..., u_s)]_R \end{aligned}$$

$$+ [\sigma_3(u_1, \ldots, u_k), [\sigma_4(u_{k+1}, u_{k+2}, \ldots, u_s), \sigma_2(u_{s+1}, u_{s+2}, \ldots, u_m)]_R]_R$$

$$= \sum_{j=1}^{r_1} \xi_j[[u_{\tau_j(1)}, \ldots, u_{\tau_j(k)}, u_{\tau_j(s+1)}, u_{\tau_j(s+2)}, \ldots, u_{\tau_j(m)}]_R,$$

$$\sigma_4(u_{k+1}, u_{k+2}, \ldots, u_s)]_R$$

$$+ \sum_{j=r_1+1}^{r_2} [\sigma_3(u_1, \ldots, u_k), \xi_j[u_{\tau_j(k+1)}, u_{\tau_j(k+2)}, \ldots, u_{\tau_j(s)}, u_{\tau_j(s+1)},$$

$$u_{\tau_j(s+2)}, \ldots, u_{\tau_j(m)}]_R]_R$$

(by first inductive assumption, where $\tau_j \in \mathbb{S}_{\{1, 2, 3, \ldots, k, s+1, s+2, \ldots, m\}}$

for $1 \le j \le r_1$; $\tau_j \in \mathbb{S}_{\{k+1, k+2, \ldots, m\}}$ for $r_1 + 1 \le j \le r_2$)

$$= \sum_{j=r_2+1}^{r_3} \xi_j[u_{\tau_j(1)}, \ldots, u_{\tau_j(m)}]_R$$

(by second inductive assumption, where $\tau_j \in \mathbb{S}_{\{1, 2, \ldots, m\}}$

for $r_2 + 1 \le j \le r_3$).

Therefore, $(*)$ holds. □

Proposition 4.3. *If* $\mathfrak{B}(V)$ *is a Nichols algebra of diagonal type, then* $\mathfrak{L}(V) = \mathfrak{L}(V)_R$ *if and only if* $p_{i,i}^2 = 1$, $p_{i,j}p_{j,i} = 1$, $p_{i,j} \in R_3$ *for* $1 \le i \ne j \le n$. *In this case*, $\mathfrak{L}(V) = \mathfrak{L}(V)_R = V$.

Proof. The sufficiency: By [20, Corollary 2.5], $\mathfrak{L}(V) = V$. By Lemma 2.5, $[x_i, x_j]_R = 0$ for $i \ne j$ and $\mathfrak{L}(V)_R = V$.

The necessity: We show this by following three steps:

(i) $p_{i,i}^2 = 1$ since $x_i^k \in \mathfrak{L}(V)$ for $k \le \operatorname{ord}(p_{i,i})$ and $x_i^m \notin \mathfrak{L}(V)_R$ when $0 \ne x_i^m$ and $m > 1$.

(ii) If $p_{i,j}p_{j,i} = 1$ and $p_{i,j} \notin R_3$ with $i \neq j$, then $[x_i, x_j]_R \neq 0$ and $[x_i, x_j] = 0$. Consequently, $[x_i, x_j]_R \in \mathfrak{L}(V)_R - \mathfrak{L}(V)$, which is a contradiction.

(iii) If $p_{i,j}p_{j,i} \neq 1$ with $i < j$, then $0 \neq [x_i, x_j] \in \mathfrak{L}(V) = \mathfrak{L}(V)_R$ and $[x_i, x_j] = k[x_i, x_j]_R$ with $k \in F^*$. By [20, Corollary 2.5], $x_jx_i \in \mathfrak{L}(V)$, which implies $x_jx_i = k'[x_i, x_j]_R$ with $k' \in F^*$. This is a contradiction since x_jx_i and $[x_i, x_j]$ are linearly independent. □

Proposition 4.4. *If $\mathfrak{B}(V)$ is a Nichols algebra of diagonal type, then the following conditions are equivalent:*

(1) $\mathfrak{L}(V) = \mathfrak{L}(V)_L$.

(2) $\mathfrak{L}(V) = \mathfrak{L}^-(V)$.

(3) $p_{i,i}^2 = 1$, $p_{i,j} = p_{j,i} = 1$ *for* $1 \leq i \neq j \leq n$. *In this case,* $\mathfrak{L}(V) = \mathfrak{L}(V)_L = V$.

Proof. By [20, Proposition 6.3], $(2) \Leftrightarrow (3)$. The proof of $(1) \Leftrightarrow (3)$ is similar to the proof of $(2) \Leftrightarrow (3)$.

Proposition 4.5. *Assume that $\mathfrak{B}(V)$ is a Nichols algebra of diagonal type. Then $\mathfrak{B}(V) = F \oplus \mathfrak{L}(V)_R$ if and only if $p_{i,i} = -1$, $p_{i,j}p_{j,i} = 1$ for all $1 \leq i \neq j \leq n$ and there exists $\tau \in \mathbb{S}_m$ such that*

$$\prod_{j=1}^{m-1}(p^3_{h_{\tau(j)}, h_{\tau(m)}, \ldots, h_{\tau(j+1)}} - 1) \neq 0$$

for all $h_1 > h_2 > \cdots > h_m$ with $h_i \in \{x_1, \ldots, x_n\}$, $1 \leq i \leq m$.

