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# GENERALIZED ALMOST JORDAN NILALGEBRAS OF NILINDEX FOUR AND DIMENSION AT MOST FIVE

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## Abstract

We study the generalized almost-Jordan nilalgebras of nilindex four that are not power associative. We first give some interesting equalities in these algebras. These equalities enable to find the possible multiplication tables relative to the dimension. We have one table in dimension three and two in dimension four. In dimension five, we show that the dimension of the square of an algebra satisfying these conditions is either 2 or 3, which shows four possible multiplication tables in dimension five.

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## 1. Introduction

Among the four degree-four identities exhibited by Carini et al. [1], $\beta x(x^2 y) + \gamma x^3 y - (\beta + \gamma)x(x(xy)) = 0$ defines a class of commutative non-associative algebras called *generalized almost-Jordan algebras*. This is one of the polynomial identities that has recently attracted the interest of many workers. Here, we are interested in the possible multiplication tables for generalized almost-Jordan nilalgebras of nilindex four, and dimension at most five, which are not power associative.

A non-associative algebra $A$ is said to be a *nilalgebra* of nilindex $n$, if $x^n = 0$ for all $x \in A$ and there exists $a \in A$ such that $a^{n-1} \neq 0.$ $A$ is called *power associative*, if the subalgebra of $A$ generated by any element $x \in A$ is associative. It is said to be a *nilpotent algebra* if for some natural integer $p$, the product of any $n$ elements from the algebra with $n \geq p$ and any arrangement of parenthesis vanishes. The least such integer is called the *index* of nilpotency of the algebra (see [6]).

## 2. Preliminaries

In what follows, $K$ is an infinite commutative field of characteristics not 2, 3, 5 and $A$ is a generalized almost-Jordan nilalgebra of nilindex 4, which is not power associative. The dimension of an algebra $A$ is denoted by $\dim A$.

**Lemma 2.1** [4]**.** *Let $A$ be a generalized almost-Jordan nilalgebra of nilindex* 4. *If* $3\beta + \gamma \neq 0$, *then $A$ satisfies*

$$x(x^2 y) + tx^3 y = 0, \tag{1}$$

*with* $t = \dfrac{\beta + 3\gamma}{3\beta + \gamma}$.

This previous lemma allows us to reduce to just one, the number of parameters in the study of generalized almost-Jordan nilalgebras of nilindex four. The particular case $t = 0$ corresponds to the well known Lie triple algebras, so we assume in this study that $t \neq 0$. In addition, if $t \neq 1$, then we have the following results, where $L_x : A \to A$, $a \mapsto xa$ is the left multiplication operator by $x$.

**Lemma 2.2.** *Let A be a generalized almost-Jordan nilalgebra of nilindex* 4, *satisfying* $0 \notin \{t; t-1\}$. *Then, for all* $x, y, z \in A$,

(i) $(t-1)x^3y - 2x(x(xy)) = 0$,

(ii) $(t-1)x(x^2y) + 2tx(x(xy)) = 0$,

(iii) $x^2x^3 = 0$,

(iv) $x(x^2x^2) = 0$,

(v) $(t-1)x^3(yx) - 2L_x^4(y) = 0$,

(vi) $x(x^3y) - x^3(xy) = 0$,

(vii) $x^3(x(xy)) + x(x^3(xy)) + x(x(x^3y)) = 0$,

(viii) $x^3(x(xy)) + 2x(x^3(xy)) = 0$,

(ix) $(t-1)x^3(x(xy)) + 4L_x^5(y) = 0$,

(x) $L_x^5(y) = 0$.

**Proof.** *A* being a generalized almost-Jordan nilalgebra of nilindex 4 satisfies the following identities:

$$x^4 = 0, \tag{2}$$

$$x(x^2y) + tx^3y = 0. \tag{3}$$

We use the linearization technique specified in [5, p. 174]. This technique applied to (2) gives

$$yx^3 + x(yx^2) + 2x(x(xy)) = 0. \tag{4}$$

From the difference between (4) and (3), we have (i).

Taking $y = x^2$ in (i), we get (iii).

And for $y = x^2$ in (3), we just need to use (iii) to have (iv).

Now, multiply (4) by $t$ and make the difference with (3) to have (ii).

To have (v), we just need to replace $y$ by $yx$ in (i).

