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PERMUTATION GROUPS AND FRIEZE PATTERNS OF TYPE $\mathbb{A}$

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Abstract

In this article, we establish a link between permutation groups and frieze patterns of Conway and Coxeter. We define a geometrical mutation of these friezes. We also discuss cluster algebra of type $\mathbb{A}$ associated with a special permutation called canonical reading.

1. Introduction

Frieze patterns were first introduced by Coxeter in [11] and studied by Conway and Coxeter in [9, 10]. After the development of cluster algebra in 2001 by Fomin and Zelevinsky [16, 17], frieze patterns regained interest [1, 2, 3, 7]. Various relationships are known between friezes and cluster algebras, see [1, 2, 3, 5, 7, 12, 19, 20, 23, 24].

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Cluster algebras are a class of commutative algebras generated by a set of variables, called cluster variables, obtained recursively by a combinatorial process known as mutation starting from a set of initial cluster variables. One may use friezes to compute cluster variables [1, 4, 12, 19, 20].

Consider a triangulation T of a convex polygon, which is the partition of its interior into triangles by non-intersecting diagonals [8, 9, 10, 13, 14, 18]. Each diagonal d in the triangulation T is the diagonal of some quadrilateral. A new triangulation T' is obtained by replacing the diagonal d with the other diagonal of that quadrilateral. This well-known process is called a flip. The set of triangulations of a convex polygon is in natural bijection with a class of permutations in S_n, which are called canonical readings, and flip of diagonals corresponds to the mutation of permutations [21].

The present work is motivated first by the correspondence between friezes of Conway-Coxeter and triangulations on polygons [9, 10] and secondly by the correspondence between triangulations and permutations in [21, 25].

The aim of this paper is to establish the relation between permutation groups and frieze patterns of Conway-Coxeter via triangulations. We also define a geometrical mutation of frieze patterns of Conway-Coxeter using quiddities, which are entries in their second rows.

The article is organized as follows. In Section 2, we present the preliminaries in which we recall the notion of mutation in the permutation group S_n and discuss permutations and triangulations. Section 3 is devoted to the connection between permutations and frieze patterns, while in Section 4 we define a geometrical mutation of friezes of type $\mathbb{A}$, thus leading to the link between frieze patterns of type $\mathbb{A}_{n-1}$ and permutation group S_n in the sense of cluster algebras.

2. Preliminaries

2.1. Mutation in permutation group S_n

Let us denote by S_n the group of permutations of $\{1, 2, ..., n\}$, and by the elementary transpositions $(i, i+1)$ for $1 \leq i \leq n-1$. Let w be the set of words on $\{1, 2, ..., n\}$. A word in w is said to be standard if its letters are pairwise distinct. The set of standard words of length n in w will be identified with S_n. In this way, a permutation σ in S_n will be represented by the word such that $\sigma = a_1 a_2 \cdots a_n \in w$, where $a_i = \sigma(i)$ for all $1 \leq i \leq n$. Note that the left multiplication of the word σ by elementary transposition $\tau = (i, i+1)$ results in the word where ith and $(i+1)$th letters of σ are permuted. Indeed, if $\sigma = u a_i a_{i+1} v$, where $u = a_1 a_2 \cdots a_{i-1}$ and $v = a_{i+2} \cdots a_n$ (with convention that u or v is empty if $i = 1$ or $i = n-1$, respectively), then $\tau\sigma = u a_{i+1} a_i v$, see [25].

Example 2.1. Let σ be in S_6 such that

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 1 & 5 & 2 & 4 \end{pmatrix}.$$

The word associated with the permutation σ is $\sigma = 361524$.

Now we define the notion of passivity classes, as in [25], which will allow us to state how to mutate a permutation.

Definition 2.2. Let $\sigma = u a_i a_{i+1} v$ and $\gamma = u a_{i+1} a_i v$ be two permutations of S_n. The words σ and γ are in the same passivity class if and only if the factor v contains a letter p in w which is between the letters a_i and a_{i+1}, that is, $a_i < p < a_{i+1}$ or $a_{i+1} < p < a_i$.

Example 2.3. Let $\sigma_1 = 356214$ and $\sigma_2 = 352614$ be in S_6. Taking $u = 35$, $v = 14$ and $p = 4$, we can see that σ_1 and σ_2 are in the same passivity class.

The following definition describes how to mutate a word σ in the permutation group S_n.

Definition 2.4. Let $\sigma = uxyv$ and $\gamma = uyxv$ be two permutations of S_n. The permutation γ is called a *mutation* of σ, $\mu_{xy}(\sigma) = \gamma$, if and only if there is no letter *p* in *v* which is between the letters *x* and *y*.

Then to mutate a permutation is to exchange its two consecutive letters under the above definition condition.

Example 2.5. Let $\sigma = 1234$ in S_n. We have $\mu_{23}(\sigma) = 1324$ and $\mu_{13}(\mu_{23}(\sigma)) = \mu_{13}(1324) = 3124$.

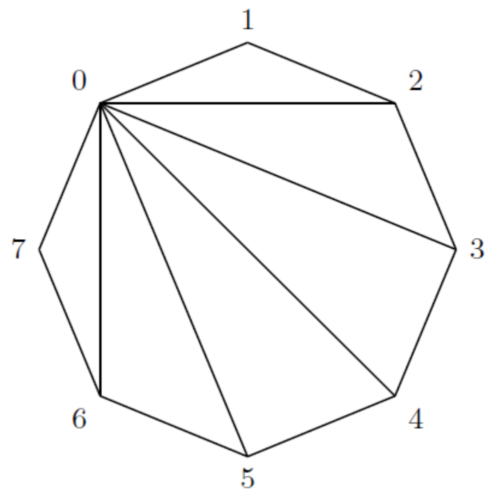
Remark 2.6. We will see in the further sections that, being in the same passivity class, some mutations are not possible.

2.2. Permutations and triangulations

In this part, we establish a link between a permutation σ in S_n and a triangulation of polygon of $(n+2)$ vertices.

Let *n* be an integer $n \geq 1$. Let P_{n+2} be a convex polygon with $(n+2)$ vertices labeled $0, 1, 2, \ldots$ to $(n+1)$ in clockwise order. The partition of the interior of P_{n+2} into triangles by non-crossing diagonals is called a *triangulation* of P_{n+2}. This partition uses $(n-1)$ diagonals. The set of triangulations of P_{n+2} will be denoted by T_n and its cardinality t_n which is Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$, see [18].

Example 2.7. Let $n = 6$ and P_8 be a convex octagon with vertices labelled 0, 1, 2, 3, 4, 5, 6, 7. We can have the following triangulation:



Taking a permutation we are going to bring out the diagonals composing its associated triangulation.

