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DIFFERENTIAL OPERATORS ON NONCOMMUTATIVE ALGEBRA: EQUIVALENCE AND SOME ALGEBRAIC OPERATIONS

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Abstract

In this work, we show the link between the definition given by Hazewinkel in [6] and that by given Lunts and Rosenberg in [8] on the notion of the differential operators algebra on a noncommutative algebra. We obtain some results for Cartesian and tensorial products of differential operator algebras on noncommutative algebras.

1. Introduction

The concept of differential operators on commutative algebras, initially defined by Grothendieck and Dieudonné in [1], has been extended to noncommutative algebras through various approaches. Indeed, in 1997, Lunts and Rosenberg introduced a new definition of the ring of differential operators on noncommutative algebra [8] to find a localization construction for quantum enveloping algebras described in [7]. This new approach allowed him to show that if the enveloping algebra $U(g)$ of a Lie algebra g acts on R as a Hopf algebra, then it acts by differential operators. In addition, a discussion took place on a recursive description of algebras of differential operators on diagrams (see [1, pp. 42-43] and in [2]). Accordingly, on commutative algebras in terms of commutators, Hazewinkel wanted to know whether something more or less similar can be done for noncommutative algebras, because this would be of particular interest for homotopy algebras, higher derivative algebras [3, 4], and for the theory of deformation of algebras and diagrams [5]. Then, focusing on the approach in [8], he proposed in 2011, a new definition of the ring of differential operators on a noncommutative algebra [6].

In this paper, we show that these two previous definitions, given separately by these authors, are equivalent. We are also interested in some algebraic operations of differential operator algebras on noncommutative algebras.

This paper is organized as follows: In Section 2, we give some definitions, notations and properties of algebras of differential operators. In Section 3, we first prove that with Hazewinkel's definition [6], the algebra of differential operators on a noncommutative algebra has some properties similar to those obtained in the commutative case. Secondly, we show that this Hazewinkel's definition and the definition given by Lunts and Rosenberg [8] are equivalent. In Section 4, we use Hazewinkel Michiel's definition to demonstrate that "the cartesian product of two algebras of differential operators on noncommutative algebras is an algebra of differential operators". From Lunts and Rosenberg definition, we determine exactly the algebra of differential operators on $M_n(k)$, the algebra of square matrices of order n, and prove that "the tensor product of algebras of differential operators on the algebras of square matrices is also an algebra of square matrices".

2. Preliminaries

Notation 1. Let A be a unitary algebra over a field k, M be an (A, A)-bimodule and $a \in A$. Then

(1) $L_A := \{l_a \in End_k(A), a \in A\}$, where $l_a(x) = ax$, for all $x \in A$.

(2) $R_A := \{r_a \in End_k(A), a \in A\}$, where $r_a(x) = xa$, for all $x \in A$.

(3) A^o denotes the opposite algebra of A.

(4) The elements of A^o will be denoted by a^o, where $a \in A$.

(5) $A^e := A \otimes_k A^o$.

(6) $M_n(k)$ denotes the algebra of square matrices of order n.

(7) $\mathfrak{Z}_A(M) := \{m \in M / ma = am, \forall a \in A\}$.

Definition 1. Let A be a commutative and unitary algebra over a field k. Then the *ring of differential operators* on A, denoted by $\mathcal{D}(A)$, is defined as follows:

$$\mathcal{D}(A) = \bigcup_{n \in \mathbb{N}} \mathcal{D}^n(A),$$

where $\mathcal{D}^{-1}(A) = 0$ and for $n \in \mathbb{N}$,

$$\mathcal{D}^n(A) = \{u \in End_k(A) : [u, a] = ua - au \in \mathcal{D}^{n-1}(A), \forall a \in A\},$$

with $ua(m) = u(am)$ and $au(m) = a(u(m))$, for all $m \in M$.

Any element $u \in \mathcal{D}^n(A)$ is called the *differential operator* of order n on A.

