JP Journal of Algebra, Number Theory and Applications
© 2026 Pushpa Publishing House, Prayagraj, India
https://pphmjopenaccess.com
https://doi.org/10.17654/0972555526009 P-ISSN: 0972-5555
Volume 65, Number 1, January 2026, Pages 165-183 E-ISSN: 3049-2084

ON POWERS OF IDEALS $(I^m J^n)_{(m,n) \in \mathbb{N}^2}$ AND σ-SPORADIC PRIME DIVISORS OF A PAIR OF IDEALS (I, J)

Kouadjo Pierre BROU*, **Edjabrou Ulrich Blanchard KABLAM** and **Neangnehi Jean Roland GUIYE**

UFR-SFA
Laboratory of Mathematics and Computer Science
NANGUI ABROGOUA University
02 BP 801 Abidjan 02, Abidjan
Ivory Coast
e-mail: brou.kouadjopierre@yahoo.fr
bkablam11@gmail.com
guiyeneangnehijeanroland@gmail.com

Received: August 12, 2025; Revised: September 20, 2025; Accepted: October 18, 2025
2020 Mathematics Subject Classification: 13A15.
Keywords and phrases: bifiltration, prime operation, σ-sporadic ideals.
*Corresponding author
How to cite this article: Kouadjo Pierre BROU, Edjabrou Ulrich Blanchard KABLAM and Neangnehi Jean Roland GUIYE, On powers of ideals $(I^m J^n)_{(m,n) \in \mathbb{N}^2}$ and σ-sporadic prime divisors of a pair of ideals (I, J), JP Journal of Algebra, Number Theory and Applications 65(1) (2026), 165-183. https://doi.org/10.17654/0972555526009

Published Online: December 26, 2025

Abstract

Let I and J be two non-zero ideals of a Noetherian ring A. Let σ be a semi-prime operation on the set of all ideals of A. Let m_0 and n_0 be two fixed positive integers, and

$$\mathbb{E}_{m_0, n_0} = \{(\beta m_0, \beta n_0)/\beta \in \mathbb{N}\}.$$

In first time, we show that:

(a) for $e \in \mathbb{E}_{m_0, n_0} \setminus \{(0, 0)\}$, $\{\sigma(\mathcal{K}^{e+t}) : \sigma(\mathcal{K}^t)\}_{t \in \mathbb{E}_{m_0, n_0}}$ is an increasing sequence and stabilizes at $\mathcal{K}_\sigma^e$,

(b) the sequence $(Ass(A/\sigma(\mathcal{K}^e)))_{e \in \mathbb{E}_{m_0, n_0}}$ is increasing and stabilizes at $A_\sigma(I, J) = A_\sigma(\mathcal{K})$ under certain conditions for e large enough.

First, we show

$$S_\ell^\sigma(I, J) = Ass(A/\sigma(\mathcal{K}^\ell)) - A_\sigma(\mathcal{K}),$$

for ℓ small enough in $\mathbb{E}_{m_0, n_0}$. The elements of $S_\ell^\sigma(I, J)$ are called σ-sporadic prime divisors of (I, J).

We give some properties on the set $S_\ell^\sigma(I, J)$.

To finish, we conclude this paper with the equality $\mathcal{K}_\sigma^\ell = [\sigma(\mathcal{K}^\ell)]_\Delta$ under certain conditions, where L_Δ is the Δ-closure of the ideal L of A.

1. Introduction

Let A be a Noetherian ring as in [12], and let I be a non-zero ideal of A. Let σ be a semi-prime operation on the set of all ideals of A [9, 16]. Associated prime ideals have been widely studied in the literature [1, 10, 11, 13-15]. As an extension of this study, the paper [3] (see also [4, 6]) examined asymptotic σ-prime divisors. Indeed, the paper [3] showed that the sequence $(Ass(A/\sigma(I^n)))_{n \in \mathbb{N}}$ is increasing under certain conditions and stabilizes at $A_\sigma(I)$. That is, $Ass(A/\sigma(I^n)) = A_\sigma(I)$ for all n large enough in $\mathbb{N}$. But for n small enough, there exist prime ideals P in $Ass(A/\sigma(I^n)) - A_\sigma(I)$. Such ideals are called σ-*sporadic prime divisors* (see [5]). $S^\sigma(I) = \bigcup_{n \in \mathbb{N}} S_n^\sigma(I)$ is the set of all σ-sporadic prime divisors, where

$$S_n^\sigma(I) = Ass(A/\sigma(I^n)) - A_\sigma(I).$$

The paper [5] revealed interesting results about the σ-sporadic prime divisors of an ideal.

This paper is a part of the extension of certain results from [5] to adic bifiltrations [2, 8, 17]. But for this, it is important to consider the study that was done in [7]. Let A be a Noetherian ring, and I and J be two non-zero ideals of A. Let σ be a semi-prime operation on A. We consider the product order on $\mathbb{Z} \times \mathbb{Z}$, denoted by $\prec$, such that for all $m_1, m_2, n_1, n_2 \in \mathbb{Z}$, $(m_1, n_1) \prec (m_2, n_2)$ if and only if $m_1 \leq m_2$ and $n_1 \leq n_2$. We set $(m_1, n_1) + (m_2, n_2) = (m_1 + m_2, n_1 + n_2)$ and for any $k \in \mathbb{Z}$, $k(m_1, n_1) = (km_1, kn_1)$. Let $\mathcal{K}^{(m, n)} = I^m J^n$, $\forall (m, n) \in \mathbb{N}^2$ and $F_{\mathcal{K}} = (\mathcal{K}^\alpha)_{\alpha \in \mathbb{N}^2}$. $F_{\mathcal{K}}$ is an (I, J)-adic bifiltration. Let m_0 and n_0 be two fixed positive integers. Then the set

$$\mathbb{E}_{m_0, n_0} = \{(\beta m_0, \beta n_0)/\beta \in \mathbb{N}\}$$

is a totally ordered subset of $\mathbb{N}^2$.

We show that the sequence $(Ass(A/\sigma(\mathcal{K}^\alpha)))_{\alpha \in \mathbb{E}_{m_0, n_0}}$ is increasing from a certain rank and stabilizes in $\mathbb{E}_{m_0, n_0}$. We note that the sequence $(Ass(A/\sigma(\mathcal{K}^\alpha)))_{\alpha \in \mathbb{E}_{m_0, n_0}}$ stabilizes at $A_\sigma(I, J) = A_\sigma(\mathcal{K})$. Moreover, if $e \in \mathbb{E}_{m_0, n_0} - \{(0, 0)\}$, then the sequence $\{\sigma(\mathcal{K}^{e+t}) : \sigma(\mathcal{K}^t)\}_{t \in \mathbb{E}_{m_0, n_0}}$ is increasing and stabilizes at $\mathcal{K}_\sigma^e$.