Let $\sigma = a_1 a_2 \cdots a_n$ be a permutation of S_n. We are going to enumerate all diagonals composing its associated triangulation. For this end, we go step by step from left to right in the reading of σ. Recall that a diagonal joins two vertices of the convex polygon P_{n+2}. Then we denote a diagonal joining the vertices i and j by $\{i, j\}$. Each a_k in σ is an element of the set $\{1, 2, ..., n\}$. Remember that the vertices of the convex polygon P_{n+2} are $0, 1, 2, ..., (n+1)$ in clockwise manner. Take a_k in σ and rewrite σ as follows: $\sigma = U a_k V$, where U and V are viewed as $U = \{a_1, a_2, ..., a_{k-1}\}$ and $V = \{a_{k+1}, a_{k+2}, ..., a_n\}$.

Considering a_k as an element of the ordered set $L = \{0, 1, ..., (n+1)\}$, we construct two subsets of L as follows:

W_1 is the subset of L such that for all $p \in W_1$, we have $p < a_k$ and W_2 is the subset of L such that for all $q \in W_2$, we have $q > a_k$.

For each a_k in σ we can construct the vertices of the corresponding diagonal as follows:

The label of the first vertex is equal to

$$\begin{cases} \max(W_1 \cap V) & \text{if } W_1 \cap V \text{ is not empty,} \\ 0 & \text{if } W_1 \cap V \text{ is empty.} \end{cases}$$

The label of the second vertex is equal to

$$\begin{cases} \min(W_2 \cap V) & \text{if } W_2 \cap V \text{ is not empty,} \\ (n+1) & \text{if } W_2 \cap V \text{ is empty.} \end{cases}$$

This procedure allows us to get two vertices ending the diagonal. Indeed, taking the permutation $\sigma = a_1 a_2 \cdots a_n$ and starting the procedure by a_1 and ending by a_{n-1}, we obtain all diagonals composing the triangulation T associated with the permutation σ. With all $(n-1)$ diagonals we construct uniquely the triangulation T.

Example 2.8. Let σ_1 and σ_2 be in S_6 such that $\sigma_1 = 321564$ and $\sigma_2 = 314265$.

Considering σ_1, we give all diagonals composing the triangulation T_1 associated with σ_1. Here we have the set $L = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and $a_1 = 3, a_2 = 2, a_3 = 1, a_4 = 5, a_5 = 6$.

For $a_1 = 3$, we have $U = \varnothing$, $V = \{1, 2, 4, 5, 6\}$, $W_1 = \{0, 1, 2\}$, $W_2 = \{4, 5, 6, 7\}$, $W_1 \cap V = \{0, 1, 2\} \cap \{1, 2, 4, 5, 6\} = \{1, 2\}$, then the first vertex of the diagonal is 2 because $\max(W_1 \cap V) = 2$.

$$W_2 \cap V = \{4, 5, 6, 7\} \cap \{1, 2, 4, 5, 6\} = \{4, 5, 6\},$$
$$\min(W_2 \cap V) = \min\{4, 5, 6\} = A,$$

then the second vertex of the diagonal is 4. Thus the diagonal obtained is $\{2, 4\}$.

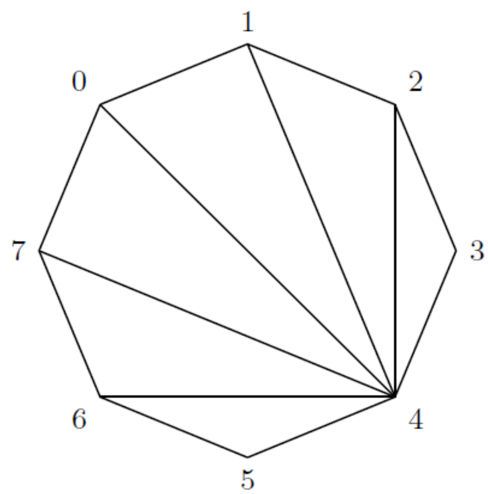
For $a_2 = 2$, we have $U = \{3\}$, $V = \{1, 4, 5, 6\}$, $W_1 = \{0, 1\}$, $W_2 = \{3, 4, 5, 6, 7\}$, $W_1 \cap V = \{0, 1\} \cap \{1, 4, 5, 6, 7\} = \{1\}$, then the first vertex of the diagonal is 1. $W_2 \cap V = \{3, 4, 5, 6, 7\} \cap \{1, 4, 5, 6\} = \{4, 5, 6\}$, $\min(W_2 \cap V) = \min\{4, 5, 6\} = 4$, then the second vertex of the diagonal is 4. Thus the diagonal obtained is $\{1, 4\}$,

For $a_3 = 1$, we get the diagonal $\{0, 4\}$.

For $a_4 = 5$, we get the diagonal $\{4, 6\}$.

For $a_5 = 6$, we get the diagonal $\{4, 7\}$.

Then the diagonals composing the triangulation T_1 are $\{2, 4\}$, $\{1, 4\}$, $\{0, 4\}$, $\{4, 6\}$ and $\{4, 7\}$.



Now we give the diagonals composing the triangulation T_2 associated with $\sigma_2 = 314265$. Here we have $L = \{0, 1, 2, 3, 4, 5, 6, 7\}$, $a_1 = 3$, $a_2 = 1$, $a_3 = 4$, $a_4 = 2$, $a_5 = 6$. Starting with $a_1 = 3$, we have $V = \{1, 2, 4, 5, 6\}$, $W_1 = \{0, 1, 2\}$ and $W_2 = \{4, 5, 6, 7\}$.

$W_1 \cap V = \{1, 2\} \neq \emptyset$, $\max(W_1 \cap V) = 2$, then the first vertex of the diagonal is 2.

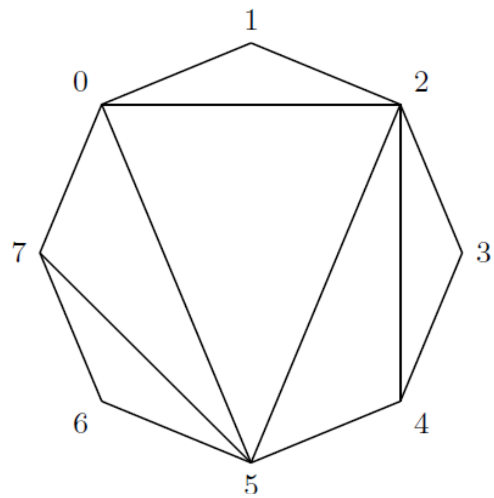
$W_2 \cap V = \{4, 5, 6\}$, $\min(W_2 \cap V) = 4$, then the second vertex of the diagonal is 4. So the diagonal obtained is $\{2, 4\}$, For $a_2 = 1$, we get the diagonal $\{0, 2\}$.