Multiplying (i) by $x$ and making the difference with (v), we have (vi).

A linearization of (i) gives us

$$(t-1)[(zx^2)y + 2y(x(xz))] - 2[z(x(xy)) + x(z(xy)) + x(x(zy))] = 0. \quad (5)$$

In (5), put $z = x^3$ and use (iii) to have (vii).

To have (viii), we just need to use (vi) in (vii).

Now, multiply (v) by $x$ and use (viii) to have (ix).

Finally, to have (x), we replace $y$ by $yx$ in (v) and make the difference with (ix). □

Since $L_x^5(y) = 0$, if $xy = \alpha y$ for a non-zero $y$ in $A$, then $\alpha^5 y = 0$ and $\alpha = 0$.

Now, we have the following lemma:

**Lemma 2.3.** *If* $x$ *and* $y$ *are two non-zero elements of* $A$ *such that* $xy = \alpha y$, *then* $\alpha = 0$.

**Theorem 2.4.** *Let* $A$ *be a generalized almost-Jordan nilalgebra of nilindex* 4 *satisfying* $0 \notin \{t; t-1\}$. *If there are elements* $x$ *and* $y$ *of* $A$ *such that* $L_x^4(y) \neq 0$, *then* $y$, $L_x(y)$, $L_x^2(y)$, $L_x^3(y)$, $L_x^4(y)$, $x$, $x^2$ *and* $x^3$ *are linearly independent.*

**Proof.** (i) Assume that $L_x^4(y) \neq 0$ and consider

$$\sum_{i=0}^{4} \alpha_i L_x^i(y) + \sum_{j=1}^{3} \beta_j x^j = 0. \quad (6)$$

Applying $L_x^4$ to it, we have $\alpha_0 = 0$.

Applying $L_x^3$ to it, we have $\alpha_1 = 0$.

Applying $L_x^2$ to it and using (v) of Lemma 2.2, we have $\alpha_2 L_x^4(y) + \beta_1 x^3 = \frac{1}{2}\alpha_2(t-1)x^3(yx) + \beta_1 x^3 = 0$, and hence by Lemma 2.3, $\alpha_2 = \beta_1 = 0$.

Applying $L_x$ to the remaining terms and using (v) of Lemma 2.2, we have $\frac{1}{2}\alpha_3(t-1)x^3(yx) + \beta_2 x^3 = 0$, and hence by Lemma 2.3, $\alpha_3 = \beta_2 = 0$.

We have $\alpha_4 L_x^4(y) + \beta_3 x^3 = 0$. Using the same argument as above gives us $\beta_3 = \alpha_4 = 0$, and we have the result. □

From Theorem 2.4, we have the following corollary:

**Corollary 2.5.** *Let* $A$ *be a generalized almost-Jordan nilalgebra of nilindex* 4 *satisfying* $0 \notin \{t; t-1\}$. *Assume that for all* $a, b \in A$, $L_a^4(b) = 0$. *If there are two elements* $x, y \in A$ *such that* $L_x^3(y) \neq 0$, *then* $\{y, x, x^2, x^3, L_x(y), L_x^2(y), L_x^3(y)\}$ *is a linearly independent family. In this case*, $\dim A \geq 7$.

**Remark 2.6.** With the conditions of Corollary 2.5, if $\dim A \leq 6$, then for all $a, b \in A$, $L_a^3(b) = 0$.

Since the nilindex is 4, there exists $x \in A$ satisfying $x^3 \neq 0$. Moreover, by considering $\alpha, \beta, \gamma \in K$ such that $\alpha x + \beta x^2 + \gamma x^3 = 0$, we just need to apply $L_x$ twice successively to this equality to obtain $\alpha = 0$. Next, we apply $L_x$ once to have $\beta = 0$. This gives $\gamma = 0$. Then $x$, $x^2$ and $x^3$ are linearly independent that means the dimension of this algebra is at least three.

In [2], Elduque and Labra studied commutative nilalgebras of nilindex four and dimension at most four. Therefore, their results also apply to generalized almost-Jordan nilalgebras under the conditions of our study, when the dimension is 3 or 4.

In the rest of our work, for a set $S$, the sub-vector space generated by $S$ is denoted by $\langle S\rangle$, and the subalgebra generated by $S$ is denoted by $alg(S)$.