We assume that for ℓ small enough in $\mathbb{E}_{m_0, n_0}$, there exist prime ideals P in $S_\ell^\sigma(I, J) = Ass(A/\sigma(\mathcal{K}^\ell)) - A_\sigma(\mathcal{K})$. The prime ideals P of A in $S^\sigma(I, J) = \bigcup_{\ell \in \mathbb{E}_{m_0, n_0}} S_\ell^\sigma(I, J)$ are called σ-*sporadic prime divisors* of (I, J). Let $\alpha_0 \in \mathbb{E}_{m_0, n_0}$ be such that for all $k \in \mathbb{E}_{m_0, n_0}$ with $k \succ \alpha_0$ and for all $e \in \mathbb{E}_{m_0, n_0}$, $\sigma(\mathcal{K}^{k+e}) : \mathcal{K}^e = \sigma(\mathcal{K}^k)$. Proposition 3.3 shows that $S_\ell^\sigma(I, J)$ is included in $Ass(\mathcal{K}_\sigma^\ell/\sigma(\mathcal{K}^\ell))$, for all $\ell \succ (1, 1)$ in $\mathbb{E}_{m_0, n_0}$. If A is an Artinian ring and if the conditions of Proposition 3.3 are satisfied, then Proposition 3.5 reveals that $S_\ell^\sigma(I, J)$ is contained in $Ass(\mathcal{K}_\sigma^{\ell-r}/\sigma(\mathcal{K}^{\ell+r}))$, for all ℓ large enough and for all r such that $\ell \succ r \succ (1, 1)$ in $\mathbb{E}_{m_0, n_0}$.

In Proposition 3.6, we show that if $e \in \mathbb{E}_{m_0, n_0} - \{(0, 0)\}$ and L is an ideal of A such that $\mathcal{K}^e \subseteq L \subseteq \mathcal{K}_\sigma^e$, then any element of $Ass(\mathcal{K}_\sigma^e/\sigma(L))$ contains $(\sigma(\mathcal{K}^e) : \mathcal{K}_\sigma^e)$. For other results in this section, we assume that σ is a prime operation on the set of all ideals of A. Thus, let u and v be two regular elements of A. Then, Proposition 3.7 shows that for all $\alpha = (\alpha_1, \alpha_2) \in \mathbb{E}_{m_0, n_0} - \{(0, 0)\}$,

$$Ass(A/\sigma(I^{\alpha_1} J^{\alpha_2})) \subseteq Ass(A/\sigma((uI)^{\alpha_1} (vJ)^{\alpha_2})).$$

In Proposition 3.8, we assume that the ideals I and J are regular. Let x be a regular element of I and y a regular element of J such that (x) is a reduction of I and (y) is a reduction of J. Then there exists $r = (r_1, r_2) \succ (1, 1)$ in $\mathbb{N}^2$ such that

$$Ass(A/\sigma(\mathcal{K}^{(r\alpha_1, r\alpha_2)})) \subseteq Ass(A/\sigma(\mathcal{K}^{((r_1+1)\alpha_1, (r_2+1)\alpha_2)})),$$

for all $(\alpha_1, \alpha_2) \succ (1, 1)$ in $\mathbb{E}_{m_0, n_0}$. Under the same hypotheses as Proposition 3.8, Proposition 3.9 reveals that there exist two integers $r_1 \geq 1$

and $r_2 \geq 1$ such that for all integers $n_1 \geq 1$ and $n_2 \geq 1$, $\sigma(\mathcal{K}^{r+n}) = x^{n_1} y^{n_2} \sigma(\mathcal{K}^r)$, where $r = (r_1, r_2)$, $n = (n_1, n_2)$ and $\mathcal{K}^{\ell} = I^{\ell_1} J^{\ell_2}$ for all $\ell = (\ell_1, \ell_2)$.

We consider that I and J are two ideals of the Noetherian ring A. The set $\Delta = \{\mathcal{K}^n \mid \mathcal{K}^n = I^{n_1} J^{n_2}\}$, with $n = (n_1, n_2) \in \mathbb{E}_{m_0, n_0} - \{(0, 0)\}$, is a non-empty multiplicatively closed set of non-zero finitely generated ideals of A. For any ideal L of A, we denote the Δ-closure of L by

$$L_{\Delta} = \bigcup_{V \in \Delta} (LV : V).$$

Let $(\sigma(\mathcal{K}^n))_{n \in \mathbb{E}_{m_0, n_0}}$ be a bifiltration such that for $s \in \mathbb{E}_{m_0, n_0}$ and s large enough, $\mathcal{K}^m \sigma(\mathcal{K}^s) = \sigma(\mathcal{K}^{m+s})$, for all $m \in \mathbb{E}_{m_0, n_0}$. Then Proposition 3.10 shows that there exists e in $\mathbb{E}_{m_0, n_0} - \{(0, 0)\}$ such that $\mathcal{K}_{\sigma}^{\ell} = [\sigma(\mathcal{K}^{\ell})]_{\Delta}$ for all $\ell \in \mathbb{E}_{m_0, n_0}$ and $\ell \succ e$.

2. Preliminaries

Throughout this paper, the letter A denotes a commutative ring with unity. In this section, we provide some definitions and notations.

(1) A filtration on the ring A is a sequence $f = (I_n)_{n \in \mathbb{Z}}$ of ideals of A such that $I_0 = A$, $I_{n+1} \subseteq I_n$ and $I_n I_m \subseteq I_{n+m}$ for all $n, m \in \mathbb{Z}$. It follows that for all $n \leq 0$, $I_n = A$.

(2) Let I be an ideal of A. Then a filtration $f = (J_n)_{n \in \mathbb{Z}}$ on A is said to be *I-adic* if $J_n = I^n$, for all $n \in \mathbb{N}$.

Definition 2.1. Let $\mathcal{I}(A)$ be the set of all ideals of a ring A. Then we consider the following properties of a map $\sigma : \mathcal{I}(A) \to \mathcal{I}(A)$:

(a) $I \subseteq \sigma(I)$,

(b) if $I \subseteq J$, then $\sigma(I) \subseteq \sigma(J)$,

(c) $\sigma(\sigma(I)) = \sigma(I)$,

(d) $\sigma(I)\sigma(J) \subseteq \sigma(IJ)$ and

(e) $\sigma(bI) = b\sigma(I)$ for any regular element $b \in A$,

for all $I, J \in \mathcal{I}(A)$. Then σ is a *semi-prime operation* on $\mathcal{I}(A)$ if (a)-(d) are satisfied for all $I, J \in \mathcal{I}(A)$, it is a prime operation if (a)-(e) are satisfied for all $I, J \in \mathcal{I}(A)$ and every regular element b of A.