Proof. The necessity: If there exists $1 \leq i \leq n$ such that $p_{i,i} \neq -1$, then $0 \neq x_i^2 \in \mathfrak{B}(V)$ and $x_i^2 \notin \mathfrak{L}(V)_R$, which is a contradiction. If there exist

i, j such that $p_{i,j}p_{j,i} \neq 1$ with $1 \leq i < j \leq n$, then $[x_i, x_j] \neq 0$ and $[x_i, x_j]_R \neq 0$. Since $\mathfrak{B}(V) = F \oplus \mathfrak{L}(V)_R$, we have that there exist $k, k' \in F^*$ such that $[x_i, x_j] = k[x_i, x_j]_R$, and $x_j x_i = k'[x_i, x_j]_R$, which contradicts to that $[x_i, x_j]$ and $x_j x_i$ are linearly independent. Therefore, V is a quantum linear space.

If there exist $h_1 > h_2 > \cdots > h_m$ with $h_i \in \{x_1, ..., x_n\}$, $1 \leq i \leq m$, such that $\prod_{j=1}^{m-1}(p^3_{h_{\tau(j)}, h_{\tau(m)}, ..., h_{\tau(j+1)}} - 1) = 0$ for any $\tau \in \mathbb{S}_m$. By [20, Lemma 3.2], Lemma 4.1 and Lemma 4.2, $\sigma(h_{\tau(1)}, h_{\tau(2)}, ..., h_{\tau(m)}) = 0$ for any $\tau \in \mathbb{S}_m$ and any method σ of adding bracket $[\cdot, \cdot]_R$ on $h_{\tau(1)}, h_{\tau(2)}, ..., h_{\tau(m)}$. Consequently, $0 \neq h_1 h_2 \cdots h_m \notin \mathfrak{L}(V)_R$, which is a contradiction.

The sufficiency: Obviously, V is a quantum linear space. For any $h_1 > h_2 > \cdots > h_m$ with $h_i \in \{x_1, ..., x_n\}$, $1 \leq i \leq m$, there exists $\tau \in \mathbb{S}_m$ such that $\prod_{j=1}^{m-1}(p^3_{h_{\tau(j)}, h_{\tau(m)} \cdots h_{\tau(j+1)}} - 1) \neq 0$. By Lemma 4.1, we have that there exists $a \in F^*$ such that $h_1 h_2 \cdots h_m = a h_{\tau(m)} h_{\tau(m-1)} \cdots h_{\tau(1)} \in \mathfrak{L}(V)_R$. □

Lemma 4.6. *Assume that* u_i *is a homogeneous element and* $u_i u_j = p_{u_i, u_j} u_j u_i$ *with* $p_{u_i, u_j} \in F^*$ *and* $p_{u_i, u_j} p_{u_j, u_i} = 1$ *for* $1 \leq i, j \leq k$ *and* $p_{u_i, u_j} = 1$ *when* $u_i = u_j$. *Then*

$$[u_1, ..., u_m]_L = \prod_{j=1}^{m-1} p_{u_m, ..., u_{j+1}, u_j}(p_{u_j, u_m, ..., u_{j+1}} - 1) u_m, ..., u_2 u_1,$$

where

$$[u_1, ..., u_m]_L := [u_1, [u_2, ..., [u_{m-1}, u_m]_L \cdots]_L]_L.$$

Lemma 4.7. *Let* $(L_0, [\cdot, \cdot]_L)$ *be a Lie algebra and* $u_1, u_2, \ldots, u_m \in L_0$. *If* σ *is a method of adding bracket* $[\cdot, \cdot]_L$ *on* $u_1, u_2, \ldots, u_m$, *then there exist some* $\tau_j \in \mathbb{S}_m$, $\xi_j = 1$ *or* -1 *such that*

$$(*)\ \sigma(u_1, u_2, \ldots, u_m) = \sum_{j=1}^{r} \xi_j [u_{\tau_j(1)}, \ldots, u_{\tau_j(m)}]_L.$$

Proposition 4.8. *Assume that* $\mathfrak{B}(V)$ *is a Nichols algebra of diagonal type. Then the following conditions are equivalent*:

(1) $\mathfrak{B}(V) = F \oplus \mathfrak{L}(V)_L$.

(2) $\mathfrak{B}(V) = F \oplus \mathfrak{L}^-(V)$.

(3) $p_{i,i} = -1$, $p_{i,j}p_{j,i} = 1$ *for all* $1 \leq i \neq j \leq n$ *and there exists* $\tau \in \mathbb{S}_m$ *such that* $\prod_{j=1}^{m-1}(p_{h_{\tau(j)}, h_{\tau(m)}, \ldots, h_{\tau(j+1)}} - 1) \neq 0$ *for all* $h_1 > h_2 > \cdots > h_m$ *with* $h_i \in \{x_1, \ldots, x_n\}$, $1 \leq i \leq m$.

Proof. By [20, Proposition 6.4], $(2) \Leftrightarrow (3)$. The proof of $(1) \Leftrightarrow (3)$ is similar to the proof of $(2) \Leftrightarrow (3)$. □

Conjecture 4.9. Assume that $\mathfrak{B}(V)$ is a Nichols algebra of diagonal type. Then $\mathfrak{L}(V)_L = \mathfrak{L}^-(V)$.

Question 4.10. Assume that $\mathfrak{B}(V)$ is a Nichols algebra of diagonal type. Then give the sufficient and necessary conditions for $\mathfrak{L}(V)_L = \mathfrak{L}(V)_R$.

