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SOME COMMON FIXED-POINT THEOREMS FOR FUZZY MAPPINGS LOCALLY DEFINED ON CLOSED BALL IN COMPLETE *G*-METRIC SPACES

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Abstract

This article presents new generalizations of common fixed-point theorems for fuzzy mappings defined locally on closed balls within a complete G-metric space. Our results extend several existing theorems in the literature and provide deeper insights into the behavior of fuzzy mappings in G-metric spaces.

1. Introduction

The beginning of the new millennium marked significant advancements in “computational intelligence” or “soft computing.” These terms were coined in the early 1990s, coinciding with the development of fuzzy set

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theory by Zadeh [11]. Prior to the 20th century, uncertainty was often viewed as undesirable in scientific work, with efforts focused on eliminating it. However, this perspective began to shift in the early 1900s, influenced by the emergence of statistical mechanics. Zadeh's seminal paper introduced a new framework for understanding uncertainty, which laid the foundation for what we now know as fuzzy set theory.

The fixed-point theorem, commonly known as the "Banach Contraction Principle," has been extensively studied for many years. These studies have led to various generalizations and adaptations of the principle, particularly for single-valued and multi-valued mappings. Notable contributions to the study of single-valued mappings include the works of Akram and Mazhar [2], Kalinda [8], Kannan [9], Khan and Kubiaczyk [10], Liu [12], and Rhoades [18]. For multi-valued mappings, significant contributions have been made by Akram and Ajmal [1], Ciric [4], Fisher and Iseki [6], Markin [14], Nadler [16], Nosheen and Akram [17], and Sing and Whitefield [22].

In this paper, we first review key concepts related to fuzzy sets, fuzzy mappings, and the fixed-point theory of fuzzy mappings, building upon previous works [2, 4, 12, 18, 19, 20, 23]. Additionally, we present new results, proving common fixed-point theorems for pairs of fuzzy mappings defined locally on closed balls within complete *G*-metric spaces.

2. Preliminaries

This section presents some basic definitions and results on *G*-metric spaces. Most of the definitions and results are taken from [7, 13, 15].

Definition 2.1 [15]**.** Let X be a nonempty set, and let $G : X \times X \times X \to \mathbb{R}_+$ be a function satisfying the following properties,

(i) $G(x, y, z) = 0$ if $x = y = z$,

(ii) $G(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$,

(iii) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$, with $z \neq y$,

(iv) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (Symmetry in all three variables),

(v) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangular inequality).

The function G is called a *generalized metric* or, more specifically, a *G-metric* on X, and the pair (X, G) is called a *G-metric space*.

Definition 2.2 [15]**.** A G-metric space (X, G) is called *symmetric G-metric* if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Definition 2.3 [15]**.** Let (X, G) be a G-metric space, and (x_n) be a sequence of points of X. Then a point $x \in X$ is said to be the *limit* of the sequence (x_n) if $\lim_{n, m \to \infty} G(x, x_n, x_m) = 0$. If the limit exists, we say that the sequence (x_n) is *G-convergent* to x.

Proposition 2.4 [15]**.** *Let* (X, G) *be a G-metric space. Then the following are equivalent*:

(i) (x_n) *is G-convergent to* x,

(ii) $G(x_n, x_n, x) \to 0$, *as* $n \to \infty$,

(iii) $G(x_n, x, x) \to 0$, *as* $n \to \infty$,

(iv) $G(x_m, x_n, x) \to 0$, *as* $m, n \to \infty$.

Definition 2.5 [13]**.** A sequence (x_n) in a G-metric space (X, G) is said to be *G-Cauchy* if, for every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$, for $n, m, l \geq N$; that is if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Proposition 2.6 [13]**.** *Let* (X, G) *be a G-metric space. Then the following are equivalent*,

(i) (x_n) *is a G-Cauchy*,

(ii) *for* $\varepsilon > 0$, *there exists* $N \in \mathbb{N}$ *such that* $G(x_n, x_m, x_m) < \varepsilon$, *for all* $n, m \geq N$.

Definition 2.7 [15]**.** A G-metric space (X, G) is said to be *G-complete* if every G-Cauchy sequence in (X, G) is G-convergent in X.