Proposition 2.2. *Let A be a commutative ring with unity and σ a semi-prime operation on $\mathcal{I}(A)$. Then*

$$\sigma(\sigma(I)\sigma(J)) = \sigma(IJ), \quad \forall I, J \in \mathcal{I}(A).$$

Proof. Let I and J be two ideals of A. Then using properties (a), (b), (d) and (c) of Definition 2.1, respectively, we have

$$I \subseteq \sigma(I) \quad \text{and} \quad J \subseteq \sigma(J) \Rightarrow IJ \subseteq \sigma(I)\sigma(J).$$

Hence, by applying σ to both sides,

$$\sigma(IJ) \subseteq \sigma(\sigma(I)\sigma(J)). \tag{2.1}$$

However, we know that

$$\sigma(I)\sigma(J) \subseteq \sigma(IJ).$$

Applying σ again and using the idempotence property $(\sigma(\sigma(X)) = \sigma(X))$,

$$\sigma(\sigma(I)\sigma(J)) \subseteq \sigma(\sigma(IJ)) = \sigma(IJ).$$

Combining this with inequality (2.1), we obtain

$$\sigma(IJ) \subseteq \sigma(\sigma(I)\sigma(J)) \subseteq \sigma(\sigma(IJ)) = \sigma(IJ).$$

Thus,

$$\sigma(IJ) = \sigma(\sigma(I)\sigma(J)). \qquad \square$$

(3) If $f = (I_n)_{n\in\mathbb{Z}}$ is a filtration on A and σ is a semi-prime operation on $\mathcal{I}(A)$, then $\sigma(f) = (\sigma(I_n))_{n\in\mathbb{Z}}$ is a filtration on A, called the σ-*closure of f*.

We consider the product order on $\mathbb{Z}\times\mathbb{Z}$ which is denoted by $\prec$ such that for all $m, n, p, q \in \mathbb{Z}$, $(m, n) \prec (p, q)$ if and only if $m \leq p$ and $n \leq q$. We set $(m, n) + (p, q) = (m + p, n + q)$ and for any $k \in \mathbb{Z}$, $k(m, n) = (km, kn)$.

Definition 2.3. (i) A *bifiltration* on the ring A is a decreasing family $F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2}$ of ideals of A (with respect to the product order on $\mathbb{Z}\times\mathbb{Z}$) such that $I_{0,0} = A$ and for all integers $m, n, p, q \in \mathbb{Z}$,

$$I_{m,n}I_{p,q} \subseteq I_{m+p,n+q}.$$

It follows that for all $m, n \leq 0$, $I_{m,n} = A$.

(ii) The set of all bifiltrations on the ring A is ordered by

$$F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2} \leq H = (J_{m,n})_{(m,n)\in\mathbb{Z}^2}$$

if $I_{m,n} \subseteq J_{m,n}$ for all $m, n \in \mathbb{Z}$.

(4) Let I and J be two ideals of A. Then a bifiltration $F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2}$ on A is said to be (I, J)-*adic* if $I_{m,n} = I^m J^n$, for all $m, n \in \mathbb{N}$.

(5) Let $F = (I_{m,n})_{(m,n)\in\mathbb{Z}^2}$ be a bifiltration on a ring A and σ a semi-prime operation on $\mathcal{I}(A)$. Then $\sigma(F) = (\sigma(I_{m,n}))_{(m,n)\in\mathbb{Z}^2}$ is a bifiltration on A, called the σ-*closure* of F.

Remark 2.4. Let I be an ideal of a ring A and let P be a prime ideal of A such that $I \subseteq P$. Then for any $x \in A$, we have

$$P/I = Ann(\bar{x}) \Leftrightarrow P = I : x.$$

3. Results

3.1. (I, J)-adic bifiltration and σ-sporadic prime ideals

Let A be a Noetherian ring and σ a semi-prime operation on the set of all ideals of A. Then we assume that for every prime ideal P, there exists a semi-prime operation $\widehat{\sigma_P}$ on the set of all ideals of A_P such that $\widehat{\sigma_P}(IA_P) = \sigma(I)A_P$ for every ideal I of A (see [5]).

Let m_0 and n_0 be two fixed positive integers. Then the set

$$\mathbb{E}_{m_0, n_0} = \{(\beta m_0, \beta n_0)/\beta \in \mathbb{N}\}$$

is a totally ordered subset of $\mathbb{N}^2$. Let I and J be two non-zero ideals of the ring A. Then for any $e = (e_1, e_2) \in \mathbb{E}_{m_0, n_0}$, we set $\mathcal{K}^e = I^{e_1}J^{e_2}$. $\mathcal{K}^e$ is an ideal of A.

Lemma 3.1. *Let A be a Noetherian ring and I, J be two non-zero ideals of A. Let σ be a semi-prime operation on I. Then for every element e of $\mathbb{E}_{m_0, n_0} \setminus \{(0, 0)\}$, the sequence*

$$\{\sigma(\mathcal{K}^{e+t}) : \sigma(\mathcal{K}^t)\}_{t \in \mathbb{E}_{m_0, n_0}}$$

is increasing and stationary.

Proof. Let $e \in \mathbb{E}_{m_0, n_0} \setminus \{(0, 0)\}$. Let t and ℓ be two elements of $\mathbb{E}_{m_0, n_0}$ such that $t \preceq \ell$ (in the product order).

Let x be an element of $\sigma(\mathcal{K}^{e+t}) : \sigma(\mathcal{K}^t)$. Then

$$x\sigma(\mathcal{K}^t) \subseteq \sigma(\mathcal{K}^{e+t}). \qquad (*)$$

Using property (d) of Definition 2.1, we have

$$\sigma(\mathcal{K}^{\ell-t})\sigma(\mathcal{K}^t) \subseteq \sigma(\mathcal{K}^{\ell-t}\mathcal{K}^t) = \sigma(\mathcal{K}^\ell).$$

Using property (a) of Definition 2.1, we have

$$x\sigma(\mathcal{K}^{\ell-t})\sigma(\mathcal{K}^{t}) \subseteq \sigma[x\sigma(\mathcal{K}^{\ell-t})\sigma(\mathcal{K}^{t})].$$

Hence

$$x\sigma(\mathcal{K}^{\ell}) \subseteq \sigma[\sigma(\mathcal{K}^{\ell-t})(x\sigma(\mathcal{K}^{t}))].$$

Then, from relation $(*)$, we have

$$x\sigma(\mathcal{K}^{\ell}) \subseteq \sigma[\sigma(\mathcal{K}^{\ell-t})\sigma(\mathcal{K}^{e+t})].$$

According to Proposition 2.2,

$$\sigma[\sigma(\mathcal{K}^{\ell-t})\sigma(\mathcal{K}^{e+t})] = \sigma(\mathcal{K}^{\ell-t}\mathcal{K}^{e+t}) = \sigma(\mathcal{K}^{\ell+e}).$$

Thus,

$$x\sigma(\mathcal{K}^{\ell}) \subseteq \sigma(\mathcal{K}^{\ell+e}).$$

Therefore, $x \in \sigma(\mathcal{K}^{\ell+e}) : \sigma(\mathcal{K}^{\ell})$.