5. Classification of $\mathfrak{L}(V)_L$ and $\mathfrak{L}(V)_R$

In this section, it is proved that if $\mathfrak{B}(V)$ is a connected Nichols algebra of diagonal type with $\dim V > 1$, then $\mathfrak{B}(V)$ is finite-dimensional if and only if $\mathfrak{L}(V)_L$ is finite-dimensional, or equivalently $\mathfrak{L}(V)_R$ is finite-dimensional.

Let $|u|$ denote the length of word u. Then a word $u \in W$ is called a *Lyndon word* if $|u| = 1$ or $|u| \geq 2$, and for each representation $u = u_1 u_2$, where u_1 and u_2 are nonempty words, the inequality $u < u_2 u_1$ holds (see [14, Definition 1]). Any word $u \in W$ has a unique decomposition into the product of non-increasing sequence of Lyndon words by [15, Theorem 5.1.5]. If u is a Lyndon word with $|u| > 1$, then there uniquely exist two Lyndon words v and w such that $u = vw$ and v is shortest (see [15, Proposition 5.1.3]) (the composition is called the *Shirshov decomposition* of u).

We call u is a *standard word* with respect to $\mathfrak{B}(V)$ if u cannot be written as a linear combination of strictly greater words in $\mathfrak{B}(V)$.

Let $S(\mathfrak{B}(V)) := \{u \in W \mid u$ is a standard word with respect to $\mathfrak{B}(V)\}$, written as S in short; $L := \{u \in W \mid u$ is a Lyndon word$\}$. Let $H := \{u \in L \mid [u]$ is a hard super-letter$\}$. Notice that often we assume for convenience that L and H are in $\mathfrak{B}(V)$.

Let $D =: \{[u] \mid [u]$ be a hard super-letter$\}$, and

$$\Delta^+(\mathfrak{B}(V)) := \{\deg(u) \mid [u] \in D\}.$$

Then $\Delta(\mathfrak{B}(V)) := \Delta^+(\mathfrak{B}(V)) \bigcup \Delta^-(\mathfrak{B}(V))$, which is called the *root system* of V. If $\Delta(\mathfrak{B}(V))$ is finite, then it is called an *arithmetic root system*.

5.1. $\Delta(\mathfrak{B}(V))$ is not an arithmetic root system

Lemma 5.1 (See [22, Lemma 2.2])**.** (i) *S is a basis of $\mathfrak{B}(V)$.*

(ii) *Any factor of a standard word is a standard word.*

(iii) *If u is a standard word, then $u = u_1 u_2 \cdots u_r$ with $u_1 \geq u_2 \geq \cdots \geq u_r$ and $u_i \in S \bigcap L$ for $1 \leq i \leq r$.*

Lemma 5.2 (See [22, Theorem 2.4])**.** *If* $\mathfrak{B}(V)$ *is a Nichols algebra of diagonal type*, *then* $S(\mathfrak{B}(V)) \bigcap L = H(\mathfrak{B}(V))$.

Lemma 5.3. (i) *If* $l \in L$, *then* $[l]_L = a_l l + \sum_{w>l, |l|=|w|} a_w w$ *in* $\mathfrak{B}(V)$, *where* $a_l, a_w \in F$ *with* $a_l \neq 0$.

(ii) *If* $l \in L$, *then* $[l]_R = b_l l + \sum_{w>l, |l|=|w|} b_w w$ *in* $\mathfrak{B}(V)$, *where* $b_l, b_w \in F$ *with* $b_l \neq 0$.

Proof. (i) We show this by induction on $|l|$. It is clear when $|l| = 1$ since $[l]_L = l$. Assume that $l = uv$ is the Shirshov decomposition of l. If $u' > u$ and $v' > v$ with $|u'| = |u|$ and $|v'| = |v|$, then $u'v' > uv = l$ and $v'u' > vu > l$,

$$\begin{aligned}[l]_L &= p_{v,u}[u][v] - p_{u,v}[v][u] \\ &= p_{v,u}\left(a'_u u + \sum_{u'>u, |u'|=|u|} a'_{u'}u'\right)\left(a'_v v + \sum_{v'>v, |v'|=|v|} a'_{v'}v'\right) \\ &\quad - p_{u,v}\left(a'_v v + \sum_{v'>v, |v'|=|v|} a'_{v'}v'\right)\left(a'_u u + \sum_{u'>u, |u'|=|u|} a'_{u'}u'\right) \\ &\qquad\qquad \text{(by inductive hypothesis)} \\ &= a_l l + \sum_{w>l, |l|=|w|} a_w w.\end{aligned}$$

(ii) The proof is similar to the proof of (i). □

Theorem 5.4. *If* $\mathfrak{B}(V)$ *is a Nichols algebra of diagonal type and* $\Delta(\mathfrak{B}(V))$ *is not an arithmetic root system*, *then* $\dim \mathfrak{L}(V)_L = \infty$, $\dim \mathfrak{L}(V)_R = \infty$.

5.2. $\Delta(\mathfrak{B}(V))$ is an arithmetic root system

For our study of Nichols algebras we need some non-standard formulas for quantum integers and Gaussian binomial coefficients.