For $a_3 = 4$, we get the diagonal $\{2, 5\}$.

For $a_4 = 2$, we get the diagonal $\{0, 5\}$.

For $a_5 = 6$, we get the diagonal $\{5, 7\}$.

Then the diagonals composing the triangulation T_2 are $\{2, 4\}$, $\{0, 2\}$, $\{2, 5\}$, $\{0, 5\}$, $\{5, 7\}$.



Now we know how to construct entirely the triangulation T associated with a permutation σ. The question that for a triangulation T, the permutation is uniquely associated is answered in Lemma 13 and Proposition 16 of [25] where it was shown that the map $t : S_n \to T_n$ is surjective and the fiber $t^{-1}(T)$ represents a passivity class.

Let us show how to associate a triangulation T with a permutation σ.

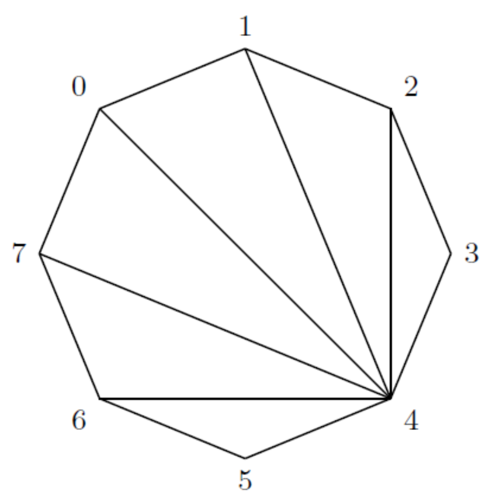
Consider a triangulation T of a convex polygon P_{n+2}. The degree of a vertex i in P_{n+2} is the number of edges of T which are incident to i. A vertex of degree exactly 2 is called an *ear* in T.

According to Theorem 1 of [22] any triangulation of convex polygon has at least two ears. Indeed, we have at least two choices of an ear.

Assume that the vertex $a_1 \in \{1, 2, ..., n\}$ (without loss of generality) is an ear. Then the word of the permutation σ begins with a_1. The next step is to cut in the triangulation T the triangle having a_1 as one of their vertices. We get a triangulation T' of polygon P_{n+1}. Applying the same procedure to T' we construct the second letter a_2 of the word of the permutation σ. Going on, we obtain the word $\sigma = a_1 a_2 \cdots a_n$ which corresponds to the permutation σ. Through this procedure we can see that at each step we have multiple choices of the letter a_k in the word of the permutation σ. It is clear that for a fixed triangulation one can obtain at least one associated permutation.

The words in S_n obtained with the cutting procedure will be the readings of the triangulation *T*. The following example gives the readings of such triangulation *T* in T_n.

Example 2.9. Consider the following triangulation *T*,



The corresponding words are

$$\sigma = 321564, \qquad \sigma_5 = 532164,$$

$\sigma_1 = 356214$, $\sigma_6 = 536214$,

$\sigma_2 = 352614$, $\sigma_7 = 532614$,

$\sigma_3 = 325614$, $\sigma_8 = 563214$.

$\sigma_4 = 325164$

Definition 2.10. Let $T \in T_n$ and $t : S_n \to T_n$ be a surjective map. The canonical reading of T is the *lexicographically smallest word* in the fiber $t^{-1}(T)$.

Example 2.11. In Example 2.9, the canonical reading of T is $\sigma = 321564$.

Remark 2.12. The set of readings of T is exactly the fiber $t^{-1}(T)$ (Lemma 15 of [25]). This shows that the canonical reading is unique in $t^{-1}(T)$. It is clear that each canonical reading represents a passivity class.

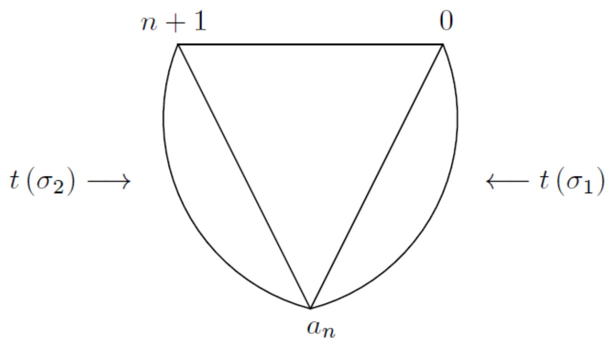
Let us give the definition of a separation of permutation as in [25] which will allow us to determine the number of canonical readings in S_n.

Definition 2.13. Let $\sigma = a_1 a_2 \cdots a_n \in S_n$. We say that σ is a separation if it may be factored as $\sigma = \sigma_1 \sigma_2 a_n$, where the subwords σ_1, σ_2 have the property that $a_i < a_n$ for every letter a_i in σ_1 and $a_j > a_n$ for every letter a_j in σ_2.

Example 2.14. Let $\sigma = 321564 \in S_6$. Then σ is a separation with $\sigma_1 = 321$, $\sigma_2 = 56$ and $a_n = a_6 = 4$.

Remark 2.15. The separation of permutation is unique if it exists.

Let $\sigma = \sigma_1 \sigma_2 a_n$ be a separation. Then the triangulation $T = t(\sigma)$ associated with σ is of the form:



Due to the fact that $a_n \in \{1, 2, ..., n\}$, we can rewrite a separation as follows: $\sigma = \sigma_1\sigma_2 i$ with $i \in \{1, 2, ..., n\}$.

The following theorem as in [21] (Theorem 2.11) gives the number of canonical readings in S_n.

Theorem 2.16. *Let* $n \geq 1$ *be an integer,* $i \in \{1, ..., n\}$ *and* S_n *be the permutation group. Let* $\sigma = \sigma_1\sigma_2 i$ *be a canonical reading. Then*:

(1) *The number of canonical readings in* S_n *ended by i is* Δ_i *such that*

$$\Delta_i = \frac{1}{i} \cdot \frac{1}{n-i+1}\binom{2(i-1)}{i-1}\binom{2(n-i)}{n-i}.$$

(2) $\sum_{i=1}^{n} \Delta_i = \frac{1}{n+1}\binom{2n}{n}$.

3. Permutations and Friezes

In this section we are going to establish a link between permutations and friezes of Conway and Coxeter. Before this, we need to state the notion of frieze patterns.