Consequently, $\sigma(\mathcal{K}^{e+t}) : \sigma(\mathcal{K}^{t}) \subseteq \sigma(\mathcal{K}^{\ell+e}) : \sigma(\mathcal{K}^{\ell})$. Thus the sequence is increasing.

Since $\sigma(\mathcal{K}^{e+t}) : \sigma(\mathcal{K}^{t})$ is an ideal of A for each $t \in \mathbb{E}_{m_0, n_0}$ and A is Noetherian, the sequence is stationary. □

Lemma 3.2. *Let A be a Noetherian ring*, σ *a semi-prime operation on I*, *and I*, *J two non-zero ideals of A*. *Suppose that there exists* $e \in \mathbb{E}_{m_0, n_0}$ *large enough such that*

$$\sigma(\mathcal{K}^{t+e}) : \mathcal{K}^{t} = \sigma(\mathcal{K}^{e}), \text{ for } e \preceq t.$$

Then the sequence

$$\{Ass(A/\sigma(\mathcal{K}^{\ell}))\}_{\ell\in\mathbb{E}_{m_0, n_0}}$$

is increasing.

Proof. By taking $F = (\mathcal{K}^k)_{k \in \mathbb{E}_{m_0, n_0}}$ as the bifiltration of A and applying Theorem 3.8 of [7], we obtain the result. □

Proposition 3.3. *Let* $\alpha_0 \in \mathbb{E}_{m_0, n_0}$ *be such that for all* $k \in \mathbb{E}_{m_0, n_0}$ *with* $k \succ \alpha_0$ *and for all* $e \in \mathbb{E}_{m_0, n_0}$, $\sigma(\mathcal{K}^{k+e}) : \mathcal{K}^e = \sigma(\mathcal{K}^k)$. *Then for all* $\ell \succ (1, 1)$ *in* $\mathbb{E}_{m_0, n_0}$, $S_\ell^\sigma(I, J) \subseteq Ass(\mathcal{K}_\sigma^\ell / \sigma(\mathcal{K}^\ell))$.

Proof. Let $\ell \succ (1, 1)$ in $\mathbb{E}_{m_0, n_0}$ and assume that there exists

$$P \in S_\ell^\sigma(I, J) = Ass(A/\sigma(\mathcal{K}^\ell)) - \mathcal{A}_\sigma(I, J).$$

Assume that $(A, \mathfrak{M})$ is a local ring, by Theorem 4.5 of [7], $\mathfrak{M} \in Ass(A/\sigma(\mathcal{K}^\ell)) - \mathcal{A}_\sigma(I, J)$. Then there exists $x \in A - \sigma(\mathcal{K}^\ell)$ such that $\mathfrak{M} = \sigma(\mathcal{K}^\ell) : x$. Assume that $\mathcal{K}_\sigma^\ell : x$ is a proper ideal of A. Since $\sigma(\mathcal{K}^\ell) \subseteq \mathcal{K}_\sigma^\ell$, we have

$$\mathfrak{M} = \sigma(\mathcal{K}^\ell) : x \subseteq \mathcal{K}_\sigma^\ell : x \subseteq \mathfrak{M}.$$

Then $\mathfrak{M} = \mathcal{K}_\sigma^\ell : x$ and $\mathfrak{M} \in Ass(A/\mathcal{K}_\sigma^\ell)$. Since the sequence $\{Ass(A/\mathcal{K}_\sigma^\ell)\}_{\ell \in \mathbb{E}_{m_0, n_0}}$ is increasing and stabilizes at $\mathcal{A}_\sigma(I, J)$ (see [7, Theorem 4.5]), $\mathfrak{M} \in \mathcal{A}_\sigma(I, J)$, which is a contradiction. Thus $\mathcal{K}_\sigma^\ell : x = A$ and $x \in \mathcal{K}_\sigma^\ell$. Hence $\mathfrak{M} \in Ass(\mathcal{K}_\sigma^\ell / \sigma(\mathcal{K}^\ell))$.

Assume that A is not local and $P \in S_\ell^\sigma(I, J)$. Consider (A_P, PA_P) which is local. Then by Theorem 4.5 of [7],

$$PA_P \in Ass_{A_P}\left(A_P / \widehat{\sigma_P}\left[(\mathcal{K}A_P)^\ell\right]\right)$$

for α large enough in $\mathbb{E}_{m_0, n_0}$. Since A is a Noetherian ring, A_P is local and with the above, we have

$$PA_P \in Ass_{A_P}\left((\mathcal{K}A_P)^{\ell}_{\widehat{\sigma_P}} / \widehat{\sigma_P}[(\mathcal{K}A_p)^{\ell}]\right).$$

Since

$$(\mathcal{K}A_P)^{\ell}_{\widehat{\sigma_P}} / \widehat{\sigma_P}[(\mathcal{K}A_p)^{\ell}] \simeq (\mathcal{K}^{\ell}_{\sigma} / \sigma(K^{\ell}))A_P,$$

it follows that $P \in Ass(\mathcal{K}^{\ell}_{\sigma} / \sigma(\mathcal{K}^{\ell}))$. □

Proposition 3.4. *Let* σ *be a semi-prime operation. Assume that* $\sigma(\mathcal{K}^{\alpha+\ell}) : \mathcal{K}^{\ell} = \sigma(\mathcal{K}^{\alpha})$ *for all* $\alpha \in \mathbb{E}_{m_0, n_0}$ *large enough and for all* $\ell \in \mathbb{E}_{m_0, n_0}$. *Let* $e = (e_1, e_2) \succ (1, 1)$ *in* $\mathbb{E}_{m_0, n_0}$. *If L is an ideal of A such that* $\mathcal{K}^e \subseteq L \subseteq \mathcal{K}^e_{\sigma}$, *then there exists an integer* $s_0 \geq 1$ *such that* $\sigma(\mathcal{K}^{s_0 e}) = \sigma(L^{s_0})$.