## 3. Cases of Dimensions 3 and 4

**Theorem 3.1** [2]. *Let $A$ be a* 3*-dimensional generalized almost-Jordan nilalgebra of nilindex* 4 *satisfying* $0 \notin \{t; t-1\}$ *which is not power associative. Then there exists an element* $a \in A$ *such that* $\{a, a^2, a^3\}$ *is a basis of $A$ with the following multiplication table* (*in which those not written products are zero*) $a.a = a^2$, $a.a^2 = a^3$, $a^2.a^2 = a^3$.

**Proof.** Let $x$ be an element of $A$ such that $x^3 \neq 0.$ Then $x$, $x^2$, $x^3$ are linearly independent. Since $\dim(A) = 3,$ we can write $A = alg(x, x^2, x^3).$ Put $x^2x^2 = \alpha x + \beta x^2 + \gamma x^3.$ Apply $L_x^2$ to this equality and use Lemma 2.2, to have $\alpha = 0$ which gives $x^2x^2 = \beta x^2 + \gamma x^3,$ and leads to $0 = x(x^2x^2) = \beta x^3$ and finally $\beta = 0.$ So we write $x^2x^2 = \gamma x^3.$ We can choose $x$ such that $\gamma \neq 0$, otherwise, for all $x \in A,$ we should have $x^4 = 0 = x^2x^2$ and $A$ would be power associative. Put $a = \gamma^{-1}x.$ Then $a^2a^2 = \gamma^{-4}x^2x^2 = \gamma^{-3}x^3 = a^3.$ Thus $A = alg(a, a^2, a^3)$ which admits the following multiplication table (all not written products being zero) $a.a = a^2,$ $a.a^2 = a^3,$ $a^2.a^2 = a^3.$ □

**Theorem 3.2** [2]. *Let $A$ be a* 4*-dimensional generalized almost-Jordan nilalgebra of nilindex* 4 *satisfying* $0 \notin \{t; t-1\}$, *which is not power associative. Then, $A$ admits one of the following multiplication tables in which the not written products are zero.*

*A basis of A is* $\{a, a^2, a^3, a^2a^2\}$ *with the following multiplication table*: $a.a = a^2$, $a.a^2 = a^3$, $a^2.a^2 = a^2a^2$.

*A basis of A is* $\{a, a^2, a^3, b\}$ *with the following multiplication table*: $a.a = a^2$, $a.a^2 = a^3$, $a^2.a^2 = a^3$, $b^2 = \beta_1 a^2 + \beta_2 a^3$.

**Proof.** We consider two cases:

**Case 1.** If there is an $a \in A$ such that $a^3 \neq 0$ and $a^2a^2 \notin \langle a, a^2, a^3\rangle$, then $alg(a) = \langle a, a^2, a^3, a^2a^2\rangle$, with $\dim A = 4$, and we conclude that $A = \langle a, a^2, a^3, a^2a^2\rangle$ and $A$ admits the following multiplication table (in which those products which not written are zero) $a.a = a^2$, $a.a^2 = a^3$, $a^2.a^2 = a^2a^2$.

**Case 2.** Otherwise, for all $x \in A$ such that $x^3 \neq 0$, we have $x^2x^2 \in \langle x, x^2, x^3\rangle$ and $alg(x) = \langle x, x^2, x^3\rangle$. For all $y \in A - alg(x)$, $alg(x) \bigcap alg(y)$ is a proper subalgebra of $alg(x)$, otherwise $alg(x) \subset alg(y) = A$. This is a contradiction. Thus $alg(x) \bigcap alg(y) \subseteq (alg(x))^2$. Since $alg(x)$ has codimension 1, $alg(x) \bigcap alg(y)$ also has codimension 1 in $alg(y)$, and we can write $(alg(y))^2 \subseteq alg(x) \bigcap alg(y) \subseteq (alg(x))^2$. Therefore, $y^2 \in (alg(x))^2$. So $A^2 = (alg(x))^2 = \langle x^2, x^3\rangle$ has codimension 2. Note that for all $y \in A$, $L_y^3(A) = 0$. Particularly, $L_x^3(A) = 0$ and $\operatorname{Ker} L_x \neq \{0\}$. Hence there exists $y_0 \in A - alg(x)$ such that $xy_0 = 0$. We have $A^3 = \langle x^3\rangle$, and then $x^2y_0 = \alpha_0 x^3$, $x^3y_0 = 0$, $y_0^2 = \beta x^2 + \lambda x^3$, $x^2x^2 = \gamma x^3$. Taking $a = \gamma^{-1}x$ on $a^2a^2 = a^3$, we have $a^2y_0 = \alpha_1 a^3$, $y_0^2 = \beta_0 a^2 + \beta_0' a^3$. From $a^2y_0 = \alpha_1 a^3$, we have $a^2(y_0 - \alpha_1 a) = 0$, and setting $b = y_0 - \alpha_1 a$, $a^2b = 0$. In the basis $\{a, a^2, a^3, b\}$, $b^2 = \beta_1 a^2 + \beta_2 a^3$, $ab = 0$ and $a^2b = 0$. Finally,