In the ring $\mathbb{Z}[a]$, let $(0)_a = 0$ and for any $m \in \mathbb{N}$, $(m)_a = 1 + a + a^2 + \cdots + a^{m-1}$. Then the polynomials $(m)_a$ with $m \in \mathbb{Z}$ are also known as *quantum integers*. Moreover, let $(0)_a^! = 1$, and for any $m \in \mathbb{Z}$ let $(m)_a^! = \prod_{k=1}^{m}(k)_a$. For any $k, m \in \mathbb{Z}$ with $0 \leq k \leq m$, the rational function $\binom{m}{k}_a = \frac{(m)_a^!}{(k)_a^!(m-k)_a^!}$, is in fact an element of $\mathbb{Z}[a]$ and is called a *Gaussian binomial coefficient*. For $m \in \mathbb{N}_0$, $k \in \mathbb{Z}$ with $k < 0$ or $k > m$ one defines $\binom{m}{k}_a = 0$. The Gaussian binomial coefficients satisfy the following formulas:

$$\binom{m}{k}_a = \binom{m}{m-k}_a, \tag{1}$$

$$a^{k-1}\binom{m}{k}_a + a^m\binom{m}{k-1}_a = a^{k-1}\binom{m+1}{k}_a \tag{2}$$

for $m \in \mathbb{N}$, $1 \leq k \leq m$.

Lemma 5.5. *For* $m \in \mathbb{N}$, $i \neq j$,

(i) $\sum_{k=0}^{m}(-1)^k\binom{m}{k}(m-k)_a = (a-1)^{m-1}$.

(ii) $l_i^m[j]_L = p_{i,i}^{\frac{m(m-1)}{2}}\sum_{k=0}^{m}(-1)^k\binom{m}{k}p_{i,j}^k p_{j,i}^{m-k} x_i^{m-k} x_j x_i^k$.

(iii) $l_i^m[j]_R = p_{i,i}^{\frac{m(m-1)}{2}} \sum_{k=0}^{m} (-1)^k \binom{m}{k} p_{i,j}^{m-k} p_{j,i}^k x_i^{m-k} x_j x_i^k.$

(iv) $l_i^m[j] = \sum_{k=0}^{m} (-1)^k p_{i,i}^{\frac{k(k-1)}{2}} p_{j,i}^k \binom{m}{k}_{p_{i,i}} x_i^k x_j x_i^{m-k}.$

Proof. It can be proved by induction. □

It is clear that $l_i^m[j]_L = l_i^m[j]_R = p_{i,i}^{\frac{m(m-1)}{2}} p_{i,j}^m \bar{l}_i^m[j]^-$ if $p_{\cdot,\cdot}$ is a symmetrical bicharacter.

Lemma 5.6. *Assume that* $\mathfrak{B}(V)$ *is a Nichols algebra of diagonal type and* $m \in \mathbb{N}$, $i \neq j$. *Then*:

(i) *If* $p_{i,j} = p_{j,i}^2$ *and* $p_{j,i} = p_{i,j}^2$, *i.e.*, $p_{i,j} p_{j,i} = 1$, $p_{i,j} \in R_3$, *then* $l_i^m[j]_R = 0$.

(ii) (1) $\langle y_j, l_i^m[j]_R \rangle = p_{i,i}^{\frac{m(m-1)}{2}} (p_{i,j} p_{j,i}^{-1} - p_{j,i})^m x_i^m.$

(2) $\langle y_i^m y_j, l_i^m[j]_R \rangle = (p_{i,j} p_{j,i}^{-1} - p_{j,i})^m (m)_{p_{i,i}}^!.$

(iii) (1)

$$\langle y_i^k, l_i^m[j]_R \rangle = p_{i,i}^{m-1} \{ (p_{i,j} - p_{j,i} p_{i,j}^{-1} p_{i,i}^{k-m}) (k)_{p_{i,i}^{-1}} \langle y_i^{k-1}, l_i^{m-1}[j]_R \rangle$$

$$+ p_{i,j} p_{i,i}^{-k} x_i \langle y_i^k, l_i^{m-1}[j]_R \rangle - p_{j,i} \langle y_i^k, l_i^{m-1}[j]_R \rangle x_i \}$$

for $1 \leq k \leq m$.

(2) $\langle y_i^m, l_i^m[j]_R \rangle = (p_{i,j} - p_{j,i} p_{i,j}^{-1})^m (m)_{p_{i,i}}^! x_j.$

(3) $\langle y_j y_i^m, l_i^m[j]_R \rangle = (p_{i,j} - p_{j,i} p_{i,j}^{-1})^m (m)_{p_{i,i}}^!.$

(iv) *Assume that* $p_{i,i} = 1$. *Then* $l_i^m[j]_R \neq 0$, *when* $p_{i,j} \neq p_{j,i}^2$ *or* $p_{j,i} \neq p_{i,j}^2$.

(v) *Assume that* $p_{i,i} \neq 1$. *Then* $l_i^m[j]_R \neq 0$, *when* $\operatorname{ord}(p_{i,i}) > m$ *with* $p_{i,j} \neq p_{j,i}^2$ *or* $p_{j,i} \neq p_{i,j}^2$.