Definition 2.8 [15]**.** Let (X, G) and (X', G') be G-metric spaces and let $f : (X, G) \to (X', G')$ be a function. Then f is said to be *G-continuous* at a point $a \in X$ if and only if, given $\varepsilon > 0$, there exists $\delta > 0$ such that $x, y \in X$; and $G(a, x, y) < \delta$ implies $G'(f(a), f(x), f(y)) < \varepsilon$. A function f is G-continuous at X if and only if it is G-continuous at all $a \in X$.

Let us gather some preliminaries. For a nonempty set X, we define the following collections.

$$2^X = \{A : A \subseteq X\}$$

$$CL(2^X) = \{A \in 2^X : A \text{ is nonempty and closed}\}$$

$$C(2^X) = \{A \in 2^X : A \text{ is nonempty and compact}\}$$

$$CB(2^X) = \{A \in 2^X : A \text{ is nonempty, closed and bounded}\}.$$

Let A and B be two bounded and closed elements of $CB(2^X)$,

$$D(A, B, B) = \inf_{a \in A, b \in B} G(a, b, b), \qquad D(a, B, B) = \inf_{b \in B} G(a, B, B). \tag{1}$$

Then the Hausdorff metric is defined as

$$H(A, B, B) = \max\{\sup_{a \in A} D(a, B, B), \sup_{b \in B} D(A, b, b)\}. \tag{2}$$

A fuzzy set A in X is a function with a domain, a universal set X and range $[0, 1]$. We denote by I^X, the collection of all fuzzy sets in X, where the universal set X is a crisp set. If A is a fuzzy set and $x \in X$, then $A(x)$ is the grade of membership of x in A. The α-level subset of a fuzzy set A is denoted by $[A]_\alpha$ and is defined by

$$[A]_\alpha = \{x : A(x) \geq \alpha \text{ if } \alpha \in (0, 1]\},$$

and

$$[A]_0 = \overline{\{x : A(x) > 0\}}.$$

We now define a subcollection $W(X)$ of I^X by

$$W(X) = \{A \in I^X : \text{with } [A]_1 \text{ nonempty and closed}\}.$$

For $A, B \in I^X$, $A \subset B$ means $A(x) \leq B(x)$ for each $x \in X$, and for $A, B \in W(X)$, we define

$$D_1(A, B, B) = H([A]_1, [B]_1, [B]_1). \tag{3}$$

Open ball: Let (X, G) be a G-metric space and $x_0 \in X$. For $r > 0$, we define a G-*open ball* with the center x_0 and having radius r as follows:

$$B_G(x_0, r) = \{x \in X; G(x_0, x, x) < r\}$$

and the *closed ball* with a center x_0 and having radius r as

$$\overline{B_G(x_0, r)} = \{x \in X; G(x_0, x, x) \leq r\}.$$

Fixed point: Let (X, G) be a G-metric space and $T : X \to F(X)$ be a fuzzy mapping. A point $x^* \in X$ is said to be a *fixed point* of T if $\{x^*\} \subseteq T(x^*)$. If $H : X \to F(X)$ is another fuzzy mapping, then a point $x^* \in X$ is said to be a *common fixed point* of T and H provided $\{x^*\} \subseteq T(x^*) \cap H(x^*)$.

Lemma 2.1 [15]**.** *Let (X, G) be a G-metric space, and A, B be two nonempty closed and bounded subsets of X. Then for each $a \in A$,*

$$G(a, b, b) \leq H(A, B, B).$$

Lemma 2.2 [13]**.** *Let* (X, G) *be a G-metric space, and A, B be two nonempty closed and bounded subsets of X. Then for each* $a \in A$, $\varepsilon > 0$, *there exists an element* $b \in B$ *such that*

$$G(a, b, b) \leq H(A, B, B) + \varepsilon.$$

Lemma 2.3 [15]**.** *Let* $x \in X$ *and* $A \in F(X)$. *Then* $\{x\} \subset A$ *if and only if* $D(x, A, A) = 0$.

Lemma 2.4 [15]**.** *Let* $x, y \in X$ *and* $A \in F(X)$. *Then* $D(x, A, A) \leq G(x, y, y) + D(y, A, A)$.