On a commutative algebra, we have the following properties:

Proposition 1 [9]**.** (1) *$End_k(A)$ is a $(\mathcal{D}(A), \mathcal{D}(A))$-bimodule.*

(2) *Let $n \geq 0$ be an integer, $\mathcal{D}^n(A)$ be an (A, A)-subbimodule of $End_k(A)$.*

(3) *$\mathcal{D}(A)$ is an (A, A)-subbimodule of $End_k(A)$.*

(4) *$\mathcal{D}(A)$ is a subalgebra of $End_k(A)$.*

Proposition 2 [9]**.** *Let $m, n \in \mathbb{N}$. Then*

(1) $\mathcal{D}^0(A) = End_A(A)$.

(2) $\mathcal{D}^n(A) \subseteq \mathcal{D}^{n+1}(A)$.

(3) $\mathcal{D}^m(A) \cdot \mathcal{D}^n(A) \subseteq \mathcal{D}^{m+n}(A)$.

(4) $[\mathcal{D}^m(A), \mathcal{D}^n(A)] \subseteq \mathcal{D}^{m+n-1}(A)$.

(5) *$\mathcal{D}(A)$ is a subalgebra of $End_k(A)$.*

In the following, the algebra A is noncommutative.

In this case, Hazewinkel defined the ring of differential operators as follows:

Definition 2 ([6] by Hazewinkel)**.** The *ring of differential operators* on A is defined by

$$\mathcal{D}(A) = \bigcup_{n \in \mathbb{N}} \mathcal{D}_n(A),$$

where $\mathcal{D}_{-1}(A) = 0$ and for $n \in \mathbb{N}$,

$$\mathcal{D}'_n(A) = \{u \in End_k(A) : [u, a] = ua - au \in \mathcal{D}_{n-1}(A), \forall a \in A\},$$

$$\mathcal{D}_n(A) = L_A \mathcal{D}'_n(A) L_A.$$

Any element $u \in \mathcal{D}_n(A)$ is called the *differential operator* of order n on A.

For any A-bimodule M, Lunts and Rosenberg have defined an A-subbimodule, the differential part of M denoted by M_{diff} in [8]. The A-bimodule M_{diff} has a filtration $M_0 \subset M_1 \subset \ldots$, where M_i is called the *ith differential part* of M. In particular, the elements of differential part of the A-bimodule $End_k(A)$ are called the *differential operators* on A. Denoted by $\mathcal{D}(A)$, this A-bimodule, which is an algebra, is called the *algebra* of differential operators on A.

Definition 3. $\mathcal{D}(A) = \bigcup_{n \in \mathbb{N}} \mathcal{D}iff^{\,n}(A)$, where $\mathcal{D}iff^{\,-1}(A) = 0$ and for $n \in \mathbb{N}$, $\mathcal{D}iff^{\,n}(A) = A^o \mathcal{C}_n(A)$, with

$$\mathcal{C}_n(A) = \left\{ u \in End_k(A) / \overline{u} \in \mathfrak{Z}_A\left(\frac{End_k(A)}{\mathcal{D}iff^{\,n-1}(A)} \right) \right\}.$$

Any element $u \in \mathcal{D}iff^{\,n}(A)$ is called the *differential operator* of order n on A.

Proposition 3 [10]**.** *Let M and N be two A-modules.*

Furthermore assume that M and N are two free A-modules. If $(m_i)_{i \in I}$ *and* $(n_j)_{j \in J}$ *are two bases of M and N, respectively, then* $(m_i \otimes n_j)_{i \in I;\, j \in J}$ *is a basis of* $M \otimes_k N$.

Proposition 4 (Corollary of the Azumaya-Nakayama theorem)**.** *If A is a simple k-algebra with center k and B a simple k-algebra, then* $A \otimes_k B$ *is a simple k-algebra.*

In the next section, *A* and *B* are noncommutative algebras over *k*.