$A$ admits the following multiplication table (all not written products being zero): $a.a = a^2,\ a.a^2 = a^3,\ a^2.a^2 = a^3,\ b^2 = \beta_1 a^2 + \beta_2 a^3.$ $\square$

## 4. Case of Dimension 5

**Lemma 4.1.** *Every* 5*-dimensional generalized almost-Jordan nilalgebra of nilindex* 4 *satisfying* $0 \notin \{t;\ t-1\}$ *satisfies* $2 \leq \dim A^2 \leq 3$.

**Proof.** Since $A$ is a nilalgebra of nilindex 4, we necessarily have $\dim A^2 \geq 2$.

Now, we show that $\dim A^2 \leq 3$. Since, for some $x \in A$, $L_x^3 = 0$, according to [3], the algebra $A$ is nilpotent, that means there is $p \geq 4$ such that $A^p = 0$. We easily see that $A^2 \neq A$. This leads to $\dim A^2 \leq 4$. Now, we prove that $\dim A^2 < 4$. Assume that $\dim A^2 = 4$. Then there should exist $y \in A$ such that $A = Ky + A^2$, providing $A^2 = Ky^2 + A^3$, and $A = \langle y, y^2 \rangle + A^3$. Therefore, $A = \langle y, y^2, y^3, y^2y^2 \rangle + A^4$. In the same way, we have $A = \langle y, y^2, y^3, y^2y^2 \rangle + A^5$, that gives $A^5 = A^4$. Since $A$ is nilpotent, we have $A^5 = A^4 = 0$, which gives $A = alg(y)$ and $\dim A \leq 4$; a contradiction. Hence, $\dim A^2 \leq 3$. $\square$

**Theorem 4.2.** *Let $A$ be a generalized almost-Jordan nilalgebra of nilindex* 4 *satisfying* $0 \notin \{t;\ t-1\}$ *and dimension* 5. *If there is* $a \in A$ *such that* $alg(a) = \langle a, a^2, a^3, a^2a^2 \rangle$, *then there exists* $b_0 \in A \backslash alg(a)$ *such that* $ab_0 = 0,\ a^2b_0 = \alpha_1 a^3 + \alpha_2 a^2a^2$ *and* $b_0^2 = \beta_1 a^2 + \beta_2 a^3 + \beta_3 a^2a^2$.

**Proof.** Consider $a \in A$ such that $alg(a) = \langle a, a^2, a^3, a^2a^2 \rangle$. Then for every $b \in A \backslash alg(a)$, $alg(a) \cap alg(b)$ is a proper subalgebra of $alg(a)$, otherwise, we should have $alg(a) \subset alg(b)$ and $A = alg(b)$. It is impossible. Thus $alg(a) \cap alg(b) \subset alg(a)^2$.