3. Main Theorems

This section presents some fixed-point theorems of fuzzy mappings defined locally on closed balls in *G*-metric spaces.

Theorem 3.1. *Let* (X, G) *be a complete G-metric space,* $x_0 \in X$ *and* $\overline{B_G(x_0, r)}$ *be a closed ball with the center at* x_0 *and radius r. Define* $T : \overline{B_G(x_0, r)} \to W(X)$ *be a fuzzy mapping that satisfies the following conditions*

$$D_1(Tx, Ty, Ty) \leq kG(x, y, y) \quad \text{for all } x, y \in \overline{B_G(x_0, r)}$$

and

$$D(x_0, [Tx_0]_1, [Tx_0]_1) < (1 - k)r \quad \text{for some } k \in [0, 1).$$

Then T possess a fuzzy fixed point, that is, there exists a point $x \in \overline{B_G(x_0, r)}$ *such that* $x \in Tx$.

Proof. Choose $x_1 \in X$ such that $\{x_1\} \subseteq Tx_0$ and $G(x_0, x_1, x_1) < (1 - k)r$. This is possible because $[Tx_0]_1 \neq \emptyset$ and $D(x_0, [Tx_0]_1, [Tx_0]_1) < (1 - k)r$.

Choose $\varepsilon > 0$ such that

$$kG(x_0, x_1, x_1) + \varepsilon < k(1 - k)r.$$

Now choose $x_2 \in X$ such that $\{x_2\} \subseteq Tx_1$ and

$$\begin{aligned} G(x_1, x_2, x_2) &\leq D_1([Tx_0]_1, [Tx_1]_1, [Tx_1]_1) + \varepsilon \\ &\leq kG(x_0, x_1, x_1) + \varepsilon \\ &< k(1 - k)r. \end{aligned}$$

Now, we show that $x_2 \in \overline{B_G(x_0, r)}$. For this, we use property (V) of G-metric.

$$\begin{aligned} G(x_0, x_2, x_2) &\leq G(x_0, x_1, x_1) + G(x_1, x_2, x_2) \\ &\leq (1 - k)r + k(1 - k)r \\ &\leq (1 - k)r[1 + k + k^2 + \cdots] \\ &\leq (1 - k)r\left[\frac{1}{1 - k}\right] = r. \end{aligned}$$

This implies $x_2 \in \overline{B_G(x_0, r)}$. Continuing this way, we obtain a sequence $\{x_n\} \subseteq Tx_{n-1}$ with

$$G(x_n, x_{n-1}, x_{n-1}) < k^{n-1}(1 - k)r \quad \text{for } n = 3, 4, \ldots.$$

This sequence $\{x_n\}$ is a Cauchy in $\overline{B_G(x_0, r)}$. Since $\overline{B_G(x_0, r)}$ is complete, so there exists $x \in \overline{B_G(x_0, r)}$ with $G(x_n, x, x) \to 0$ as $n \to \infty$.

Further, notice that,

$$\begin{aligned} D(x, [Tx]_1, [Tx]_1) &\leq G(x, x_n, x_n) + D(x_n, [Tx]_1, [Tx]_1) \\ &\leq G(x, x_n, x_n) + kG(x_n, x, x) \to 0 \text{ as } n \to \infty \end{aligned}$$

which implies $x \in [Tx]_1$ that means $\{x\} \subseteq Tx$.

Theorem 3.2. *Let* (X, G) *be a complete G-metric space,* $x_0 \in X$ *and* $\overline{B_G(x_0, r)}$ *be a closed ball with the center at* x_0 *and radius* $r > 0$. *Let*

$F, T : \overline{B_G(x_0, r)} \to W(X)$ be two fuzzy mappings. Suppose there exists a constant $k \in [0, 1/2)$ such that

$$D_1(Fx, Ty, Ty) \leq k[D(x, [Ty]_1, [Ty]_1) + D(y, [Fx]_1, [Fx]_1)] \tag{1}$$

for all $x, y \in \overline{B_G(x_0, r)}$ and

$$D(x_0, [Fx_0]_1, [Fx_0]_1) \leq \left(\frac{1-2k}{1-k}\right)r. \tag{2}$$

Then F and T possess a common fuzzy fixed point in closed ball $\overline{B_G(x_0, r)}$, that is, there exists $x^* \in \overline{B_G(x_0, r)}$ such that $\{x^*\} \subseteq Fx^* \bigcap Tx^*$.