Proof. Let $e = (e_1, e_2) \in \mathbb{E}_{m_0, n_0}$. Then for any integer $s \geq 1$, we have

$$(\mathcal{K}^e)^s = (I^{e_1} J^{e_2})^s = I^{e_1 s} J^{e_2 s} = \mathcal{K}^{s(e_1, e_2)} = \mathcal{K}^{se} \subseteq L^s.$$

Thus $\sigma(\mathcal{K}^{se}) \subseteq \sigma(L^s)$. Conversely, we know that $L \subseteq \mathcal{K}^e_{\sigma}$, so for any integer $s \geq 1$, we have $L^s \subseteq (\mathcal{K}^e_{\sigma})^s \subseteq \mathcal{K}^{se}_{\sigma}$ (see [7, Proposition 4.2]). Since $\mathcal{K}^{se}_{\sigma}$ is σ-closed, we have $\sigma(L^s) \subseteq \mathcal{K}^{se}_{\sigma}$ for any integer $s \geq 1$. We know that if $\sigma(\mathcal{K}^{\alpha+e}) : \mathcal{K}^e = \sigma(\mathcal{K}^{\alpha})$ with $\alpha \in \mathbb{E}_{m_0, n_0}$ and large enough, then $\sigma(\mathcal{K}^{\alpha}) = \sigma(\mathcal{K}^{\alpha}_{\sigma})$ (we refer to Proposition 3.3) and $\sigma(\mathcal{K}^{\alpha}) = \mathcal{K}^{\alpha}_{\sigma}$. Let $s_0 \geq 1$ be an integer such that $s_0 e$ is large enough in $\mathbb{E}_{m_0, n_0}$. Thus $\sigma(\mathcal{K}^{s_0 e}) = \mathcal{K}^{s_0 e}_{\sigma}$. Hence $\sigma(L^{s_0}) \subseteq \sigma(\mathcal{K}^{s_0 e})$. Therefore, there exists an integer $s_0 \geq 1$ such that $\sigma(L^{s_0}) = \sigma(\mathcal{K}^{s_0 e})$. □

Proposition 3.5. *Assume that A is an Artinian ring and* σ *is a semi-prime operation. Let* $\alpha_0 \in \mathbb{E}_{m_0, n_0}$ *be such that for all* $k \in \mathbb{E}_{m_0, n_0}$, $k \succ \alpha_0$

and for all $e \in \mathbb{E}_{m_0, n_0}$, $\sigma(\mathcal{K}^{k+e}) : \mathcal{K}^e = \sigma(\mathcal{K}^k)$. *Then*

$$S_\ell^\sigma(I, J) \subseteq Ass(\mathcal{K}_\sigma^{\ell-r}/\sigma(\mathcal{K}^{\ell+r})),$$

for all ℓ *large enough and for all r such that* $\ell \succ r \succ (1, 1)$ *in* $\mathbb{E}_{m_0, n_0}$.

Proof. For all $\ell \succ (1, 1)$ in $\mathbb{E}_{m_0, n_0}$, we have

$$S_\ell^\sigma(I, J) \subseteq Ass(\mathcal{K}_\sigma^\ell/\sigma(\mathcal{K}^\ell))$$

(we refer to Proposition 3.3). If $P \in Ass(\mathcal{K}_\sigma^\ell/\sigma(\mathcal{K}^\ell))$, then there exist $x \in \mathcal{K}_\sigma^\ell$ and $x \notin \sigma(\mathcal{K}^\ell)$ such that $P = \sigma(\mathcal{K}^\ell) : x$. Since $\mathcal{K}_\sigma^\ell \subseteq \mathcal{K}_\sigma^{\ell-r}$ for all $r \in \mathbb{E}_{m_0, n_0}$ and $\ell \succ r \succ (1, 1)$ (see [7, Theorem 4.6]), it follows that $x \in \mathcal{K}_\sigma^{\ell-r}$ and $x \notin \sigma(\mathcal{K}^\ell)$ such that $P = \sigma(\mathcal{K}^\ell) : x$. Then $P \in Ass(\mathcal{K}_\sigma^{\ell-r}/\sigma(\mathcal{K}^\ell))$ and

$$Ass(\mathcal{K}_\sigma^\ell/\sigma(\mathcal{K}^\ell)) \subseteq Ass(\mathcal{K}_\sigma^{\ell-r}/\sigma(\mathcal{K}^\ell)),$$

for all $r \in \mathbb{E}_{m_0, n_0}$ and $\ell \succ r \succ (1, 1)$. Let $P \in Ass(\mathcal{K}_\sigma^{\ell-r}/\sigma(\mathcal{K}^\ell))$. Then there exist $z \in \mathcal{K}_\sigma^{\ell-r}$ and $z \notin \sigma(\mathcal{K}^\ell)$ such that $P = \sigma(\mathcal{K}^\ell) : z$. Since $\sigma(\mathcal{K}^s) : z \subseteq \sigma(\mathcal{K}^t) : z$ for all $t \prec s$ in $\mathbb{E}_{m_0, n_0}$ and since the ring A is Artinian, we have $P = \sigma(\mathcal{K}^\ell) : z = \sigma(\mathcal{K}^{\ell+r}) : z$ for all ℓ large enough and for all *r* in $\mathbb{E}_{m_0, n_0}$. Thus $P \in Ass(\mathcal{K}_\sigma^{\ell-r}/\sigma(\mathcal{K}^{\ell+r}))$, for all ℓ large enough and for all *r* such that $\ell \succ r \succ (1, 1)$ in $\mathbb{E}_{m_0, n_0}$. □

Proposition 3.6. *Let* $e \in \mathbb{E}_{m_0, n_0} - \{(0, 0)\}$. *Let L be an ideal of A such that* $\mathcal{K}^e \subseteq L \subseteq \mathcal{K}_\sigma^e$. *If* $P \in Ass(\mathcal{K}_\sigma^e/\sigma(L))$, *then* $(\sigma(\mathcal{K}^e) : \mathcal{K}_\sigma^e) \subseteq P$.