Proof. (ii) (1)

$$\begin{aligned}\langle y_j, l_i^m[j]_R\rangle &= \left\langle y_j,\ p_{i,i}^{\frac{m(m-1)}{2}} \sum_{k=0}^{m} (-1)^k \binom{m}{k} p_{i,j}^{m-k} p_{j,i}^{k} x_i^{m-k} x_j x_i^k \right\rangle \\ &= p_{i,i}^{\frac{m(m-1)}{2}} p_{i,j}^{m} p_{j,i}^{-m} \sum_{k=0}^{m} (-1)^k \binom{m}{k} p_{i,j}^{-k} p_{j,i}^{2k} x_i^m \\ &= p_{i,i}^{\frac{m(m-1)}{2}} p_{i,j}^{m} p_{j,i}^{-m} (1 - p_{i,j}^{-1} p_{j,i}^{2})^m x_i^m \\ &= p_{i,i}^{\frac{m(m-1)}{2}} (p_{i,j} p_{j,i}^{-1} - p_{j,i})^m x_i^m.\end{aligned}$$

(2) It can be proved by induction.

(iii) We show (1) by induction on k for $1 \leq k \leq m$. We have

$$\begin{aligned}\langle y_i, l_i^m[j]_R\rangle &= \langle y_i,\ p_{i,i}^{m-1} p_{i,j} x_i l_i^{m-1}[j]_R - p_{i,i}^{m-1} p_{j,i} l_i^{m-1}[j]_R x_i\rangle \\ &= p_{i,i}^{m-1}\{(p_{i,j} - p_{j,i} p_{i,j}^{-1} p_{i,i}^{1-m}) l_i^{m-1}[j]_R \\ &\quad + p_{i,j} p_{i,i}^{-1} x_i \langle y_i, l_i^{m-1}[j]_R\rangle - p_{j,i}\langle y_i, l_i^{m-1}[j]_R\rangle x_i\}.\end{aligned}$$

Thus, equation (1) holds when $k = 1$. Assume $k > 1$. We have

$$\langle y_i^k, l_i^m[j]_R\rangle$$

$$= \langle y_i,\ p_{i,i}^{m-1}\{(p_{i,j} - p_{j,i} p_{i,j}^{-1} p_{i,i}^{k-1-m})(k-1)_{p_{i,i}^{-1}} \langle y_i^{k-2}, l_i^{m-1}[j]_R\rangle$$

$$+\, p_{i,j}p_{i,i}^{-k+1}x_i\langle y_i^{k-1},\, l_i^{m-1}[j]_R\rangle - p_{j,i}\langle y_i^{k-1},\, l_i^{m-1}[j]_R\rangle x_i\}\rangle$$

(by inductive hypothesis)

$$= p_{i,i}^{m-1}\{(p_{i,j} - p_{j,i}p_{i,j}^{-1}p_{i,i}^{k-1-m})(k-1)_{p_{i,i}^{-1}}\langle y_i^{k-1},\, l_i^{m-1}[j]_R\rangle$$

$$+\, p_{i,j}p_{i,i}^{-k+1}\langle y_i^{k-1},\, l_i^{m-1}[j]_R\rangle + p_{i,j}p_{i,i}^{-k}x_i\langle y_i^{k},\, l_i^{m-1}[j]_R\rangle$$

$$-\, p_{j,i}\langle y_i^{k},\, l_i^{m-1}[j]_R\rangle x_i - p_{j,i}p_{i,j}^{-1}p_{i,i}^{k-m}\langle y_i^{k-1},\, l_i^{m-1}[j]_R\rangle\}$$

$=$ the right hand side of (1). Consequently, (1) holds.

Now we show (2) by induction on m. (2) is clear when $m = 1$. By (1), we have

$$\langle y_i^m,\, l_i^m[j]_R\rangle = p_{i,i}^{m-1}(p_{i,j} - p_{j,i}p_{i,j}^{-1})(m)_{p_{i,i}^{-1}}\langle y_i^{m-1},\, l_i^{m-1}[j]_R\rangle$$

$$= (p_{i,j} - p_{j,i}p_{i,j}^{-1})(m)_{p_{i,i}}\langle y_i^{m-1},\, l_i^{m-1}[j]_R\rangle$$

$$= (p_{i,j} - p_{j,i}p_{i,j}^{-1})(m)_{p_{i,i}}(p_{i,j} - p_{j,i}p_{i,j}^{-1})^{m-1}(m-1)!_{p_{i,i}}x_j$$

(by inductive hypothesis)

$=$ the right hand side of (2). Therefore, (2) and (3) hold.

(iv) If $p_{i,i} = 1$, then $\langle y_i^m y_j,\, l_i^m[j]_L\rangle = (p_{i,j}p_{j,i}^{-1} - p_{j,i})^m(m)!$ and $\langle y_j y_i^m,\, l_i^m[j]_L\rangle = (p_{i,j} - p_{j,i}p_{i,j}^{-1})^m(m)!$ by (ii) and (iii).

(v) It follows from (ii) and (iii). □

Lemma 5.7. *Assume that* $\mathfrak{B}(V)$ *is a Nichols algebra of diagonal type and* $m \in \mathbb{N}$, $i \neq j$. *Then*:

(i) *If* $p_{i,j} = p_{j,i} = 1$, *then* $l_i^m[j]_L = 0$.