3.1. Friezes patterns

Frieze patterns were introduced by Coxeter [11] and studied by Conway and Coxeter [9, 10]. A frieze pattern consists of a finite number of rows such that the first and last rows consist of ones and for every four entries of the form $\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$ the relation $ad - bc = 1$ must be satisfied. This relation is called *unimodular rule*.

Example 3.1. Here we give an example of frieze pattern of integers.

$$\begin{array}{cccccccccccccccccc}
\cdots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots \\
\cdots & & 1 & & 4 & & 1 & & 2 & & 2 & & 2 & & 1 & & 4 & \cdots \\
\cdots & 1 & & 3 & & 3 & & 1 & & 3 & & 3 & & 1 & & 3 & & \cdots \\
\cdots & & 2 & & 2 & & 2 & & 1 & & 4 & & 1 & & 2 & & 2 & \cdots \\
\cdots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots
\end{array}$$

The order of a frieze pattern is defined to be one up than its number of rows. The frieze in the above example is of order 6. In [9, 10] Conway and Coxeter showed that every frieze is determined by its second row. They also proved that all frieze patterns are periodic, that is, there is an integer $h > 0$ such that $B_k = B_{k+h}$, where B_k is an entry of such row. Thus, if $\mathcal{F}$ is a frieze pattern of order h, then $\mathcal{F}$ is h-periodic ([9, 10], point (21)).

Example 3.2. In Example 3.1, the frieze is 6-periodic.

3.2. Friezes and triangulations

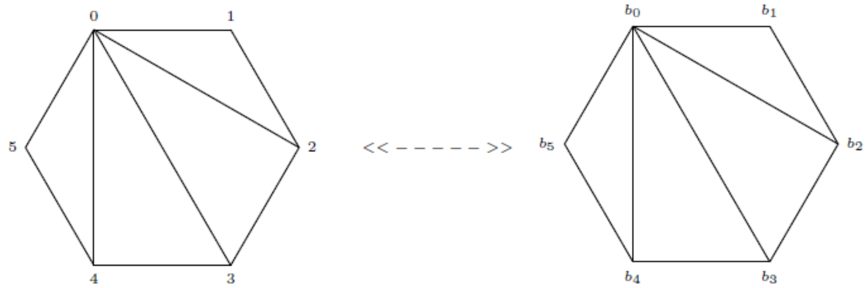
Conway and Coxeter gave in [9, 10] a geometric interpretation of frieze patterns via triangulations of polygons. They associated a frieze pattern of order N with each triangulation of a regular polygon with N sides. They also showed that every frieze pattern arises in this way.

Consider a convex polygon P_{n+2} with $(n+2)$ vertices, labelled clockwise by $\{0, 1, ..., (n+1)\}$. Taking a triangulation of P_{n+2}, let b_k be the number matchings of triangles incident with vertex k. Then we have the quiddity sequence $(b_0, b_1, ..., b_{n+1})$ of a frieze pattern of order $(n+2)$. Now we can construct the first two rows of the frieze pattern $\mathcal{F}$ as follows:

$$\begin{array}{ccccccccccccccccccccc}
\cdots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \cdots & 1 & & 1 & & 1 & & 1 & \cdots \\
\cdots & & b_0 & & b_1 & & b_2 & & b_3 & & b_4 & & \cdots & & b_{n+1} & & b_0 & & b_1 & & \cdots
\end{array}$$

Next we construct the other rows of the frieze by using the unimodular rule. Doing this, we obtain the entire frieze pattern $\mathcal{F}$ of order $(n+2)$.

Example 3.3. Consider a triangulation of a polygon P_6,



We have the quiddity sequence

$$(b_0, b_1, b_2, b_3, b_4, b_5) = (4, 1, 2, 2, 2, 1).$$

Due to the fact that P_6 is of 6 sides, the associated frieze with the triangulation is 6 periodic.

The first two rows of $\mathcal{F}$ are

$$\begin{array}{ccccccccccccccccccccccc}
\cdots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \cdots \\
\cdots & & 4 & & 1 & & 2 & & 2 & & 2 & & 1 & & 4 & & 1 & & 2 & & \cdots
\end{array}$$

Using the unimodular rule we construct the entire frieze $\mathcal{F}$,

$$\begin{array}{ccccccccccccccccccccccc}
\cdots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \cdots \\
\cdots & & 4 & & 1 & & 2 & & 2 & & 2 & & 1 & & 4 & & 1 & & 2 & & \cdots \\
\cdots & 3 & & 3 & & 1 & & 3 & & 3 & & 1 & & 3 & & 3 & & 3 & & 1 & \cdots \\
\cdots & & 2 & & 2 & & 1 & & 4 & & 1 & & 2 & & 2 & & 2 & & 1 & & \cdots \\
\cdots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \cdots
\end{array}$$

Now consider a frieze $\mathcal{F}$ of order $(n+2)$.

Let $\mathcal{F}$ be bordered by two rows of P^k. Thus, we have

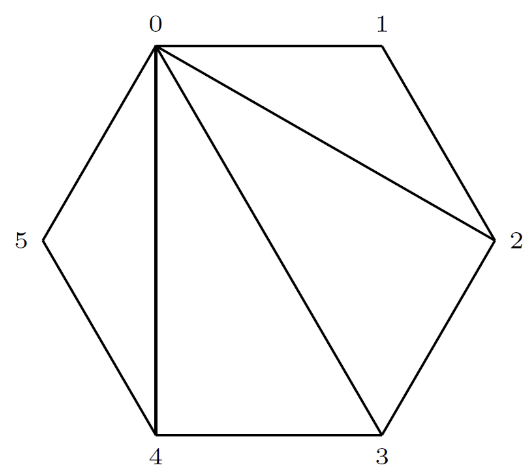
$$\mathcal{F}: \begin{array}{cccccccccccccccc}
\cdots & & P^0 & & P^1 & & P^2 & & \cdots & & P^{n+1} & & P^0 & & P^1 & \cdots \\
\cdots & 1 & & 1 & & 1 & & & \cdots & & & 1 & & 1 & & \cdots \\
\cdots & & b_0 & & b_1 & & b_2 & & \cdots & & b_{n+1} & & b_0 & & b_1 & \cdots \\
\cdots & \ddots & & \ddots & & \ddots & & \ddots & & \ddots & & \ddots & & \ddots & & \cdots \\
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \cdots \\
\cdots & P^2 & \cdots & p^n & & P^{n+1} & & P^0 & & P^1 & & P^2 & & \cdots & P^{n+1} &
\end{array}$$

and let $\{r, s\}$ denote the entry where a South-East (SE) diagonal $P^r P^r$ meets a North-East (NE) diagonal $P^s P^s$ or vice versa. For $\{r, s\}$, $s \notin \{r-1, r+1\}$ corresponding to the value 1 we have in the triangulation T associated with the $(n+2)$-gon P_{n+2} an arc joining the vertices labelled r and s. Then we construct easily all diagonals composing the triangulation T of P_{n+2}.