3. Equivalence between Two Definitions of Differential Operators on Noncommutative Algebras

3.1. Algebra of differential operators on a noncommutative algebra according to Hazewinkel

With Hazewinkel's definition, we get the following properties, similar to Propositions 1 and 2.

Proposition 5. *Let* $n \in \mathbb{N}$. *Then*

(1) $End_k(A)$ *is an* (A, A)*-bimodule.*

(2) $End_k(A)$ *is a* $(\mathcal{D}(A), \mathcal{D}(A))$*-bimodule.*

(3) $\mathcal{D}_n(A)$ *is an* (A, A)*-subbimodule of* $End_k(A)$.

(4) $\mathcal{D}(A)$ *is an* (A, A)*-subbimodule of* $End_k(A)$.

Proof. (1) Since $End_k(A)$ is an additive group, it is an (A, A)-bimodule with the external laws

$$\varphi_1 : A \times End_k(A) \to End_k(A)$$

$$(a, u) \mapsto l_a \circ u$$

and

$$\varphi_2 : End_k(A) \times A \to End_k(A)$$

$$(u, a) \mapsto u \circ l_a.$$

(2) The following laws

$$\varphi_3 : \mathcal{D}(A) \times End_k(A) \to End_k(A)$$

$$(u, v) \mapsto u \circ v$$

and

$$\varphi_4 : End_k(A) \times \mathcal{D}(A) \to End_k(A)$$

$$(v, u) \mapsto v \circ u$$

justify that $\mathcal{D}(A)$ is an (A, A)-subbimodule of $End_k(A)$.

(3) Let $(n, a) \in \mathbb{N} \times A$ and $u \in \mathcal{D}_n(A)$. In fact that $au = l_a \circ u \in \mathcal{D}_n(A)$ and $ua = u \circ l_a \in \mathcal{D}_n(A)$, then $\mathcal{D}_n(A)$ is an (A, A)-subbimodule of $End_k(A)$.

(4) From (3), we obtain (4). □

Proposition 6. *Let* $m, n \in \mathbb{N}$. *Then*

(1) $id_A \in \mathcal{D}_0(A)$.

(2) $\mathcal{D}_n(A) \subseteq \mathcal{D}_{n+1}(A)$.

(3) $\mathcal{D}_m(A) \cdot \mathcal{D}_n(A) \subseteq \mathcal{D}_{m+n}(A)$.

(4) $\mathcal{D}(A)$ *is a subalgebra of* $End_k(A)$.

Proof. (1) $id_A \in \mathcal{D}_0(A)$ because $id_A = r_{1_A}$.

(2) By induction on $n \in \mathbb{N}$.

Let $u \in \mathcal{D}_n(A)$.

► For $n = 0$, $\mathcal{D}_0(A) = L_A R_A L_A$. Therefore, there exist $a, b, c \in A$ such that $u = l_a \circ r_c \circ l_b$. Then, for all $s \in A$, we have

$$[u, s] = l_a \circ [g, s] \circ l_b + l_{as-sa} \circ g \circ l_b + l_a \circ g \circ l_{bs-sb} \in \mathcal{D}_0(A).$$

It follows that $u \in \mathcal{D}'_1(A) \subseteq \mathcal{D}_1(A)$.

Hence, $\mathcal{D}_0(A) \subseteq \mathcal{D}_1(A)$.

► Now, assume that the result is true for an integer $n \geq 0$ and prove it for $n + 1$.

► Let $u \in \mathcal{D}_{n+1}(A) = L_A \mathcal{D}'_{n+1}(A) L_A$.

Thus, there exist $a, b, c \in A$ and $g \in \mathcal{D}'_{n+1}(A)$ such that $u = l_a \circ g \circ l_b$.