Proof. Let *P* be a prime ideal of *A* such that $P \in Ass(\mathcal{K}_\sigma^e/\sigma(L))$. Then there exists $x \in \mathcal{K}_\sigma^e - \sigma(L)$ such that $P = \sigma(L) : x$. We know that $\mathcal{K}^e \subseteq$

$L \subseteq \mathcal{K}^e_\sigma$. Since σ is a semi-prime operation and $\mathcal{K}^e_\sigma$ is σ-closed, it follows that $\sigma(\mathcal{K}^e) \subseteq \sigma(L) \subseteq \sigma(\mathcal{K}^e_\sigma) = \mathcal{K}^e_\sigma$. Let $z \in (\sigma(\mathcal{K}^e) : \mathcal{K}^e_\sigma)$. Then $z\mathcal{K}^e_\sigma \subseteq \sigma(\mathcal{K}^e) \subseteq \sigma(L)$. Since $x \in \mathcal{K}^e_\sigma$ and $x \notin \sigma(L)$, it follows that $zx \in \sigma(L)$ and $z \in \sigma(L) : x = P$.

Thus $z \in P$ and therefore $(\sigma(\mathcal{K}^e) : \mathcal{K}^e_\sigma) \subseteq P$. □

Proposition 3.7. *Let σ be a prime operation on the set of all ideals of A. Let I and J be two ideals of A. Let u and v be two regular elements of A. Then for all $\alpha = (\alpha_1, \alpha_2) \in \mathbb{E}_{m_0, n_0} - \{(0, 0)\}$,*

$$Ass(A/\sigma(I^{\alpha_1}J^{\alpha_2})) \subseteq Ass(A/\sigma((uI)^{\alpha_1}(vJ)^{\alpha_2}))$$

$$= Ass(A/\sigma((uI)^{\alpha_1}(vJ)^{\alpha_2})).$$

We set $I^{\alpha_1}J^{\alpha_2} = \mathcal{K}^\alpha$ and $u^{\alpha_1}v^{\alpha_2}I^{\alpha_1}J^{\alpha_2} = (uI)^{\alpha_1}(vJ)^{\alpha_2} = (uv\mathcal{K})^\alpha$. Thus

$$Ass(A/\sigma(\mathcal{K}^\alpha)) \subseteq Ass(A/\sigma[(uv\mathcal{K})^\alpha]).$$

Proof. Let $P \in Ass(A/\sigma(I^{\alpha_1}J^{\alpha_2})) = Ass(A/\sigma(\mathcal{K}^\alpha))$. Then there exists $x \in A - \sigma(I^{\alpha_1}J^{\alpha_2})$ such that $P = \sigma(I^{\alpha_1}J^{\alpha_2}) : x$.

If $a \in \sigma(I^{\alpha_1}J^{\alpha_2}) : x$, then $ax \in \sigma(I^{\alpha_1}J^{\alpha_2})$ and

$$axu^{\alpha_1}v^{\alpha_2} \in u^{\alpha_1}v^{\alpha_2}\sigma(I^{\alpha_1}J^{\alpha_2}) \subseteq \sigma(u^{\alpha_1}v^{\alpha_2}I^{\alpha_1}J^{\alpha_2}).$$

Hence $a \in \sigma(u^{\alpha_1}v^{\alpha_2}I^{\alpha_1}J^{\alpha_2}) : xu^{\alpha_1}v^{\alpha_2}$.

Conversely, if $b \in \sigma(u^{\alpha_1}v^{\alpha_2}I^{\alpha_1}J^{\alpha_2}) : xu^{\alpha_1}v^{\alpha_2}$, then

$$bxu^{\alpha_1}v^{\alpha_2} \in \sigma(u^{\alpha_1}v^{\alpha_2}I^{\alpha_1}J^{\alpha_2}).$$

Since $\alpha = (\alpha_1, \alpha_2) \in \mathbb{E}_{m_0, n_0} - \{(0, 0)\}$, $u^{\alpha_1}v^{\alpha_2}$ is a regular element of A and σ is a prime operation, it follows that $bx \in \sigma(I^{\alpha_1}J^{\alpha_2})$ and $b \in$

$\sigma(I^{\alpha_1}J^{\alpha_2}) : x$. Therefore,

$$P = \sigma(I^{\alpha_1}J^{\alpha_2}) : x = \sigma(u^{\alpha_1}v^{\alpha_2}I^{\alpha_1}J^{\alpha_2}) : xu^{\alpha_1}v^{\alpha_2}$$

and

$$P \in Ass(A/\sigma(u^{\alpha_1}v^{\alpha_2}I^{\alpha_1}J^{\alpha_2})). \qquad \square$$

Proposition 3.8. *Let* σ *be a prime operation. Let x be a regular element of I and y a regular element of J such that* (x) *is a reduction of I and* (y) *is a reduction of J. Then there exists* $r = (r_1, r_2) \succ (1, 1)$ *in* $\mathbb{N}^2$ *such that*

$$Ass(A/\sigma(\mathcal{K}^{(\eta\alpha_1, r_2\alpha_2)})) \subseteq Ass(A/\sigma(\mathcal{K}^{((\eta+1)\alpha_1, (r_2+1)\alpha_2)})),$$

for all $(\alpha_1, \alpha_2) \succ (1, 1)$ *in* $\mathbb{E}_{m_0, n_0}$.

Proof. We know that (x) is a reduction of *I*, so there exists an integer $r_1 \geq 1$ such that $(x)I^{r_1} = I^{r_1+1}$, and that (y) is a reduction of *J*, so there exists an integer $r_2 \geq 1$ such that $(y)J^{r_2} = J^{r_2+1}$. We set $L = I^{r_1}$ and $V = J^{r_2}$. Let $\alpha = (\alpha_1, \alpha_2) \succ (1, 1)$ in $\mathbb{E}_{m_0, n_0}$. Then

$$\begin{aligned}
& Ass(A/\sigma(\mathcal{K}^{(\eta\alpha_1, r_2\alpha_2)})) \\
&= Ass(A/\sigma(I^{\eta\alpha_1}J^{r_2\alpha_2})) \\
&= Ass(A/\sigma(L^{\alpha_1}V^{\alpha_2})) \\
&\subseteq Ass(A/\sigma(x^{\alpha_1}L^{\alpha_1}y^{\alpha_2}V^{\alpha_2})) \\
&= Ass(A/\sigma((xL)^{\alpha_1}(yV)^{\alpha_2})) \\
&= Ass(A/\sigma((xI^{r_1})^{\alpha_1}(yJ^{r_2})^{\alpha_2})) \\
&= Ass(A/\sigma(I^{(r_1+1)\alpha_1}J^{(r_2+1)\alpha_2})) \\
&= Ass[A/\sigma(\mathcal{K}^{((r_1+1)\alpha_1, (r_2+1)\alpha_2)})]. \qquad \square
\end{aligned}$$

Proposition 3.9. *Let* σ *be a prime operation. Let I and J be two regular ideals of A. Let x* $($*resp.* $y)$ *be a regular element of I* $($*resp.* $J)$ *such that* (x) $($*resp.* $(y))$ *is a reduction of* I $($*resp.* $J)$. *Then there exist two integers* $r_1 \geq 1$ *and* $r_2 \geq 1$ *such that for all integers* $n_1 \geq 1$ *and* $n_2 \geq 1$, $\sigma(\mathcal{K}^{r+n}) = x^{n_1} y^{n_2} \sigma(\mathcal{K}^r)$, *where* $r = (r_1, r_2)$, $n = (n_1, n_2)$ *and* $\mathcal{K}^\ell = I^{\ell_1} J^{\ell_2}$ *for all* $\ell = (\ell_1, \ell_2)$.