Example 3.4. Consider the frieze $\mathcal{F}$ of order 6 of Example 3.3. We border $\mathcal{F}$ with two rows of P^k $k \in \{0, 1, 2, 3, 4, 5\}$ as follows:

$\cdots$		P^0		P^1		P^2		P^3		P^4		P^5		P^0		P^1		P^2		$\cdots$
$\cdots$	1		1		1		1		1		1		1		1		1		1	$\cdots$
$\cdots$		4		1		2		2		2		1		4		1		2		$\cdots$
$\cdots$	3		3		1		3		3		1		3		3		3		1	$\cdots$
$\cdots$		2		2		1		4		1		2		2		2		1		$\cdots$
$\cdots$	1		1		1		1		1		1		1		1		1		1	$\cdots$
$\cdots$		P^3		P^4		P^5		P^0		P^1		P^2		P^3		P^4		P^5		$\cdots$

We can see easily that the entries $\{r, s\}$ corresponding to 1 are $\{0, 3\}, \{0, 4\}, \{0, 2\}$. Then we get the diagonals composing the triangulation T of the polygon P_6.



By doing this, Conway and Coxeter give in [9, 10] the following theorem.

Theorem 3.5. *There is a bijection between frieze patterns of order* $(n + 2)$ *and triangulations of convex polygons with* $(n + 2)$ *sides.*

3.3. Permutations and friezes

Let $\sigma \in S_n$, $n \geq 1$, be a canonical reading, $\sigma = a_1 a_2 \cdots a_n$. It was proved in Section 2 that σ corresponds to a triangulation T of a convex polygon P_{n+2}. According to Theorem 2.16 there is a bijection between the set of canonical readings of S_n and the set of triangulations of a convex polygon P_{n+2} with $(n + 2)$ sides. Considering the bijection of Theorem 3.5 we state the following theorem.

Theorem 3.6. *Let* $n \geq 1$ *be an integer and* S_n *be the permutation group. There is a bijection between the canonical readings of* S_n *and frieze patterns of integers of order* $(n + 2)$.

Proof. Let $\sigma = a_1 a_2 \cdots a_n \in S_n$ be a canonical reading. Let $\{i, j\}$, $0 \leq i, j \leq (n+1)$ be diagonal obtained by the permutation σ. These diagonals correspond to those of a triangulation T of a $(n + 2)$-gon. It is well known that the second row of a frieze pattern of integers $\mathcal{F}$ determines entirely the frieze. The quiddity sequence $(b_0, b_1, ..., b_{n+1})$ which is $(n + 2)$-periodic composes the second row of the frieze pattern $\mathcal{F}$. It suffices to determine the quiddity sequence to obtain the corresponding frieze pattern of order $(n + 2)$. For this end, let us consider all diagonals occurred by the canonical reading σ. The value $(b_k - 1)$ is the number of appearances of the vertex k in the diagonals $\{i, j\}$, $0 \leq i, j \leq (n+1)$. This is because, in a triangulation, the number of incident diagonals to a vertex is one less than that of its incident triangles. By this procedure we construct the quiddity sequence $(b_0, b_1, ..., b_{n+1})$. By periodicity we obtain the second row of the frieze pattern of integers. Using the unimodular rule we construct entirely and uniquely the corresponding frieze pattern.

Conversely let $\mathcal{F}$ be a frieze pattern of integers of order $(n+2)$ and T be its associated triangulation.

Consider the first two rows of the frieze pattern, that is,

$$\begin{array}{cccccccccccccccccccc} \cdots & 1 & & 1 & & 1 & & 1 & & 1 & \cdots & 1 & & 1 & & 1 & & 1 & \cdots \\ \cdots & & b_0 & & b_1 & & b_2 & & b_3 & & \cdots & & b_{n+1} & & b_0 & & b_1 & & \cdots \end{array}$$

Consider the quiddity sequence $(b_0, b_1, ..., b_{n+1})$ which is a period in the second line below the 1's. According to the fact that each triangulation of polygon has at least two ears [22], we have at least two b_k corresponding to one. Knowing that canonical reading is the lexicographically smallest word in the fiber $t^{-1}(T)$ (Definition 2.10), we choose the smallest k such that $b_k = 1,\ 1 \leq k \leq n.$ Without loss of generality, we assume that the vertex j corresponds to $b_j = 1$ as desired. We have the following quiddity sequence $(b_0, b_1, ..., b_{j-1}, 1, b_{j+1}, ..., b_{n+1})$. In this way we have constructed the first value $a_1 = j$ of the canonical reading σ. Next, we delete $b_j = 1$ in the above quiddity sequence. This deletion consists of removing, in the triangulation T, the triangle such that the vertex j is an ear. Consequently the two neighbours vertices of the vertex j have its number of incident triangles decrease by one. Then we get the new quiddity sequence $(b_0, b_1, ..., (b_{j-1} - 1), (b_{j+1} - 1), b_{j+2}, ..., b_{n+1})$ which corresponds to triangulation T' of $(n+1)$-gon. Now we consider the new quiddity sequence $(b_0, b_1, ..., (b_{j-1} - 1), (b_{j+1} - 1), b_{j+2}, ..., b_{n+1})$ and repeat the previous procedure. We construct the second value a_2 of the canonical reading σ. Going on, by induction we construct uniquely $a_3, a_4, ..., a_n$. Thus the corresponding canonical reading is $\sigma = a_1 a_2 \cdots a_n$. □

Example 3.7. (1) Consider in S_4 the canonical reading $\sigma = 1234$. The occurred diagonals associated with σ are $\{0, 2\}, \{0, 3\}, \{0, 4\}$. Recall that $b_k - 1$ is the number of appearances of the vertex k in the occurred diagonals.

The number of appearances of zero is thrice, then $b_0 - 1 = 3$.

The number of appearances of one is zero, then $b_1 - 1 = 0$.