Proof. Let us pick $x_1 \in X$ satisfying $\{x_1\} \subseteq Fx_0$ such that

$$G(x_0, x_1, x_1) < \left(\frac{1-2k}{1-k}\right)r. \tag{3}$$

This is possible because $[Fx_0]_1 \neq \varnothing$ and $D(x_0, [Fx_0]_1, [Fx_0]_1) < \left(\frac{1-2k}{1-k}\right)r.$

For simplicity choose $\lambda = \frac{k}{1-k}$. This gives

$$G(x_0, x_1, x_1) < (1-\lambda)r < r.$$

Because $(1-\lambda) < 1$. This implies that $x_1 \in \overline{B_G(x_0, r)}$. Now, choose $\varepsilon > 0$,

$$\lambda G(x_0, x_1, x_1) + \frac{\varepsilon}{1-k} < \lambda(1-\lambda)r. \tag{4}$$

Next, we pick $x_2 \in E$ satisfying $\{x_2\} \subseteq Tx_1$ and

$$\begin{aligned} G(x_1, x_2, x_2) &\leq D_1(Fx_0, Tx_1, Tx_1) + \varepsilon \\ &\leq k[D(x_0, [Tx_1]_1, [Tx_1]_1) + D(x_1, [Fx_0]_1, [Fx_0]_1)] + \varepsilon \\ &\leq k[G(x_0, x_2, x_2) + G(x_1, x_1, x_1)] + \varepsilon \\ &\leq kG(x_0, x_2, x_2) + \varepsilon \quad \text{because } G(x_1, x_1, x_1) = 0. \end{aligned}$$

Using property (V) of G-metric, we get

$$G(x_1, x_2, x_2) \le k[G(x_0, x_1, x_1) + G(x_1, x_2, x_2)] + \varepsilon.$$

By rearranging, we get

$$(1-k)G(x_1, x_2, x_2) \le kG(x_0, x_1, x_1) + \varepsilon.$$

Therefore,

$$\begin{aligned} G(x_1, x_2, x_2) &\le \frac{k}{1-k} G(x_0, x_1, x_1) + \frac{\varepsilon}{1-k} \\ &\le \lambda G(x_0, x_1, x_1) + \frac{\varepsilon}{1-k}. \end{aligned}$$

Using inequality (4), we get

$$G(x_1, x_2, x_2) \le \lambda(1-\lambda)r. \tag{5}$$

Now, we show that $x_2 \in \overline{B_G(x_0, r)}$.

As

$$\begin{aligned} G(x_0, x_2, x_2) &\le G(x_0, x_1, x_1) + G(x_1, x_2, x_2) \\ &\le (1-\lambda)r + \lambda(1-\lambda)r \quad \text{by (4) and (5),} \end{aligned}$$

then we have

$$G(x_0, x_2, x_2) \le (1-\lambda)r(1+\lambda) \le (1-\lambda)r[1 + \lambda + \lambda^2 + \lambda^3 + \cdots] = r.$$

This gives that $x_2 \in \overline{B_G(x_0, r)}$.

Continuing in this way, we can pick a sequence $\{x_n\}$ in $\overline{B_G(x_0, r)}$ such that $\{x_{2k+1}\} \subseteq Fx_{2k}$ and $\{x_{2k+2}\} \subseteq Fx_{2k+1}$ with

$$G(x_{2k+1}, x_{2k+2}, x_{2k+2}) < \lambda^{2k+1}(1-\lambda)r \quad \text{for } k = 0, 1, 2, \ldots.$$