(ii) (1) $\langle y_j, l_i^m[j]_L\rangle = p_{i,i}^{\frac{m(m-1)}{2}}(1 - p_{i,j})^m x_i^m.$

(2) $\langle y_i^m y_j, l_i^m[j]_L\rangle = (1 - p_{i,j})^m (m)^!_{p_{i,i}}.$

(iii) (1)

$$\langle y_i^k, l_i^m[j]_L\rangle = p_{i,i}^{m-1}\{(p_{j,i} - p_{i,i}^{k-m})(k)_{p_{i,i}^{-1}}\langle y_i^{k-1}, l_i^{m-1}[j]_L\rangle$$

$$+ p_{j,i}p_{i,i}^{-k}x_i\langle y_i^k, l_i^{m-1}[j]_L\rangle - p_{i,j}\langle y_i^k, l_i^{m-1}[j]_L\rangle x_i\}$$

for $1 \le k \le m$.

(2) $\langle y_i^m, l_i^m[j]_L\rangle = (p_{j,i} - 1)^m (m)^!_{p_{i,i}} x_j.$

(3) $\langle y_j y_i^m, l_i^m[j]_L\rangle = (p_{j,i} - 1)^m (m)^!_{p_{i,i}}.$

(iv) *Assume that* $p_{i,i} = 1$. *Then* $l_i^m[j]_L \neq 0$, *when* $p_{i,j} \neq 1$ *or* $p_{j,i} \neq 1$.

(v) *Assume that* $p_{i,i} \neq 1$. *Then* $l_i^m[j]_L \neq 0$, *when* $\mathrm{ord}(p_{i,i}) > m$ *with* $p_{i,j} \neq 1$ *or* $p_{j,i} \neq 1$.

Proof. (ii) (1)

$$\langle y_j, l_i^m[j]_L\rangle = \left\langle y_j, p_{i,i}^{\frac{m(m-1)}{2}} \sum_{k=0}^{m} (-1)^k \binom{m}{k} p_{i,j}^k p_{j,i}^{m-k} x_i^{m-k} x_j x_i^k \right\rangle$$

$$= p_{i,i}^{\frac{m(m-1)}{2}} \sum_{k=0}^{m} (-1)^k \binom{m}{k} p_{i,j}^k p_{j,i}^{m-k} p_{j,i}^{k-m} x_i^m$$

$$= p_{i,i}^{\frac{m(m-1)}{2}} (1 - p_{i,j})^m x_i^m.$$

(2) It can be proved by induction.

(iii) We show (1) by induction on k for $1 \leq k \leq m$. We have

$$\begin{aligned}\langle y_i,\, l_i^m[j]_L\rangle &= \langle y_i,\, p_{i,i}^{m-1}p_{j,i}x_i l_i^{m-1}[j]_L - p_{i,i}^{m-1}p_{i,j}l_i^{m-1}[j]_L x_i\rangle \\ &= p_{i,i}^{m-1}\{(p_{j,i} - p_{i,i}^{1-m})l_i^{m-1}[j]_L + p_{j,i}p_{i,i}^{-1}x_i\langle y_i,\, l_i^{m-1}[j]_L\rangle \\ &\quad - p_{i,j}\langle y_i,\, l_i^{m-1}[j]_L\rangle x_i\}.\end{aligned}$$

Thus, equation (1) holds when $k = 1$. Assume $k > 1$. We have

$$\begin{aligned}&\langle y_i^k,\, l_i^m[j]_L\rangle \\ &= \langle y_i,\, p_{i,i}^{m-1}\{(p_{j,i} - p_{i,i}^{k-1-m})(k-1)_{p_{i,i}^{-1}}\langle y_i^{k-2},\, l_i^{m-1}[j]_L\rangle \\ &\quad + p_{j,i}p_{i,i}^{-k+1}x_i\langle y_i^{k-1},\, l_i^{m-1}[j]_L\rangle - p_{i,j}\langle y_i^{k-1},\, l_i^{m-1}[j]_L\rangle x_i\}\rangle \\ &\qquad\qquad\qquad\qquad\qquad\qquad\text{(by inductive hypothesis)} \\ &= p_{i,i}^{m-1}\{(p_{j,i} - p_{i,i}^{k-1-m})(k-1)_{p_{i,i}^{-1}}\langle y_i^{k-1},\, l_i^{m-1}[j]_L\rangle \\ &\quad + p_{j,i}p_{i,i}^{-k+1}\langle y_i^{k-1},\, l_i^{m-1}[j]_L\rangle + p_{j,i}p_{i,i}^{-k}x_i\langle y_i^k,\, l_i^{m-1}[j]_L\rangle \\ &\quad - p_{i,j}\langle y_i^k,\, l_i^{m-1}[j]_L\rangle x_i - p_{i,j}p_{i,j}^{-1}p_{i,i}^{k-m}\langle y_i^{k-1},\, l_i^{m-1}[j]_L\rangle\} \\ &= \text{the right hand side of (1). Consequently, (1) holds.}\end{aligned}$$

Now we show (2) by induction on m. (2) is clear when $m = 1$. By (1), we have

$$\begin{aligned}\langle y_i^m,\, l_i^m[j]_L\rangle &= p_{i,i}^{m-1}(p_{j,i}-1)(m)_{p_{i,i}^{-1}}\langle y_i^{m-1},\, l_i^{m-1}[j]_L\rangle \\ &= (p_{j,i}-1)(m)_{p_{i,i}}\langle y_i^{m-1},\, l_i^{m-1}[j]_L\rangle \\ &= (p_{j,i}-1)(m)_{p_{i,i}}(p_{j,i}-1)^{m-1}(m-1)^!_{p_{i,i}}x_j \\ &\qquad\qquad\qquad\qquad\text{(by inductive hypothesis)} \\ &= \text{the right hand side of (2). Therefore, (2) and (3) hold.}\end{aligned}$$

(iv) If $p_{i,i} = 1$, then

$$\langle y_i^m y_j,\, l_i^m[j]_L\rangle = (1 - p_{i,j})^m (m)! \text{ and } \langle y_j y_i^m,\, l_i^m[j]_L\rangle = (p_{j,i} - 1)^m (m)!$$

by (ii) and (iii).