Then, for all $s \in A$, we get

$$[u, s] = l_a \circ [g, s] \circ l_b + l_{as-sa} \circ g \circ l_b + l_a \circ g \circ l_{bs-sb}.$$

Therefore, we have $[u, s] \in \mathcal{D}_{n+1}(A)$, for all $s \in A$. Thus, $u \in \mathcal{D}'_{n+2}(A) \subseteq \mathcal{D}_{n+2}(A)$.

► We conclude that

$$\mathcal{D}_n(A) \subseteq \mathcal{D}_{n+1}(A), \quad \forall n \in \mathbb{N}.$$

(3) See in [6, pp. 9-10].

(4) From (1)-(3), we obtain (4). □

3.2. Equivalence between two definitions of differential operators on noncommutative algebras

Here, we show that the definitions of the algebra of differential operators given by Hazewinkel and by Lunts and Rosenberg are equivalent.

Lemma 1. *$End_k(A)$ is a left A^e-module.*

Proof. The morphism

$$\varphi : A^e \times End_k(A) \to End_k(A)$$

$$\left(\sum_j a_j \otimes b_j^o, u\right) \mapsto \sum_j (l_{a_j} \circ u \circ l_{b_j})$$

shows that $End_k(A)$ has an A^e-module structure on the left.

Remark 1. Let $n \in \mathbb{N}$. Then any element $u \in \mathcal{D}iff^n(A)$ is written as follows:

$$u = \sum_{j=1}^{q} (l_{a_j} \circ u \circ l_{b_j}), \text{ where } \forall j \in \overline{1, q}, (a_j, b_j^o) \in A \times A^o \text{ and } u \in C_n.$$

Proposition 7. $\mathcal{D}iff^0(A) = R_A L_A = L_A R_A$.

Proof. Let $u \in \mathcal{D}iff^0(A) = A^e \mathcal{C}_0(A)$. Then

$$\mathcal{C}_0(A) = \left\{u \in End_k(A) / \overline{u} \in \mathfrak{Z}_A\left(\frac{End_k(A)}{\mathcal{D}iff^{-1}(A)}\right)\right\} = R_A.$$

Thus,

$$\mathcal{D}iff^0(A) = A^o R_A.$$

Hence, there exist $r_s \in R_A$ and $(a_j, b_j^o)_{1 \le j \le m} \in A \times A^o$ such that

$$u = \sum_{j=1}^{m} (l_{a_j} \circ r_s \circ l_{b_j}).$$

Since $L_A R_A L_A \subseteq \mathcal{D}iff^0(A)$, so

$$\mathcal{D}iff^0(A) = L_A R_A L_A = R_A L_A = L_A R_A. \qquad \square$$

Proposition 8. $\mathcal{D}_n(A) = \mathcal{D}iff^n(A)$, *for all* $n \in \mathbb{N}^*$.

Proof. Let $n \in \mathbb{N}^*$.

► Since

$$\mathcal{C}_n(A) = \left\{ u \in End_k(A) : \overline{u} \in \mathfrak{Z}_A\left(\frac{End_k(A)}{\mathcal{D}iff^{\,n-1}(A)}\right)\right\},$$

so

$$\mathcal{C}_n(A) = \{u \in End_k(A) : [u, l_a] \in \mathcal{D}iff^{\,n-1}(A), \forall a \in A\}.$$

Then,

$$\mathcal{D}'_n(A) = \mathcal{C}_n(A), \text{ for all } n \in \mathbb{N}^*. \tag{1}$$

► Let $u \in \mathcal{D}iff^{\,n}(A)$.

According to Remark 1, there exist $f \in \mathcal{C}_n(A)$ and $(a_j, b_j^o)_{1 \le j \le m} \in A \times A^o$ such that

$$u = \sum_{j=1}^{m} (l_{a_j} \circ f \circ l_{b_j}).$$

We deduce from (1) that $u \in L_A \mathcal{D}'_n(A) L_A = \mathcal{D}_n(A)$. It follows that

$$\mathcal{D}iff^{\,n}(A) \subset \mathcal{D}_n(A). \tag{2}$$

► Let $u \in \mathcal{D}_n(A) = L_A \mathcal{D}'_n(A) L_A$.