Notice that from the Cauchy root test, the sequence $\{x_n\}$ is Cauchy in $\overline{B_G(x_0, r)}$. As $\overline{B_G(x_0, r)}$ is complete, there exists a point $x^* \in \overline{B_G(x_0, r)}$

such that $\lim_{n\to\infty} x_n = x^*$. It remains to show that $\{x^*\} \subseteq Tx^*$. Consider

$$
\begin{aligned}
& D(x^*, [Tx^*]_1, [Tx^*]_1) \\
& \leq G(x^*, x_{2n+1}, x_{2n+1}) + D(x_{2n+1}, [Tx^*]_1, [Tx^*]_1) \\
& \leq G(x^*, x_{2n+1}, x_{2n+1}) + D_1(Fx_{2n}, [Tx^*]_1, [Tx^*]_1) \\
& \leq G(x^*, x_{2n+1}, x_{2n+1}) + k[D(x_{2n}, [Tx^*]_1, [Tx^*]_1) \\
& \quad + D(x^*, [Fx_{2n}]_1, [Fx_{2n}]_1)] \\
& \leq G(x^*, x_{2n+1}, x_{2n+1}) + kD(x_{2n}, [Tx^*]_1, [Tx^*]_1) \\
& \quad + kG(x^*, x_{2n+1}, x_{2n+1}).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& D(x^*, [Tx^*]_1, [Tx^*]_1) - kD(x_{2n}, [Tx^*]_1, [Tx^*]_1) \\
& \leq G(x^*, x_{2n+1}, x_{2n+1}) + kG(x^*, x_{2n+1}, x_{2n+1}).
\end{aligned}
$$

Taking limit $n \to \infty$, we get, $(1-k)D(x^*, [Tx^*]_1, [Tx^*]_1) \leq 0$. This implies that $\{x^*\} \subseteq Tx^*$. Similarly, using

$$D(x^*, [Fx^*]_1, [Fx^*]_1) \leq G(x^*, x_{2n+1}, x_{2n+1}) + D(x_{2n+1}, [Fx^*]_1, [Fx^*]_1).$$

It can be shown that $\{x^*\} \subseteq Fx^*$. Hence, $\{x^*\} \subseteq Fx^* \cap Tx^*$. That means x^* is a common fixed point of *F* and *T*.

Corollary 3.3. *Let* (X, G) *be a complete G-metric space. Let* $x_0 \in X$ *and* $\overline{B_G(x_0, r)}$ *be a closed ball with the center at* x_0 *and radius* $r > 0$. *A fuzzy mapping* $F : \overline{B_G(x_0, r)} \to W(X)$ *possesses a fuzzy fixed point in* $\overline{B_G(x_0, r)}$ *if there exists a constant* $k \in (0, 1)$ *such that*

$$D_1(Fx, Fy, Fy) \leq k[D(x, [Fy]_1, [Fy]_1) + D(y, [Fx]_1, [Fx]_1)]$$

and

$$D(x_0, [Fx_0]_1, [Fx_0]_1) \leq \left(\frac{1-2k}{1-k}\right)r.$$

Theorem 3.4. *Let* (X, G) *be a complete G-metric space. Let* $x_0 \in X$ *and* $\overline{B_G(x_0, r)}$ *be a closed ball with the center at* x_0 *and radius* $r > 0$. *Two fuzzy mappings* $T_1, T_2 : \overline{B_G(x_0, r)} \to W(X)$ *can have a common fuzzy fixed point in* $\overline{B_G(x_0, r)}$, *if there exists a constant* $k \in (0, 1/2)$ *with*

$$D_1(T_1x, T_2y, T_2y) \leq k \max\left\{\begin{matrix} D(x, [T_1x]_1, [T_1x]_1) + D(y, [T_2y]_1, [T_2y]_1), \\ D(x, [T_1x]_1, [T_1x]_1) + G(x, y, y), \\ D(y, [T_2y]_1, [T_2y]_1) + G(x, y, y) \end{matrix}\right\}$$

for all $x, y \in \overline{B_G(x_0, r)}$ *and* $D(x_0, [T_1x_0]_1, [T_1x_0]_1) \leq (1-\lambda)r$ *holds, where* $\lambda = \max\left(2k, \dfrac{2k}{1-k}\right)$.

Proof. Let us take two points $x_0, x_1 \in X$ such that $\{x_1\} \subseteq T_1x_0$ and

$$G(x_0, x_1, x_1) < (1-\lambda)r. \tag{6}$$

This is possible because T_1x_0 is nonempty and $D(x_0, [T_1x_0]_1, [T_1x_0]_1) \leq (1-\lambda)r$.