Thus $b_2 - 1 = 1, b_3 - 1 = 1, b_4 - 1 = 1$ and $b_5 - 1 = 0$ whence the quiddity sequence is $(b_0, b_1, b_2, b_3, b_4, b_5) = (4, 1, 2, 2, 2, 1)$. The first two rows of the frieze pattern of order 6 are

$$\begin{array}{ccccccccccccccccccccc}
\cdots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \cdots \\
\cdots & & 4 & & 1 & & 2 & & 2 & & 2 & & 1 & & 4 & & 1 & & 2 & & \cdots
\end{array}$$

By unimodular rule we construct entirely the frieze $\mathcal{F}$ of order 6.

$$\mathcal{F}: \quad \begin{array}{ccccccccccccccccccccc}
\cdots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \cdots \\
\cdots & & 4 & & 1 & & 2 & & 2 & & 2 & & 1 & & 4 & & 1 & & 2 & & \cdots \\
\cdots & 3 & & 3 & & 1 & & 3 & & 3 & & 1 & & 3 & & 3 & & 3 & & 1 & \cdots \\
\cdots & & 2 & & 2 & & 1 & & 4 & & 1 & & 2 & & 2 & & 2 & & 1 & & \cdots \\
\cdots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \cdots
\end{array}$$

(2) Consider the following frieze pattern of integers $\mathcal{F}$ of order 6.

$$\begin{array}{ccccccccccccccccccccc}
\cdots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \cdots \\
\cdots & & 4 & & 1 & & 2 & & 2 & & 2 & & 1 & & 4 & & 1 & & 2 & & \cdots \\
\cdots & 3 & & 3 & & 1 & & 3 & & 3 & & 1 & & 3 & & 3 & & 3 & & 1 & \cdots \\
\cdots & & 2 & & 2 & & 1 & & 4 & & 1 & & 2 & & 2 & & 2 & & 1 & & \cdots \\
\cdots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \cdots
\end{array}$$

A subtle quiddity sequence $(b_0, b_1, b_2, b_3, b_4, b_5)$ gives $(b_0, b_1, b_2, b_3, b_4, b_5) = (4, 1, 2, 2, 2, 1)$, $a_1 = 1$, then deleting b_1 we obtain the new following quiddity sequence

$$(b_0 - 1, b_2 - 1, b_3, b_4, b_5) = (3, 1, 2, 2, 1),$$

$$(b'_0, b'_2, b_3, b_4, b_5) = (3, 1, 2, 2, 1).$$

Then we have $a_2 = 2$ because the smallest k such that $b_k = 1$, $1 \le k \le n$, is $k = 2$. Next deleting $b'_2 = 1$ we have the following quiddity $(b'_0 - 1, b_3 - 1, b_4, b_5) = (b''_0, b'_3, b_4, b_4) = (2, 1, 2, 1)$, then $a_3 = 3$. Deleting $b'_3 = 1$ we get $(b''_0, b_4 - 1, b_5) = (b_0^{(4)}, b'_4, b_5) = (1, 1, 1)$, then $a_4 = 4$. Thus $\sigma = a_1 a_2 a_3 a_4 = 1234$.

We know that a canonical reading σ represents the fiber $t^{-1}(T)$, where T is its associated triangulation. Then, according to the point (2) of Theorem 2.16, the mutation of permutations corresponds to the flip of diagonals as we have in Theorem 3.8 of [21].

Combining Theorem 3.8 of [21] and Theorem 3.6, we can see the possibility to talk about geometrical mutation of frieze pattern of Conway-Coxeter which is said to be the frieze of type $\mathbb{A}$. Notice that frieze mutation in type $\mathbb{A}$ has been studied by other authors, in particular, ([6], Theorem 6.12) precisely describes how all entries in the frieze change under mutation.

4. Geometrical Mutation of Friezes of Type $\mathbb{A}$

In this section we talk about geometrical mutation of frieze of type $\mathbb{A}$ and then give the connection with cluster algebras of type $\mathbb{A}$.

4.1. Frieze mutations

Let n be an integer $n \geq 1$. Let P_{n+2} be a convex polygon with $(n+2)$ vertices labelled $P^0, P^1, \ldots, P^{n+1}$ in clockwise order, T be a triangulation of P_{n+2} and $\mathcal{F}$ be its associated frieze. We know that $\{r, s\}$ denotes, in the bordered frieze $\mathcal{F}$, the entry where a South-East (SE) diagonal $P^r P^r$ meets a North-East (NE) diagonal $P^s P^s$ or vice versa. In the bordered frieze, the entry $\{r, s\}, s \notin \{r-1, r+1\}$ corresponding to the value 1 represents geometrically in T the diagonal joining the vertices labelled P^r and P^s. It is well known that flips are performed on the diagonals, then the entry $\{r, s\}, s \in \{r-1, r+1\}$ corresponding to the value 1 represents diagonal in the triangulation T. Thus mutation will be made on the value 1 in the bordered frieze $\mathcal{F}$.

The way to mutate a frieze pattern $\mathcal{F}$ of type $\mathbb{A}$ is described as follows:

Consider the frieze of type $\mathbb{A}$, $\mathcal{F}$ bordered by two rows of P^j, $j \in \{0, 1, ..., (n+1)\}$. Indeed, for a square of the form $\begin{matrix} & P^j & \\ 1 & & 1 \\ & b^j & \end{matrix}$, where b_j is an entry on the second row in $\mathcal{F}$ corresponding to the number of incident triangles to the vertex P^j of P_{n+2}.

Consider an entry $\{r, s\}$ corresponding to the value 1 in $\mathcal{F}$. Recall that E_i is the set of adjacent vertices to the vertex P^i in P_{n+2}.

We define the sets E_r and E_s as follows:

$$E_r = \{P^j, 0 \le j \le (n+1)/\{r, j\} = 1\},$$

$$E_s = \{P^j, 0 \le j \le (n+1)/\{s, j\} = 1\}.$$

Knowing that if $\{r, s\} = 1$, $s \notin \{r-1, r+1\}$, then $P^r P^s$ is a diagonal in the triangulation T. The diagonal $P^r P^s$ is inscribed in a quadrilateral such that it is one of its diagonals and the vertices P^r and P^s are two vertices of that quadrilateral. The other two vertices of the quadrilateral are given by the overlap of E_r and E_s, that is, $E_r \cap E_s$. This is because the triangulation T is composed of non-intersecting diagonals. Then we have $E_r \cap E_s = \{P^u, P^v\}$, $u, v \in \{0, 1, ..., (n+1)\} \setminus \{r, s\}$.

Now we are going to mutate the frieze $\mathcal{F}$ at the entry $\{r, s\} = 1$.