Since $alg(a)$ has codimension 1, $alg(a) \cap alg(b)$ also has codimension 1 in $alg(b)$. So we have $alg(b)^2 \subset alg(a) \cap alg(b) \subset alg(a)^2$. Therefore, we can write $b^2 \in alg(a)^2$. Then $A^2 = alg(a)^2 = \langle a^2, a^3, a^2a^2 \rangle$ which has codimension 2. Since for every $y \in A$, we have $R_y^3(a) = 0$, particularly $0 = R_a^3(A)$. So $Ker\, R_a \neq 0$ and there exists $b_0 \in A \backslash alg(a)$ such that $ab_0 = 0$. This leads to $A = \langle b_0, a, a^2, a^3, a^2a^2 \rangle$, $A^2 = \langle a^2, a^3, a^2a^2 \rangle$ and $A^3 = \langle a^3, a^2a^2 \rangle$. Finally, we have the following products: $ab_0 = 0$, $b_0^2 = \beta_1 a^2 + \beta_2 a^3 + \beta_3 a^2a^2$, $a^2b_0 = \alpha_1 a^3 + \alpha_2 a^2a^2$, $a^3b_0 = 0$ and $(a^2a^2)b_0 = 0$. ☐

**Remark 4.3.** If $A$ does not contain any element $x$ such that $\dim(alg(x)) = 4$, then if $a \in A$ is such that $a^3 \neq 0$, we have $alg(a) = \langle a, a^2, a^3 \rangle$. Thus $a^2a^2 \in alg(a)^3 = \langle a^3 \rangle$ which means $a^2a^2 = \alpha a^3$. We can chose $a$ such that $\alpha \neq 0$, otherwise, $A$ should be power associative. Putting $a' = \alpha^{-1}a$, $a'^2a'^2 = a'^3$.

In the remainder of this work, when $alg(a) = \langle a, a^2, a^3 \rangle$, we consider $a^2a^2 = a^3$.

**Proposition 4.4.** *Let $A$ be a 5-dimensional generalized almost-Jordan nilalgebra of nilindex 4 satisfying $0 \notin \{t; t-1\}$. Assume that $A$ does not contain any element $x$ such that $\dim(alg(x)) = 4$. If $\dim A^2 = 3$, then there are $y, x \in A$ such that $A = \langle y, y^2, x, x^2, x^3 \rangle$.*

**Proof.** Consider $x \in A$ such that $x^3 \neq 0$. Then $alg(x) = \langle x, x^2, x^3 \rangle$. Let $X = alg(x)$. Then we show that there exists $y \in A$ such that $y^2 \notin \langle y, x, x^2, x^3 \rangle$.

We proceed by negation. Assume that for all $z \in A$, we have $z^2 \in \langle z, x, x^2, x^3 \rangle$. Since $\dim A^2 = 3$, there exists $z \in A$ such that $A^2 = \langle z^2, x^2, x^3 \rangle$. (Indeed, if for all $z \in A$, we have $z^2 \in \langle x^2, x^3 \rangle$, then for all $a, b \in A$, we should have $ab = \frac{1}{4}[(a+b)^2 - (a-b)^2] \in \langle x^2, x^3 \rangle$. This leads to $A^2 \subset \langle x^2, x^3 \rangle$, contradicting $\dim A^2 = 3$.) Since $z \notin \langle x, x^2, x^3 \rangle = X$, $D = \langle z, x, x^2, x^3 \rangle$ is such that $z^2 \in D$ by hypothesis. We have $A^2 = \langle z^2, x^2, x^3 \rangle \subset D$ and then $D$ is an ideal of $A$. Thus $D$ is a nilalgebra of nilindex 4 in dimension 4. Since, by hypothesis, there is no $x \in A$ such that $\dim(alg(x)) = 4$, the algebra $D$ satisfies Table 3 of [2, Theorem 3]. Therefore, $D^2 = \langle x^2, x^3 \rangle = X^2$ and so we have $z^2 \in X^2$. This is a contradiction. So there exists $y \in A$ such that $A = \langle y, y^2, x, x^2, x^3 \rangle$. □

**Theorem 4.5.** *Let $A$ be a 5-dimensional generalized almost-Jordan nilalgebra of nilindex 4 satisfying $0 \notin \{t; t-1\}$. Assume that $A$ does not contain any element $x$ such that $\dim(alg(x)) = 4$. If $\dim A^2 = 3$ and $\dim A^3 = 1$, then there are $x, y \in A$ such that $A = \langle y, yx, x, x^2, x^3 \rangle$, $(xy)x^2 = \alpha x^3$, $(xy)x = \beta x^3$, $(xy)^2 = \gamma x^3$, $y(xy) = \lambda x^3$ and $y^2 = \varepsilon_1(yx) + \varepsilon_2 x^2 + \varepsilon_3 x^3$.*

**Proof.** Let $x \in A$ such that $x^3 \neq 0$. Then $alg(x) = \langle x, x^2, x^3 \rangle$. For $X = alg(x)$, we show that there exists $y \in A$ such that $yx \notin X^2 = \langle x^2, x^3 \rangle$. We proceed by negation.