Take a positive number $\varepsilon > 0$ such that

$$\lambda G(x_0, x_1, x_1) + \frac{\varepsilon}{1-k} < \lambda(1-\lambda)r. \tag{7}$$

Choose a point $x_2 \in X$, such that $\{x_2\} \subseteq T_2x_1$

$$G(x_1, x_2, x_2)$$

$$\leq D_1(T_1x_0, T_2x_1, T_2x_1) + \varepsilon$$

$$\leq k\max\left\{\begin{array}{l} D(x_0, [T_1x_0]_1, [T_1x_0]_1) + D(x_1, [T_2x_1]_1, [T_2x_1]_1), \\ D(x_0, [T_1x_0]_1, [T_1x_0]_1) + G(x_0, x_1, x_1), \\ D(x_1, [T_2x_1]_1, [T_2x_1]_1) + G(x_0, x_1, x_1) \end{array}\right\} + \varepsilon$$

$$\leq k\max\left\{\begin{array}{l} G(x_0, x_1, x_1) + G(x_1, x_2, x_2), \\ G(x_0, x_1, x_1) + G(x_0, x_1, x_1), \\ G(x_1, x_2, x_2) + G(x_0, x_1, x_1), \end{array}\right\} + \varepsilon.$$

Here are two possible cases:

Case 1.

$$G(x_1, x_2, x_2) \leq k[G(x_0, x_1, x_1) + G(x_1, x_2, x_2)] + \varepsilon.$$

This gives

$$(1-k)G(x_1, x_2, x_2) \leq kG(x_0, x_1, x_1) + \varepsilon$$

$$G(x_1, x_2, x_2) \leq \frac{k}{1-k}G(x_0, x_1, x_1) + \frac{\varepsilon}{1-k}$$

$$\leq \lambda G(x_0, x_1, x_1) + \frac{\varepsilon}{1-k} \leq \lambda(1-\lambda)r \text{ by (7).}$$

Case 2.

$$G(x_1, x_2, x_2) \leq 2kG(x_0, x_1, x_1) + \varepsilon$$

$$\leq \lambda G(x_0, x_1, x_1) + \varepsilon \leq \lambda G(x_0, x_1, x_1) + \frac{\varepsilon}{1-k}$$

$$\leq \lambda(1-\lambda)r \text{ by (7).}$$

Using property (V) of G-metric, and results of Case 1 and Case 2

$$G(x_0, x_2, x_2) \leq G(x_0, x_1, x_1) + G(x_1, x_2, x_2)$$

$$\leq (1-\lambda)r + \lambda(1-\lambda)r$$

$$\leq (1-\lambda)r[1+\lambda]$$

$$\leq (1-\lambda)r[1+\lambda+\lambda^2+\cdots] = r.$$

This implies $x_2 \in \overline{B_G(x_0, r)}$. Continuing in the same way, we can generate a sequence $\{x_n\}$, in $\overline{B_G(x_0, r)}$ with $\{x_{2n+1}\} \subseteq T_1 x_{2n}$ and $\{x_{2n+2}\} \subseteq T_2 x_{2n+1}$ and satisfying the inequality

$$G(x_{2n+1}, x_{2n+2}, x_{2n+2}) < \lambda^{2n+1}(1-\lambda)r, \qquad n = 0, 1, 2, \ldots.$$

From the Cauchy criteria, it is clear that $\{x_n\}$ is a Cauchy sequence in $\overline{B_G(x_0, r)}$. As $\overline{B_G(x_0, r)}$ is complete, there must exist a point z in $\overline{B_G(x_0, r)}$ such that $\{x_n\}$ converges to z.