Let us denote by $\mu_{rs}(\mathcal{F})$ the frieze obtained by mutation of the frieze $\mathcal{F}$ at the entry $\{r, s\} = 1$. We construct $\mu_{rs}(\mathcal{F})$ by starting with the first three rows of the bordered frieze $\mathcal{F}$. We have the following:

$$\begin{matrix} \cdots & P^0 & & P^1 & & P^2 & \cdots\cdots & P^n & & P^{n+1} & & \cdots \\ \cdots 1 & & 1 & & 1 & & 1 \cdots\cdots 1 & & 1 & & 1 \cdots \\ \cdots & b_0 & & b_1 & & b_2 & \cdots\cdots & b_n & & b_{n+1} & & \cdots \end{matrix}$$

Next we construct the first three rows of $\mu_{rs}(\mathcal{F})$ bordered with P^j by replacing the numbers b_j, $j \in \{r, s, u, v\}$ in above three rows with the new numbers associated with the vertices P^r, P^s, P^u and P^v in the new triangulation $\mu_{rs}(T)$, that is, $(b_r - 1), (b_s - 1), (b_u + 1)$ and $(b_v + 1)$ respectively. It is well known that b_j, $j \in \{0, 1, ..., (n + 1)\}$ corresponds to the number of incident triangles to the vertex P^j and the quadrilateral in which $P^r P^s$ is a diagonal, the second diagonal is given by the vertices of $E_r \cap E_s$. Then knowing that a flip on the diagonal $P^r P^s$ consists to replace $P^r P^s$ by the second diagonal, it is clear that the numbers b_r, b_s of the incident triangles to the vertices P^r, P^s respectively, decrease each by one. Due to the construction of the second diagonal, the numbers b_u, b_v of the incident triangles of P^u, P^v respectively, increase each by one.

Finally we construct entirely and uniquely the frieze $\mu_{rs}(\mathcal{F})$ by using the unimodular rule.

This geometrical mutation of the frieze of type $\mathbb{A}$ allows us to give the following theorem:

Theorem 4.1. *Let* $n \geq 1$ *be an integer*, P_{n+2} *be a convex polygon with* $(n + 2)$ *sides and T be its associated triangulation*. *Let* $\mathcal{F}_T$ *be the frieze of type* $\mathbb{A}$ *associated with T. Then for all* $\{r, s\} = 1, s \notin \{r - 1, r + 1\}$, $r, s \in \{0, 1, ..., (n + 1)\}$ *in the frieze* $\mathcal{F}$, *we have* $\mu_{rs}(\mathcal{F}_T) = \mathcal{F}_{\mu_{rs}(T)}$.

Proof. Let $\mathcal{F}_T$ be the frieze associated with a triangulation T of the convex polygon P_{n+2} and denoted by $\mu_{rs}(T)$ the triangulation of P_{n+2} obtained from T by flipping on diagonal $P^r P^s$. Then we replace the diagonal $P^r P^s$ in the quadrilateral $P^r P^u P^s P^v$ by the diagonal $P^u P^v$. Doing this, we decrease the numbers of the incident triangles at vertices P^r

and P^s each by one. The frieze $\mathcal{F}_{\mu_{rs}(T)}$ associated with the triangulation $\mu_{rs}(T)$ has on its second row the same values as those on the frieze $\mathcal{F}_T$ associated with T except the values b_r, b_s, b_u and b_v which are replaced by $(b_r - 1)$, $(b_s - 1)$, $(b_u + 1)$ and $(b_v + 1)$ respectively. We can see that the second rows of the friezes $\mu_{rs}(\mathcal{F}_T)$ and $\mathcal{F}_{\mu_{rs}(T)}$ coincide. According to ([6, 7], point 28) these friezes are the same. Thus $\mu_{rs}(\mathcal{F}_T) = \mathcal{F}_{\mu_{rs}(T)}$. □

Now Theorem 4.1 allows us to give the following corollaries which are the analogous of Theorem 3.8 in [21].

Corollary 4.2. *The mutation of frieze of type* $\mathbb{A}$ *corresponds to the flip diagonals of triangulations*.

Corollary 4.3. *The mutation of frieze of type* $\mathbb{A}$ *corresponds to the mutation of permutations*.

Example 4.4. Consider the following frieze $\mathcal{F}$ of order 8.

$$\mathcal{F}: \quad \begin{array}{cccccccccccccccccccccc}
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \cdots \\
\cdots & 3 & & 1 & & 4 & & 1 & & 3 & & 1 & & 4 & & 1 & & 3 & & 1 & & \cdots \\
\cdots & & 2 & & 3 & & 3 & & 2 & & 2 & & 3 & & 3 & & 2 & & 2 & & 3 & \cdots \\
\cdots & 1 & & 5 & & 2 & & 5 & & 1 & & 5 & & 2 & & 5 & & 1 & & 5 & & \cdots \\
\cdots & & 2 & & 3 & & 3 & & 2 & & 2 & & 3 & & 3 & & 2 & & 2 & & 3 & \cdots \\
\cdots & 3 & & 1 & & 4 & & 1 & & 3 & & 1 & & 4 & & 1 & & 3 & & 1 & & \cdots \\
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \cdots
\end{array}$$

Let us border $\mathcal{F}$ by two rows of vertices P^j, $j \in \{0, 1, 2, ..., 7\}$ of polygon P_8 in order. We obtain

$$\begin{array}{cccccccccccccccccccccc}
\cdots & P^7 & & P^0 & & P^1 & & P^2 & & P^3 & & P^4 & & P^5 & & P^6 & & P^7 & & P^0 & & \cdots \\
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \cdots \\
\cdots & 3 & & 1 & & 4 & & 1 & & 3 & & 1 & & 4 & & 1 & & 3 & & 1 & & \cdots \\
\cdots & & 2 & & 3 & & 3 & & 2 & & 2 & & 3 & & 3 & & 2 & & 2 & & 3 & \cdots \\
\cdots & 1 & & 5 & & 2 & & 5 & & 1 & & 5 & & 2 & & 5 & & 1 & & 5 & & \cdots \\
\cdots & & 2 & & 3 & & 3 & & 2 & & 2 & & 3 & & 3 & & 2 & & 2 & & 3 & \cdots \\
\cdots & 3 & & 1 & & 4 & & 1 & & 3 & & 1 & & 4 & & 1 & & 3 & & 1 & & \cdots \\
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \cdots \\
\cdots & P^3 & & P^4 & & P^5 & & P^6 & & P^7 & & P^0 & & P^1 & & P^2 & & P^3 & & P^4 & & \cdots
\end{array}$$

Let us mutate the frieze $\mathcal{F}$ at $\{3, 5\}$ corresponding to the value 1.