Proof. (x) is a reduction of *I*, so there exists an integer $r_1 \geq 1$ such that for any integer $n_1 \geq 1$, we have $(x)^{n_1} I^{r_1} = I^{r_1 + n_1}$. (y) is a reduction of *J*, so there exists an integer $r_2 \geq 1$ such that for any integer $n_2 \geq 1$, we have $(y)^{n_2} J^{r_2} = J^{r_2 + n_2}$. Thus $(x)^{n_1} (y)^{n_2} I^{r_1} J^{r_2} = I^{r_1 + n_1} J^{r_2 + n_2}$.

Since σ is a prime operation and $(x)^{n_1}(y)^{n_2}$ is a regular element, it follows that

$$(x)^{n_1} (y)^{n_2} \sigma(I^{r_1} J^{r_2}) = \sigma(I^{r_1 + n_1} J^{r_2 + n_2}).$$

Consequently, $x^{n_1} y^{n_2} \sigma(\mathcal{K}^r) = \sigma(\mathcal{K}^{r+n})$, where $r = (r_1, r_2)$, $n = (n_1, n_2)$, $\mathcal{K}^r = I^{r_1} J^{r_2}$ and $\mathcal{K}^{r+n} = I^{r_1 + n_1} J^{r_2 + n_2}$. □

3.2. (I, J)**-adic bifiltration and** Δ**-closure**

Let I and J be two non-zero ideals of a Noetherian ring A. Set $\mathcal{K}^\alpha = I^{\alpha_1} J^{\alpha_2}$, for all $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ and

$$\Delta = \{\mathcal{K}^n;\ n = (n_1, n_2) \in \mathbb{E}_{m_0, n_0} - \{(0, 0)\}\}.$$

Δ is a non-empty multiplicatively closed set of non-zero ideals of *A*. For any ideal *L* of *A*, we denote the Δ-closure of *L* by

$$L_\Delta = \bigcup_{V \in \Delta} (LV : V).$$

The Δ-closure is a semi-prime operation on the set of ideals of A (see Proposition 1-(v) of [11]).

Let σ be a semi-prime operation on the set of all ideals of the Noetherian ring A. In Proposition 3.10 and Corollary 3.11, we consider that the bifiltration $(\sigma(\mathcal{K}^n))_{n\in\mathbb{E}_{m_0,n_0}}$ is such that for $s\in\mathbb{E}_{m_0,n_0}$ and s large enough, $\mathcal{K}^m\sigma(\mathcal{K}^s)=\sigma(\mathcal{K}^{m+s})$, for all $m\in\mathbb{E}_{m_0,n_0}$. For the existence of such a bifiltration, one may refer to [16, Theorem 3.1.3].

Proposition 3.10. *Let* $(\sigma(\mathcal{K}^n))_{n\in\mathbb{E}_{m_0,n_0}}$ *be a bifiltration such that for* $s\in\mathbb{E}_{m_0,n_0}$ *and s large enough,*

$$\mathcal{K}^m\sigma(\mathcal{K}^s)=\sigma(\mathcal{K}^{m+s}),$$

for all $m\in\mathbb{E}_{m_0,n_0}$.

Then there exists e in $\mathbb{E}_{m_0,n_0}-\{(0,0)\}$ *such that* $\mathcal{K}_\sigma^\ell=[\sigma(\mathcal{K}^\ell)]_\Delta$ *for all* $\ell\in\mathbb{E}_{m_0,n_0}$ *and* $\ell\succ e$.

Proof. Let $\ell\in\mathbb{E}_{m_0,n_0}-\{(0,0)\}$ and

$$x\in[\sigma(\mathcal{K}^\ell)]_\Delta=\bigcup_{V\in\Delta}(\sigma(\mathcal{K}^\ell)V:V).$$

Since $\Delta=\{\mathcal{K}^n\,|\,\mathcal{K}^n=I^{n_1}J^{n_2},\text{ with } n=(n_1,n_2)\in\mathbb{E}_{m_0,n_0}-\{(0,0)\}\}$, there exists $k\in\mathbb{E}_{m_0,n_0}-\{(0,0)\}$ such that $x\mathcal{K}^k\subseteq\sigma(\mathcal{K}^\ell)\mathcal{K}^k$. Since σ is a semi-prime operation,

$$x\sigma(\mathcal{K}^k)\subseteq\sigma(x\mathcal{K}^k)\subseteq\sigma[\sigma(\mathcal{K}^\ell)\mathcal{K}^k]\subseteq\sigma(\mathcal{K}^{\ell+k}).$$

Thus $x\in\sigma(\mathcal{K}^{\ell+k}):\sigma(\mathcal{K}^k)$. We know that $\{\sigma(\mathcal{K}^{\ell+n}):\sigma(\mathcal{K}^n)\}_{n\in\mathbb{E}_{m_0,n_0}}$ is an increasing sequence that stabilizes at $\mathcal{K}_\sigma^\ell$. Consequently, $x\in\mathcal{K}_\sigma^\ell$ and it follows that $[\sigma(\mathcal{K}^{\ell})]_{\Delta} \subseteq \mathcal{K}_{\sigma}^{\ell}$. Conversely, let $x \in \mathcal{K}_{\sigma}^{\ell}$ and assume that $x \notin [\sigma(\mathcal{K}^{\ell})]_{\Delta}$. Then $x \in \mathcal{K}_{\sigma}^{\ell}$ implies $x\sigma(\mathcal{K}^{n}) \subseteq \sigma(\mathcal{K}^{\ell+n})$ for all n large enough in $\mathbb{E}_{m_0,n_0}$. Thus $x\mathcal{K}^{n} \subseteq \sigma(\mathcal{K}^{\ell+n})$. Also, $x \notin [\sigma(\mathcal{K}^{\ell})]_{\Delta}$, so $x\mathcal{K}^{m} \nsubseteq \sigma(\mathcal{K}^{\ell})\mathcal{K}^{m} = \mathcal{K}^{m}\sigma(\mathcal{K}^{\ell})$, for all $m \in \mathbb{E}_{m_0,n_0} - \{(0,0)\}$. There exists $e \in \mathbb{E}_{m_0,n_0}$ such that for all $s \succ e$ and $s \in \mathbb{E}_{m_0,n_0}$, $\mathcal{K}^{m}\sigma(\mathcal{K}^{s}) = \sigma(\mathcal{K}^{m+s})$, for all $m \in \mathbb{E}_{m_0,n_0} - \{(0,0)\}$. Thus for $\ell \succ e$ and for n large enough in $\mathbb{E}_{m_0,n_0}$, $x\mathcal{K}^{n} \nsubseteq \sigma(\mathcal{K}^{\ell+n})$ and $x\mathcal{K}^{n} \subseteq \sigma(\mathcal{K}^{\ell+n})$. This is a contradiction, therefore we have the equality. □

The following corollary is a consequence of Proposition 3.10.