In this case, there exist $g \in \mathcal{D}'_n(A)$ and $a, b \in A$ such that $u = l_n \circ g \circ l_b$.

According to (2), we have $u \in A^e \mathcal{C}_n(A)$. Thus,

$$\mathcal{D}_n(A) \subset \mathcal{D}iff^{\,n}(A).$$

From (1) and (2), we get $\mathcal{D}_n(A) \subset \mathcal{D}iff^{\,n}(A)$, for all $n \in \mathbb{N}^*$. □

Corollary 1. *The definitions given by Hazewinkel and by Lunts and Rosenberg are equivalent.*

Proof. According to Propositions 7 and 8. □

4. Cartesian Product and Tensor Product of Algebras of Differential Operators

4.1. Cartesian product of algebras of differential operators

In this sequel, we use Hazewinkel's approach to prove that "the Cartesian product of two algebras of differential operators on noncommutative k-algebras is an algebra of differential operators".

Notation 2. (1) p_1 and p_2 denote the projections defined as follows:

$$p_1 : A \times B \to A \quad (a, b) \mapsto a;$$

$$p_2 : A \times B \to B \quad (a, b) \mapsto b.$$

(2) j_1 and j_2 denote the injections defined as follows:

$$j_1 : A \to A \times B \quad a \mapsto (a, 0_B);$$

$$j_2 : B \to A \times B \quad b \mapsto (0_A, b).$$

Proposition 9. *Let* $(a, b), (a', b') \in A \times B$ *and* $f \in \mathcal{D}(A \times B)$. *Then,*

(1) $f(A, \{0_B\}) \subseteq A \times \{0_B\}$ *and* $f(\{0_A\} \times B) \subseteq \{0_A\} \times B$.

(2) $f(a, b) = (a', b')$ *is equivalent to* $f(a, 0) = (a', 0)$ *and* $f(0, b) = (0, b')$.

Proof. By induction on $n \in \mathbb{N}$. (1) Let $f \in \mathcal{D}_n(A \times B)$ and $a \in A$.

► For $n = 0$, we have

$$f = \sum_{j=1}^{l} l_{(x_j, y_j)} r_{(s_j, t_j)},$$

where for all $j \in \overline{1, l}, (x_j, y_j), (s_j, t_j) \in A \times B$. Thus, $f(a, 0) \in A \times \{0_B\}$.

► Now, assume that the result is true for an integer $n \geq 0$ and prove it for $n+1$.

► Let

$$f = \sum_{j=1}^{l} l_{(x_j, y_j)} \circ g_j \circ l_{(s_j, t_j)} \in \mathcal{D}_{n+1}(A \times B)$$

$$= L_{A \times B} \mathcal{D}'_{n+1}(A \times B) \times L_{A \times B},$$

where for all $j \in \overline{1, l}$, $g_j \in \mathcal{D}'_n(A \times B)$ and $(x_j, y_j), (s_j, t_j) \in A \times B$. For all $a \in A$, we obtain

$$f(a, 0) = \sum_{j=1}^{l} (x_j, y_j)([g_j, (s_j, 0)](a, 0) + (s_j, 0).g_j(a, 0)).$$

Since for all $j \in \overline{1, l}$, $[g_j, (s_j, 0)](a, 0) \in A \times \{0_B\}$, $f(a, 0) \in A \times \{0_B\}$.

Hence, $f(A, \{0_B\}) \subseteq A \times \{0_B\}$.

Similarly, we have $f(\{0_A\} \times B) \subseteq \{0_A\} \times B$.