Consider

$$\begin{aligned}
&D(z, [T_2 z]_1, [T_2 z]_1)\\
&\leq G(z, x_{2n+1}, x_{2n+1}) + D(x_{2n+1}, [T_2 z]_1, [T_2 z]_1)\\
&\leq G(z, x_{2n+1}, x_{2n+1}) + D_1(T_1 x_{2n}, [T_2 z]_1, [T_2 z]_1)\\
&\leq G(z, x_{2n+1}, x_{2n+1})\\
&\quad + k \max\begin{Bmatrix} D(x_{2n}, [T_1 x_{2n}]_1, [T_1 x_{2n}]_1) + D(z, [T_2 z]_1, [T_2 z]_1), \\ D(x_{2n}, [T_1 x_{2n}]_1, [T_1 x_{2n}]_1) + G(x_{2n}, z, z), \\ D(z, [T_2 z]_1, [T_2 z]_1) + G(x_{2n}, z, z) \end{Bmatrix}\\
&\leq G(z, x_{2n+1}, x_{2n+1})\\
&\quad + k \max\begin{Bmatrix} G(x_{2n}, x_{2n+1}, x_{2n+1}) + D(z, [T_2 z]_1, [T_2 z]_1), \\ D(x_{2n}, x_{2n+1}, x_{2n+1}) + G(x_{2n}, z, z), \\ D(z, [T_2 z]_1, [T_2 z]_1) + G(x_{2n}, z, z) \end{Bmatrix}.
\end{aligned}$$

Applying limit as n approaches to ∞, we have

$$\begin{aligned}
&D(z, [T_2 z]_1, [T_2 z]_1)\\
&\leq G(z, z, z) + k \max\begin{Bmatrix} G(z, z, z) + D(z, [T_2 z]_1, [T_2 z]_1), \\ G(z, z, z) + G(z, z, z), \\ D(z, [T_2 z]_1, [T_2 z]_1) + G(z, z, z) \end{Bmatrix}.
\end{aligned}$$

As $G(z, z, z) = 0$

$$D(z, [T_2z]_1, [T_2z]_1) \leq kD(z, [T_2z]_1, [T_2z]_1)$$

which implies

$$(1-k)D(z, [T_2z]_1, [T_2z]_1) \leq 0.$$

That means $D(z, [T_2z]_1, [T_2z]_1) = 0.$ Hence $\{z\} \subseteq T_2z$ and z is a fuzzy fixed point of T_2.

In the same way, by repeating for

$$D(z, [T_1z]_1, [T_1z]_1) \leq G(z, x_{2n+2}, x_{2n+2}) + D(x_{2n+2}, [T_1z]_1, [T_1z]_1).$$

We can show $D(z, [T_1z]_1, [T_1z]_1) = 0,$ which implies $\{z\} \subseteq T_1z$ and z is a fuzzy fixed point of T_1. Hence z is a common fixed point of T_1 and T_2.

Corollary 3.5. *Let (X, G) be a complete G-metric space. Let $x_0 \in X$ and $\overline{B_G(x_0, r)}$ be a closed ball with the center at x_0 and radius $r > 0$. A fuzzy mapping $T : \overline{B_G(x_0, r)} \to W(X)$ possesses a fixed point z in $\overline{B_G(x_0, r)}$ if there exists a constant $k \in \left(0, \frac{1}{2}\right)$ such that*

$$D_1(Tx, Ty, Ty) \leq k \max\begin{Bmatrix} D(x, [Tx]_1, [Tx]_1) + D(y, [Ty]_1, [Ty]_1), \\ D(x, [Tx]_1, [Tx]_1) + G(x, y, y), \\ D(y, [Ty]_1, [Ty]_1) + G(x, y, y) \end{Bmatrix}$$

for all $x, y \in \overline{B_G(x_0, r)}$ and $D(x_0, [Tx_0]_1, [Tx_0]_1) < (1-\lambda)r$ with $\lambda = \max\left(2k, \frac{k}{1-k}\right).$

Proof. In Theorem 3.4, take $T_1 = T_2 = T.$

4. Conclusion

We have demonstrated that two fuzzy mappings, defined locally on a closed ball in a *G*-metric space satisfying a generalized contractive condition possess a common fuzzy fixed point. Furthermore, Theorems 3.2 and 3.4 extend previous results, generalizing the classical fixed-point theorems from single fuzzy mappings to the case of common fixed points for pairs of fuzzy mappings defined locally in a *G*-metric space. These findings contribute to a deeper understanding of fuzzy mappings and their fixed-point properties in *G*-metric spaces.

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