Because of Proposition 4.4, we know that there exists $y \in A$ such that $A = \langle y, y^2, x, x^2, x^3 \rangle$. Because of our hypothesis, $xy \in X^2$ which means $xy = \alpha_1 x^2 + \alpha_2 x^3$. If $y' = y - (\alpha_1 x + \alpha_2 x^2)$, then $xy' = 0$ and

$A = \langle y', y'^2, x, x^2, x^3\rangle$. We can therefore assume in the initial basis $\{y, y^2, x, x^2, x^3\}$ that $xy = 0$.

Note that there is $\lambda \in K^*$ such that $(y + \lambda x)^3 \neq 0$. Then, we should have by hypothesis, $y(y + \lambda x) \in \langle (x + \lambda y)^2, (x + \lambda y)^3\rangle$ which means $y^2 = \beta_2(y + \lambda x)^2 + \beta_3(y + \lambda x)^3$. So we have $y^2 = \beta_2 y^2 + \beta_2\lambda^2 x^2 + z_0$ with $z_0 \in A^3 = X^3$. This leads to $\beta_2 = 1$ and $\lambda^2 x^2 = -z_0 \in X^3$. This is a contradiction. We therefore conclude that there exists $y \in A$ such that $xy \notin X^2 = \langle x^2, x^3\rangle$.

Since $y \notin X$, $xy \notin \langle y, x, x^2, x^3\rangle$. Indeed, assume that $xy = \gamma_0 y + \gamma_1 x + \gamma_2 x^2 + \gamma_3 x^3$. Then $x(xy) - (\gamma_0 xy + \gamma_1 x^2) = \gamma_2 x^3 \in X^3$. However, we have $x(xy) \in X^3$ (because $\dim A^3 = 1 = \dim X^3$ implies that $A^3 = X^3$), which means that $\gamma_0 xy + \gamma_1 x^2 \in X^3 \subset X^2$. This implies that $\gamma_0 = 0$ and $\gamma_1 x^2 \in X^3$, which means $\gamma_1 = 0$. So, we have $xy = \gamma_2 x^2 + \gamma_3 x^3$. This is a contradiction.

Finally, $\{y, yx, x, x^2, x^3\}$ is a basis of $A$.

Let $yx^2 = \sigma x^3$. Then, we have $(y - \sigma x)x^2 = 0$. We can then assume that in the basis $\{y, yx, x, x^2, x^3\}$, $yx^2 = 0$. The other products are written as $(yx)x = \alpha x^3$, $(yx)x^2 = \beta x^3$, $(yx)^2 = \gamma x^3$, $y(yx) = \lambda x^3$ and $y^2 = \varepsilon_1 x^2 + \varepsilon_2 x^3 + \varepsilon_3(yx)$. □

**Lemma 4.6.** *Let A be a generalized almost-Jordan nilalgebra of nilindex 4 satisfying* $0 \notin \{t; t - 1\}$ *and of dimension* $\leq 6$. *Then, for all* $x, y \in A$, $x(xy^2) = y(yx^2)$.

**Proof.** Because of Remark 2.6, in dimension $\leq 6$, $A$ satisfies $L_x^3(y) = 0$ for all $x, y \in A.$ Linearization of $x(x(xy)) = 0$ gives

$$z(x(xy)) + x(z(xy)) + x(x(zy)) = 0. \tag{7}$$

Taking $z = y$ in (7), we have

$$y(x(xy)) + x(y(xy)) + x(x(y^2)) = 0. \tag{8}$$

Interchanging $x$ and $y$ in (8), we have

$$x(y(xy)) + y(x(xy)) + y(y(x^2)) = 0. \tag{9}$$

Difference between (8) and (9) gives $x(xy^2) = y(yx^2)$. □

**Lemma 4.7.** *Let* $A$ *be a* 5*-dimensional generalized almost-Jordan nilalgebra of nilindex* 4 *satisfying* $0 \notin \{t; t-1\}$. *Then, for all* $x, y, z \in A$,

(i) $x(yx^2) = 0$,

(ii) $x^2(yx^2) = 0$,

(iii) $(zx^2)(yx^2) = 0$.