(v) It follows from (ii) and (iii). □

Proposition 5.8. *Assume that* $\mathfrak{B}(V)$ *is a Nichols algebra of diagonal type. Then*

(i) $\dim(\mathfrak{L}(V)_R) = \infty$ *when there exist i and j with* $i \neq j$, $p_{i,j} \neq p_{j,i}^2$ *or* $p_{j,i} \neq p_{i,j}^2$ *and* $\operatorname{ord}(p_{i,i}) = 1$ *or* ∞.

(ii) $\dim(\mathfrak{L}(V)_L) = \infty$ *when there exist i and j with* $i \neq j$, $p_{i,j} \neq 1$ *or* $p_{j,i} \neq 1$ *and* $\operatorname{ord}(p_{i,i}) = 1$ *or* ∞.

Lemma 5.9 (See [22, Lemma 5.1]). *The set of monomials* $x_i^s x_j x_i^t$ *with* $0 \leq s \leq m_{ij}$, $0 \leq t \leq \operatorname{ord}(p_{i,i}) - 1$ *for* $i < j$ *and the set of monomials* $x_i^s x_j x_i^t$ *with* $0 \leq t \leq m_{ij}$, $0 \leq s \leq \operatorname{ord}(p_{i,i}) - 1$ *for* $i > j$ *are linearly independent.*

We know $m_{ij} \leq \operatorname{ord}(p_{i,i}) - 1$.

Proposition 5.10. *For* $m \in \mathbb{N}$, $i \neq j$.

(i) *Assume that* $p_{i,j} = p_{j,i}^{-1} \neq 1$. *Then* $l_i^m[j]_L \neq 0$ *if and only if* $\operatorname{ord}(p_{i,i}) > m \geq 0$, $l_i^m[j]_R \neq 0$ *if and only if* $p_{j,i} \notin R_3$ *and* $\operatorname{ord}(p_{i,i}) > m \geq 0$.

(ii) *Assume that* $m_{ij} = \operatorname{ord}(p_{i,i}) - 1$, $p_{i,j} p_{j,i} \neq 1$. *Then* $l_i^m[j]_L \neq 0$ *if and only if* $0 \leq m \leq m_{ij} - 1 + \operatorname{ord}(p_{i,i})$ *if and only if* $l_i^m[j]_R \neq 0$.

Proof. (i) We show $l_i^m[j]_L = p_{i,i}^{\frac{m(m-1)}{2}}(1 - p_{i,j})^m ji^m$ and $l_i^m[j]_R = p_{i,i}^{\frac{m(m-1)}{2}} p_{i,j}^{2m}(1 - p_{j,i}^3)^m ji^m$ by induction on m since $ij = p_{i,j}ji$, the other is clear by [9, Lemma 1.3.3].

(ii) We obtain

$$l_i^m[j]_L = p_{i,i}^{\frac{m(m-1)}{2}} \sum_{k=0}^{m} (-1)^k \binom{m}{k} p_{i,j}^k p_{j,i}^{m-k} x_i^{m-k} x_j x_i^k$$

by Lemma 5.5(ii). Then

$$\begin{aligned} & l_i^{m_{ij}-1+\mathrm{ord}(p_{i,i})}[j]_L \\ & = l_i^{2\mathrm{ord}(p_{i,i})-2}[j]_L = p_{i,i}^{\frac{(2\mathrm{ord}(p_{i,i})-2)(2\mathrm{ord}(p_{i,i})-3)}{2}} \\ & \times \sum_{k=0}^{2\mathrm{ord}(p_{i,i})-2} (-1)^k \binom{2\mathrm{ord}(p_{i,i})-2}{k} p_{i,j}^k p_{j,i}^{2\mathrm{ord}(p_{i,i})-2-k} x_i^{2\mathrm{ord}(p_{i,i})-2-k} x_j x_i^k \\ & = p_{i,i}^{\frac{(2\mathrm{ord}(p_{i,i})-2)(2\mathrm{ord}(p_{i,i})-3)}{2}} (-1)^{\mathrm{ord}(p_{i,i})-1} \binom{2\mathrm{ord}(p_{i,i})-2}{\mathrm{ord}(p_{i,i})-1} \\ & \times (p_{i,j}p_{j,i})^{\mathrm{ord}(p_{i,i})-1} x_i^{\mathrm{ord}(p_{i,i})-1} x_j x_i^{\mathrm{ord}(p_{i,i})-1} \end{aligned}$$

by [9, Lemma 1.3.3(i)]. It is clear $l_i^{m_{ij}-1+\mathrm{ord}(p_{i,i})}[j]_L \neq 0$ since