Corollary 3.11. *Let I and J be two ideals of a Noetherian ring A, and* σ *be a semi-prime operation compatible with the closure operation* Δ *such that for an* $\mathbb{E}_{m_0,n_0}$ *large enough,*

$$\mathcal{K}^{m}\sigma(\mathcal{K}^{s}) = \sigma(\mathcal{K}^{m+s}), \quad \forall m \in \mathbb{E}_{m_0,n_0}.$$

Then there exists $e \in \mathbb{E}_{m_0,n_0}/\{(0,0)\}$ *such that*

(i) $\mathcal{K}_{\sigma}^{\ell} = \sigma[(\mathcal{K}^{\ell})_{\Delta}]$ *for all* $\ell \in \mathbb{E}_{m_0,n_0}$ *with* $e \prec \ell$.

(ii) $(\mathcal{K}_{\sigma}^{\ell})^{n} \subseteq \sigma[(\mathcal{K}^{n\ell})_{\Delta}]$ *for all* $\ell \in \mathbb{E}_{m_0,n_0}$, $e \prec \ell$ *and for any integer* $n \geq 1$.

Proof. (i) Since $(\sigma(\mathcal{K}^{n}))_{n\in\mathbb{E}_{m_0,n_0}}$ is a bifiltration such that for some $s \in \mathbb{E}_{m_0,n_0}$ large enough, we have

$$\mathcal{K}^{m}\sigma(\mathcal{K}^{s}) = \sigma(\mathcal{K}^{m+s}), \quad \forall m \in \mathbb{E}_{m_0,n_0}.$$

According to Proposition 3.10, there exists $e \in \mathbb{E}_{m_0,n_0}\setminus\{(0,0)\}$ such that

$$\mathcal{K}_{\sigma}^{\ell} = \sigma[(\mathcal{K}^{\ell})_{\Delta}], \quad \forall \ell \in \mathbb{E}_{m_0, n_0} \text{ with } e \prec \ell.$$

Hence the result.

(ii) Since $(\mathcal{K}_{\sigma}^{\ell})_{\ell \in \mathbb{E}_{m_0, n_0}}$ is a bifiltration, we have

$$(\mathcal{K}_{\sigma}^{\ell})^{n} \subseteq \mathcal{K}_{\sigma}^{n\ell}.$$

According to (i),

$$\mathcal{K}_{\sigma}^{n\ell} = \sigma[(\mathcal{K}^{n\ell})_{\Delta}].$$

Hence,

$$(\mathcal{K}_{\sigma}^{\ell})^{n} \subseteq \sigma[(\mathcal{K}^{n\ell})_{\Delta}]$$

for all $\ell \in \mathbb{E}_{m_0, n_0}$ with $e \prec \ell$ and for all integers $n \geq 1$. □

Acknowledgements

The authors thank the anonymous reviewers for their valuable suggestions and constructive criticisms that improved the presentation of the paper.

References

[1] M. Brodmann, Asymptotic stability of $Ass(M/I^{n}M)$, Proc. Amer. Math. Soc. 74 (1979), 16-18.

[2] H. Dichi and D. Sangare, Filtrations, asymptotic and prüferian closures, cancellation laws, Proc. Amer. Math. Soc. 113(3) (1991), 617-624.

[3] Essan K. Ambroise, Closure operations on submodules of a module, Annales Mathématiques Africaines 3 (2012), 89-100.

[4] Essan K. Ambroise, Filtrations, closure operations and the sequence $\{Ass(A/(f^{(k)})_{\sigma})\}_{k \in \mathbb{N}}$, Annales Mathématiques Africaines 4 (2013), 117-124.

[5] K. A. Essan, A. Abdoulaye, D. Kamano and E. D. Akeke, σ-sporadic prime ideal and superficial elements, Journal of Algebra and Related Topics 5(2) (2017), 35-45.

[6] K. A. Essan, Filtration, asymptotic σ-prime divisors and superficial elements, Journal of Algebra and Related Topics 9(1) (2021), 159-167.

[7] K. A. Essan and E. D. Akeke, Bifiltrations on a ring and asymptotic σ-prime divisor, JP Journal of Algebra, Number Theory and Applications 60(2) (2023), 69-84.

[8] Y. Idrissa and D. Sangare, On a reduction criterion for bifiltrations in terms of their generalized Rees rings, Annales Mathématiques Africaines 6 (2017), 45-54.

[9] D. Kirby, Closure operations on ideals and submodules, J. London Math. Soc. 44 (1969), 283-291.

[10] S. McAdam, Asymptotic prime divisors, Lecture Notes in Mathematics, Volume 1023, Springer-Verlag, New York, 1983.

[11] S. McAdam, Primes associated to an ideal, Contemporary Mathematics, Amer. Math. Soc., Providence, Vol. 102, 1989.

[12] M. Nagata, Local rings, Interscience Tracts, No. 13, Interscience, New York, 1961.

[13] L. J. Ratliff, Jr., Notes on essentially power filtration, Michigan Math. J. 26 (1973), 313-324.

[14] L. J. Ratliff, Jr., On prime divisors of I^n, n large, Michigan Math. J. 23 (1976), 337-352.

[15] L. J. Ratliff, Jr., On asymptotic prime divisors, Pacific J. Math. 111(2) (1984), 395-413.

[16] L. J. Ratliff, Jr., Residual division, semi-prime operation and a theorem of Sakuma and Okuyama, J. Algebra 140 (1991), 502-518.

[17] M. Traore, H. Dichi and D. Sangare, Bifiltrations, Hilbert-Samuel polynomials, multiplicities, Annales Mathématiques Africaines Serie 1 2 (2011), 7-21.