**Proof.** Since $\dim A \leq 6$, by (ii) of Lemma 2.2, we have $(t-1)x(yx^2) = -2tL_x^3(y) = 0$, which establishes (i).

A linearization of (i) gives

$$z(yx^2) + 2x(y(xz)) = 0. \tag{10}$$

Taking $z = x^2$ in (10), we have $x^2(yx^2) + 2x(yx^3) = 0$. But Lemma 2.2 gives $x(yx^3) = x^3(yx) = \frac{2}{t-1}L_x^4(y) = 0$, because $\dim A \leq 6$. Thus follows (ii).

To have (iii), we just need to replace $z$ by $zx^2$ in (10) and use (i). □

**Theorem 4.8.** *Let $A$ be a 5-dimensional generalized almost-Jordan nilalgebra of nilindex 4 satisfying $0 \notin \{t;\, t-1\}$. Assume that $A$ does not contain any element $x$ such that $\dim(alg(x)) = 4$. If $\dim A^2 = 3$ and $\dim A^3 = 2$, then there exist $x_0, y_0 \in A$ such that*

$$A = \langle y_0, y_0x_0^2, x_0, x_0^2, x_0^3\rangle \quad \textit{and} \quad y_0^2 = \alpha_1 y_0x_0^2 + \alpha_2 x_0^2 + \alpha_3 x_0^3.$$

**Proof.** Let $x \in A$ such that $x^3 \neq 0$. Then $alg(x) = \langle x, x^2, x^3\rangle$. Set $X = alg(x)$.

We first show that there exists $y \in A$ such that $yx^2 \notin X^3$. We proceed by negation, that means for all $z \in A$, we have $zx^2 \in X^3$.

Since $X^3 = \langle x^3\rangle$ is an ideal because $x^3y = 0$ for all $y \in A$, the hypothesis implies that $X^2 = \langle x^2, x^3\rangle$ is also an ideal and the algebra $A/X^2$ is of dimension 3. On account of Proposition 4.4, there exists $z \in A$ such that $A = \langle z, z^2, x, x^2, x^3\rangle$, which implies that $A/X^2 = \langle \bar{z}, \bar{z}^2, \bar{x}\rangle$ is a nilalgebra of nilindex 3. Thus, [2, Theorem 1] implies that $\overline{A}^3 = (A/X^2)^3 = 0$ which means $A^3 \subset X^2$. However, $\dim A^3 = 2 = \dim X^2$, which gives $A^3 = X^2$.

Since $X^3$ is an ideal, consider $\widetilde{A} = A/X^3$ which is of dimension 4. $\widetilde{A}$ is a nilalgebra of nilindex 4. Indeed, if the nilindex was 3, because of [2, Theorem 1], we should have $\widetilde{A}^3 = 0$ which means $A^3 = X^2 \subset X^3$. It is a contradiction.

The nilindex is thus 4. So there exists $\bar{y} \in \widetilde{A}$ such that $\bar{y}^3 \neq 0$, and hence $y^3 \notin X^3$. Since $y^3 \in A^3 = X^2$, $y^3 = \alpha_2 x^2 + \alpha_3 x^3$ with $\alpha_2 \neq 0$. However, $xy^3 = 0 = \alpha_2 x^3$, which implies $\alpha_2 = 0$. It is a contradiction.

So, there is $y \in A$ such that $yx^2 \notin X^3$. Since $\widetilde{A} = A/X^3$ is a nilalgebra of nilindex 4 and dimension 4, because of [2, Theorem 3], we have $\widetilde{A}^4 = 0$. Thus $A^4 \subset X^3$. However, we have $y \in A$ such that $yx^2 \notin X^3$, which implies that $y \notin X$. In addition, $yx^2 \notin \langle y, x, x^2, x^3 \rangle$. Indeed, if we assume that $yx^2 = \beta_0 y + \beta_1 x + \beta_2 x^2 + \beta_3 x^3$, then multiplying by $x^2$, we have $0 = x^2(yx^2) = \beta_0 yx^2 + (\beta_1 + \beta_2)x^3$ by (ii) of Lemma 4.7. It implies that $\beta_0 = 0$ and $\beta_1 + \beta_2 = 0$. So, we have $yx^2 = \beta_1 x + \beta_2 x^2 + \beta_3 x^3$. Multiplying by $x$, (i) of Lemma 4.7 gives $0 = x(yx^2) = \beta_1 x^2 + \beta_2 x^3$ and $\beta_1 = \beta_2 = 0$. Thus $yx^2 = \beta_3 \in X^3$. This is a contradiction.