According to (1), we obtain (2) because for all $(a, b) \in A \times B$, $f(a, b) = f(a, 0) + f(0, b)$. □

Lemma 2. *Let* $(a, b) \in A \times B$ *and* $f \in \mathcal{D}(A \times B)$. *Then*

(1) $p_1 \circ l_{(a,b)} = l_a \circ p_1$ *and* $p_2 \circ l_{(a,b)} = l_b \circ p_2$,

(2) $l_{(a,b)} \circ j_1 = j_1 \circ l_a$ *and* $l_{(a,b)} \circ j_2 = j_2 \circ l_b$,

(3) $p_1 \circ r_{(a,b)} = r_a \circ p_1$ *and* $p_2 \circ r_{(a,b)} = r_b \circ p_2$,

(4) $r_{(a,b)} \circ j_1 = j_1 \circ r_a$ *and* $l_{(a,b)} \circ j_2 = j_2 \circ r_b$,

(5) $\begin{cases} [p_1 \circ f \circ j_1, a] = p_1 \circ [f, (a, 0)] \circ j_1, \\ [p_2 \circ f \circ j_2, b] = p_2 \circ [f, (0, b)] \circ j_2. \end{cases}$

Proof. Let $(a, b), (x, y) \in A \times B$.

(1) We have $p_1 \circ l_{(a,b)}(x, y) = ax$ and $l_a \circ p_1(x, y) = ax$. Thus, $p_1 \circ l_{(a,b)} = l_a \circ p_1$. Similarly, we obtain all other equalities. □

Proposition 10. *Let* $n \in \mathbb{N}$ *and* $f \in \mathcal{D}(A \times B)$.

If $f \in \mathcal{D}_n(A \times B)$, *then* $p_1 \circ f \circ j_1 \in \mathcal{D}_n(A)$ *and* $p_2 \circ f \circ j_2 \in \mathcal{D}_n(B)$.

Proof. By induction on $n \in \mathbb{N}$.

Let $f \in \mathcal{D}_n(A \times B)$. Then

► For $n = 0$, $\mathcal{D}_0(A \times B) = L_{A \times B} R_{A \times B}$. Thus,

$$f = \sum_{q=1}^{l} l_{(x_q, y_q)} \circ r_{(s_q, t_q)},$$

where $(x_q, y_q)_{1 \leq p \leq l}, (s_q, t_q)_{1 \leq p \leq l} \in A \times B$.

From Lemma 2, we get

$$p_1 \circ f \circ j_1 = \sum_{q=1}^{l} l_{x_q} \circ r_{s_q} \in \mathcal{D}_0(A).$$

► Now, assume that the result is true for an integer $n \geq 0$. Then we prove it for $n + 1$.

Case 1.

$$f \in \mathcal{D}'_{n+1}(A \times B)$$

$$= \{u \in End_k(A \times B) : [u, (a, b)] \in \mathcal{D}_n(A \times B), \forall (a, b) \in A \times B\}.$$

Let $a \in A$. Then

$$[p_1 \circ f \circ j_1, a] = p_1 \circ [f, (a, 0)] \circ j_1.$$

Since $[f, (a, 0)] \in \mathcal{D}_n(A)$, $[p_1 \circ f \circ j_1, a] \in \mathcal{D}_n(A)$. Hence,

$$p_1 \circ f \circ j_1 \in \mathcal{D}'_{n+1}(A) \subseteq \mathcal{D}_{n+1}(A). \tag{3}$$

Case 2. If $f \in \mathcal{D}_{n+1}(A \times B)$, then

$$f = \sum_{q=1}^{l} l_{(x_q, y_q)} \circ g_q \circ l_{(s_q, t_q)},$$

where

$$(x_q, y_q)_{1 \leq p \leq l}, (s_q, t_q)_{1 \leq p \leq l} \in A \times B.$$

According to Lemma 2, we get

$$p_1 \circ f \circ j_1 = \sum_{j=1}^{l} l_{x_q} \circ (p_1 \circ g_q \circ j_1) \circ l_{s_q}. \tag{4}$$

From (3), we get

$$p_1 \circ f \circ j_1 \in L_A \mathcal{D}'_{n+1}(A) L_A = \mathcal{D}_{n+1}(A).$$

► Similarly, we obtain $p_2 \circ f \circ j_2 \in \mathcal{D}_n(B)$. □

Theorem 1. $\mathcal{D}(A) \times \mathcal{D}(B) \simeq \mathcal{D}(A \times B)$.