The triangulation T associated with $\mathcal{F}$ contains the diagonal P^3P^5. We determine the sets E_3 and E_5 of the adjacent vertices to P^3 and P^5 respectively.

We get, $E_3 = \{P^1, P^3, P^4, P^5\}$ and $E_5 = \{P^1, P^3, P^4, P^6, P^7\}$.

Thus $E_3 \cap E_5 = \{P^1, P^4\}$ and P^1P^4 is the second diagonal of the quadrilateral containing P^3P^5.

Let $\mu_{35}(T)$ be the new triangulation of P_8 obtained by flipping T on the diagonal P^3P^5. The numbers of the incident triangles to the vertices P^5, P^3, P^1 and P^4 are 4, 3, 4 and 1 respectively. These numbers are replaced in order by $(4-1), (3-1), (4+1)$ and $(1+1)$ respectively in the new triangulation $\mu_{35}(T)$. The new numbers corresponding to the vertices P^5, P^3, P^1 and P^4 are 3, 2, 5 and 2 respectively.

Then we construct the first three rows of the bordered frieze $\mu_{35}(\mathcal{F})$ by the vertices P_j, $j \in \{0, 1, ..., 7\}$ and replacing the b_j, $j \in \{1, 3, 4, 5\}$ by the new value of b_j. That is,

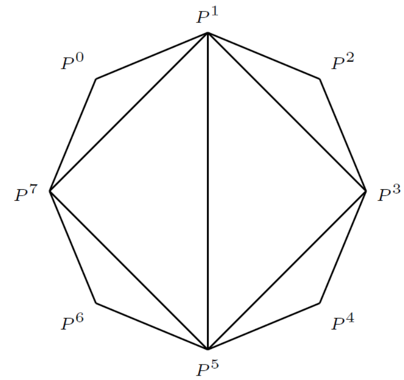
$$
\begin{array}{cccccccccccccccccccccc}
\cdots & P^7 & & P^0 & & P^1 & & P^2 & & P^3 & & P^4 & & P^5 & & P^6 & & P^7 & & P^0 & & \cdots \\
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \cdots \\
\cdots & 3 & & 1 & & 5 & & 1 & & 2 & & 2 & & 3 & & 1 & & 3 & & 1 & & \cdots
\end{array}
$$

Using the unimodular rule, we construct entirely and uniquely the frieze $\mu_{35}(\mathcal{F})$ which is

$$
\mu_{35}(\mathcal{F}) : \quad
\begin{array}{cccccccccccccccccccccc}
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \cdots \\
\cdots & 3 & & 1 & & 5 & & 1 & & 2 & & 2 & & 3 & & 1 & & 3 & & 1 & & \cdots \\
\cdots & & 2 & & 4 & & 4 & & 1 & & 3 & & 5 & & 2 & & 2 & & 2 & & 4 & \cdots \\
\cdots & 1 & & 7 & & 3 & & 3 & & 1 & & 7 & & 3 & & 3 & & 1 & & 7 & & \cdots \\
\cdots & & 3 & & 5 & & 2 & & 2 & & 2 & & 4 & & 4 & & 1 & & 3 & & 5 & \cdots \\
\cdots & 2 & & 2 & & 3 & & 1 & & 3 & & 1 & & 5 & & 1 & & 2 & & 2 & & \cdots \\
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \cdots
\end{array}
$$

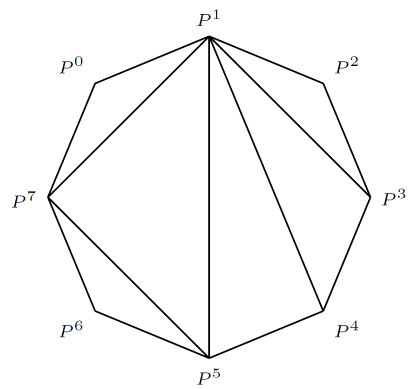
The corresponding triangulations of P_8 associated with $\mathcal{F}$ and $\mu_{35}(\mathcal{F})$ are respectively

T :



and

T' :



It is clear to see that T' is obtained from T by flipping on the diagonal P^3P^5, that is, $\mu_{35}(T) = T'$.

All these results allow us to make connection with cluster algebras of type $\mathbb{A}$.

4.2. Cluster algebras and permutations

A cluster algebra is generated by a set of variables, called cluster variables, obtained recursively by a combinatorial process known as mutation starting from a set of initial cluster variables [16, 17].

It was shown in ([3], Section 8) that a suitable choice of initial variables makes us obtain a frieze of variables and all these variables obtained are cluster variables in sense of Fomin and Zelevinsky. Therefore, combining

Theorem 4.1 and Corollaries 4.2 and 4.3, we can give the analogous of Theorem 3.10 of [21] in the following:

Theorem 4.5. *Let* $n \geq 1$ *be an integer, let* σ *be a canonical reading in the permutation group* S_n *and* $\mathcal{F}$ *be the frieze associated with* σ. *Then the cluster algebra* $\mathcal{A}(\mathcal{F})$ *of type* $\mathbb{A}_{n-1}$ *and* $\mathcal{A}(\sigma)$ *coincide, where* $\mathcal{A}(\mathcal{F})$ *and* $\mathcal{A}(\sigma)$ *are cluster algebras associated with the frieze* $\mathcal{F}$ *and permutation* σ *respectively.*

Before giving the proof of Theorem 4.5, let us give the analogous of Corollary 5.3.6 in [15].

Proposition 4.6. *Cluster variables in a seed pattern of type* $\mathbb{A}_{n-1}$ *can be labelled by diagonals of convex* $(n+2)$*-gon* P_{n+2} *so that*

- *Clusters correspond to canonical readings,*
- *Canonical readings correspond to frieze patterns of variables,*
- *Flips correspond to mutations of permutations,*
- *Mutations of permutations correspond to mutations of friezes of type* $\mathbb{A}_{n-1}$.

Cluster variables labelled by diagonals are distinct, so there are altogether $\frac{(n-1)(n+2)}{2}$ *cluster variables and* $\frac{1}{n+1}\binom{2n}{n}$ *seeds.*

Proof of Theorem 4.5. Each canonical reading produces $(n-1)$ diagonals which correspond to a seed. According to Theorem 2.16, there are $\frac{1}{n+1}\binom{2n}{n}$ canonical readings, so the same number of the frieze patterns of order $(n+2)$ by Theorem 3.6. It is well known in ([3], Section 8) that a suitable choice of initial variables in the frieze occurs the frieze of variables and all variables in the frieze are cluster variables. By Proposition 4.6, the mutations of permutations correspond to mutations of the frieze patterns of order $(n+2)$ which correspond to cluster mutation. □

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