$$x_i^{\mathrm{ord}(p_{i,i})-1} x_j x_i^{\mathrm{ord}(p_{i,i})-1} = x_i^{m_{ij}} x_j x_i^{\mathrm{ord}(p_{i,i})-1} \text{ or } x_i^{\mathrm{ord}(p_{i,i})-1} x_j x_i^{m_{ij}}$$

is a basic element by Lemma 5.5. Then $l_i^m[j]_L \neq 0$ if $0 \leq m \leq m_{ij} - 1 + \mathrm{ord}(p_{i,i})$. On the other hand,

$$\begin{aligned} & l_i^{m_{ij}+\mathrm{ord}(p_{i,i})}[j]_L \\ &= p_{i,i}^{\frac{(2\mathrm{ord}(p_{i,i})-2)(2\mathrm{ord}(p_{i,i})-3)}{2}}(-1)^{\mathrm{ord}(p_{i,i})-1}\binom{2\mathrm{ord}(p_{i,i})-2}{\mathrm{ord}(p_{i,i})-1} \\ &\quad \times (p_{i,j}p_{j,i})^{\mathrm{ord}(p_{i,i})-1}[x_i, x_i^{\mathrm{ord}(p_{i,i})-1}x_jx_i^{\mathrm{ord}(p_{i,i})-1}]_L \\ &= 0. \end{aligned}$$

The rest of the proof is similar. □

Lemma 5.11. *Assume that* $\mathfrak{B}(V)$ *is a connected Nichols algebra of diagonal type with* $\dim V > 1$. *Then*

(i) *If* $p_{i,i} = p_{j,j} = -1$ *and* $\mathrm{ord}(p_{i,j}p_{j,i}) = \infty$ *for* $i \neq j$. *Then* $\dim(\mathfrak{L}(V)_R) = \infty$, $\dim(\mathfrak{L}(V)_L) = \infty$.

(ii) *If* $\Delta(\mathfrak{B}(V))$ *is an arithmetic root system and there exists* $u \in D$ *such that* $\mathrm{ord}(p_{u,u}) = \infty$, *then there exists* $1 \leq i \leq n$ *such that* $\mathrm{ord}(p_{i,i}) = \infty$ *or* $\dim(\mathfrak{L}(V)_R) = \infty$, $\dim(\mathfrak{L}(V)_L) = \infty$.

Proof. (i) It follows from Theorem 3.3 and Theorem 3.5.

(ii) If there exists $u \in D$ such that $\mathrm{ord}(p_{u,u}) = \infty$, then there exists a $1 \leq i \leq n$ such that $\mathrm{ord}(p_{i,i}) = \infty$ except the cases of [9, Row 3 Diagram 2, Table 1], [9, Row 8, Diagram 2, Table 2; Row 9, Diagram 4, Table 2; Row 10, Diagram 3, Table 2] and [10, Row 10, Diagram 6, Appendix B; Row 12, Diagram 5, Appendix B. Row 2, Appendix C; Row 10, Diagram 2, Appendix C]. By (i), $\dim(\mathfrak{L}(V)_R) = \infty$, $\dim(\mathfrak{L}(V)_L) = \infty$. □

5.3. Finiteness of $\mathfrak{L}(V)_L$ and $\mathfrak{L}(V)_R$

Proposition 5.12. *Assume that* $\mathfrak{B}(V)$ *is a connected Nichols algebra of diagonal type. Then* $\dim(\mathfrak{B}(V)) = \infty$ *and* $\dim(\mathfrak{L}(V)_L) < \infty$ *if and only if* $\dim V = 1$ *and* $\mathrm{ord}(p_{1,1}) = 1$ *or* ∞.

Proof. The if part of the assertion is obvious. It remains to prove the only if part. Assume $\dim V > 1$. Then by Theorem 5.4, $\Delta(\mathfrak{B}(V))$ is an arithmetic root system. Consequently, there exists $u \in D$ such that $\operatorname{ord}(p_{u,u}) = \infty$. Considering Lemma 5.11(ii) and Proposition 5.8(ii) we get a contradiction. □

Proposition 5.13. *Assume that* $\mathfrak{B}(V)$ *is a connected Nichols algebra of diagonal type*. *Then* $\dim(\mathfrak{B}(V)) = \infty$ *and* $\dim(\mathfrak{L}(V)_R) < \infty$ *if and only if* $\dim V = 1$, $\operatorname{ord}(p_{1,1}) = 1$ *or* ∞.

Proof. Sufficiency is clear. Assume $\dim V > 1$. Then by Theorem 5.4, $\Delta(\mathfrak{B}(V))$ is an arithmetic root system. Consequently, there exists $u \in D$ such that $\operatorname{ord}(p_{u,u}) = \infty$. Considering Lemma 5.11(ii) and Proposition 5.8(i) we have $p_{i,j} = p_{j,i}^2$, $p_{j,i} = p_{i,j}^2$, then $p_{i,j}p_{j,i} = 1$, $p_{i,j} \in R_3$, which is a contradiction. □

Using Proposition 5.12 and Proposition 5.13, we obtain the following result.

Theorem 5.14. *Assume that* $\mathfrak{B}(V)$ *is a connected Nichols algebra of diagonal type with* $\dim V > 1$, *then the following conditions are equivalent*: (i) $\mathfrak{B}(V)$ *is finite-dimensional*; (ii) $\mathfrak{L}(V)_L$ *is finite-dimensional*; (iii) $\mathfrak{L}(V)_R$ *is finite-dimensional*.

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