Proof. Consider the morphism:

$$\theta : \mathcal{D}(A) \times \mathcal{D}(B) \to \mathcal{D}(A \times B)$$

$$(f_1, f_2) \mapsto f_{1,2} : A \times B \to A \times B$$

$$(a, b) \mapsto (f_1(a), f_2(b)).$$

► For all $(a, b) \in A \times B$ and $(f_1, f_2) \in \mathcal{D}(A) \times \mathcal{D}(B)$, we have

$$[\theta(f_1, f_2), (a, b)] = \theta([f_1, a], [f_2, b]). \tag{5}$$

Due to (5), we obtain by induction on $n = Max\{r, s\}$ that $\theta(\mathcal{D}_r(A) \times \mathcal{D}_s(B)) \subseteq \mathcal{D}_{Max\{r, s\}}(A \times B)$, for all $\{r, s\} \in \mathbb{N}^2$.

Thus, θ is well defined.

► In addition, θ is an injective algebra morphism and according to Proposition 10, it is also surjective.

Therefore, θ is an isomorphism of algebras. □

4.2. Tensor product of algebras of differential on $M_n(k)$

We prove by Lunts and Rosenberg's definition that "the tensor product of algebras of differential operators on the algebras of square matrices is also an algebra of square matrices".

Proposition 11. *For all* $n \in \mathbb{N}^*$, $\mathcal{D}(M_n(k)) \simeq M_{n^2}(k)$.

Proof. Consider the following morphism:

$$\varphi' : M_n(k) \times M_n(k)^o \to End_k(M_n(k))$$
$$(A, B^o) \mapsto \varphi'(A, B^o) : M_n(k) \to M_n(k)$$
$$C \mapsto ACB.$$

Since φ' is bilinear, it induces a morphism of k-modules:

$$\varphi : M_n(k) \otimes_k M_n(k)^o \to End_k(M_n(k))$$
$$A \otimes_k B^o \mapsto \varphi'(A, B^o)$$

which is an algebra morphism.

► According to Proposition 4, $M_n(k) \otimes_k M_n(k)^o$ is a simple k-algebra. Thus, φ is injective. Since

$$\dim_k[M_n(k) \otimes_k M_n(k)^o] = \dim_k[End_k(M_n(k))],$$

φ is an isomorphism of algebras, and hence

$$M_n(k) \otimes_k M_n(k)^o \simeq End_k(M_n(k)). \quad (6)$$

► According to Definition 3 of Lunts and Rosenberg, we have

$$M_n(k) \otimes_k M_n(k)^o \subseteq \mathcal{D}(M_n(k)) \subseteq End_k(M_n(k)). \tag{7}$$

From (6) and (7), we obtain $\mathcal{D}(M_n(k)) \simeq M_{n^2}(k)$. □

Corollary 2. *For all* $n, m \in \mathbb{N}^*$,

$$\mathcal{D}(M_n(k)) \otimes \mathcal{D}(M_m(k)) \simeq M_{n^2m^2}(k).$$

Proof. Let $n, m \in \mathbb{N}^*$.

From the previous proposition and Proposition 3, it follows that

$$\mathcal{D}(M_n(k)) \otimes \mathcal{D}(M_n(k)) \simeq M_{n^2m^2}(k). \quad \square$$

Corollary 3. *For all* $n, m \in \mathbb{N}^*$,

$$\mathcal{D}(M_n(k)) \otimes \mathcal{D}(M_m(k)) \simeq \mathcal{D}(M_n(k) \otimes M_m(k)).$$

Proof. It follows from Propositions 7, 10 and Corollary 2. □

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