Thus, $yx^2 \notin \langle y, x, x^2, x^3 \rangle$ and $A = \langle y, yx^2, x, x^2, x^3 \rangle$. Since $yx \in A^2$, $yx = \delta_1 yx^2 + \delta_2 x^2 + \delta_3 x^3$. If $x_0 = x - \delta_1 x^2$, then $yx_0 = \gamma_2 x_0^2 + \gamma_3 x_0^3$. Let $y_0 = y - (\gamma_2 x_0 + \gamma_3 x_0^2)$. Then $x_0 y_0 = 0$ and $\{y_0, y_0 x_0^2, x_0, x_0^2, x_0^3\}$ is a basis of $A$.

Consider $y_0^2 = \alpha_1 x_0^2 + \alpha_2 x_0^3 + \alpha_3 y_0 x_0^2$ and $(y_0 x_0^2)^2 = \lambda x_0^3$.

Lemma 4.6 gives

$$y_0(y_0 x_0^2) = x_0(x_0 y_0^2) = x_0\,(\alpha_3 x_0(y_0 x_0^2) + \alpha_1 x_0^3) = 0.$$

Taking $z = y = y_0$ in Lemma 4.7(iii), we have $\lambda = 0$. □

**Theorem 4.9.** *Let A be a* 5*-dimensional generalized almost-Jordan nilalgebra of nilindex* 4 *satisfying* $0 \notin \{t; t-1\}$. *Assume that A does not contain any element x such that* $\dim(alg(x)) = 4$. *If* $\dim A^2 = 2$, *then there exist* $x_0, y_0, a \in A$ *such that* $A = \langle x_0, y_0, a, a^2, a^3 \rangle$ *and* $x_0^2 = \alpha_1 a^2 + \alpha_2 a^3$, $y_0^2 = \lambda_1 a^2 + \lambda_2 a^3$, $x_0 y_0 = \beta_1 a^2 + \beta_2 a^3$.

**Proof.** Let $a \in A$ such that $a^3 \neq 0$. Since $\dim A^2 = 2$, we have $A^2 = \langle a^2, a^3 \rangle$, $A^3 = \langle a^3 \rangle$. Thus there exist $x, y \in A \backslash alg(a)$ such that $A = \langle x, y, a, a^2, a^3 \rangle$ and $x^2 = \alpha_1 a^2 + \alpha_2 a^3$, $ax = \gamma_1 a^2 + \gamma_2 a^3$, $ya = \varepsilon_1 a^2 + \varepsilon_2 a^3$, $xy = \beta_1 a^2 + \beta_2 a^3$, $xa^2 = \delta a^3$, $ya^2 = \mu a^3$ and $y^2 = \lambda_1 a^2 + \lambda_2 a^3$.

In equality $ax = \gamma_1 a^2 + \gamma_2 a^3$, taking $x' = x - \gamma_1 a - \gamma_2 a^2$, we have $ax' = 0$.

In equality $ay = \varepsilon_1 a^2 + \varepsilon_2 a^3$, $y' = y - \varepsilon_1 a - \varepsilon_2 a^2$ gives $ay' = 0$.

Thus, in the basis $\{x, y, a, a^2, a^3\}$, we can assume that $ax = 0 = ay$.

In the same way, in equalities $xa^2 = \delta a^3$ and $ya^2 = \mu a^3$, by, respectively, setting $x_0 = x - \delta a$ and $y_0 = y - \mu a$, we have $x_0 a^2 = 0 = y_0 a^2$. Thus, in the basis $\{x_0, y_0, a, a^2, a^3\}$, we have $x_0^2 = \alpha_1 a^2 + \alpha_2 a^3$, $x_0 y_0 = \beta_1 a^2 + \beta_2 a^3$, $y_0^2 = \lambda_1 a^2 + \lambda_2 a^3$. □

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