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GEOMETRIC STRUCTURE OF SUM OF THE POWERS OF A NATURAL NUMBER

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Abstract

Sum of the powers of a natural number is proved pictorially via a one-dimensional modification of the two- and three-dimensional representations proposed by Mukherjee in a past study. As examples, sum of the powers of 4 and sum of the powers of 8 are illustrated by drawing an array of rectangles. These diagrams can be regarded as reflecting the self-similarity of the geometric structures of the above-mentioned sums.

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1. Introduction

Mukherjee [1] published a visual proof of sums of the powers of 8,

$$8^n - 1 = 7 \sum_{k=0}^{n-1} 8^k,$$

using a technique in three dimensions. This proof is an extension of prior work [2] on

$$4^n - 1 = 3 \sum_{k=0}^{n-1} 4^k$$

in two dimensions. In this paper, to achieve a one-dimensional modification of the two- and three-dimensional representations, the idea of the visual proofs [3] of

$$2^n - 1 = \sum_{k=0}^{n-1} 2^k$$

and

$$3^n - 1 = 2 \sum_{k=0}^{n-1} 3^k$$

based on self-similarity is applied to the

$$4^n - 1 = 3 \sum_{k=0}^{n-1} 4^k$$

and

$$8^n - 1 = 7 \sum_{k=0}^{n-1} 8^k.$$

Contrary to a previous study [3], an array of rectangles is drawn instead of a tree diagram, since the diagrams are complicated by many branches. As a supplement to the previous proof without words (PWW) [3], this study introduces a method to explain not only sums of the powers of small natural numbers, such as 2 and 3, but also sums of the powers of large natural numbers in the same way. An array of rectangles helps us imagine sum of the powers of an arbitrary natural number, although self-similarity is clearer in a tree diagram than in an array of rectangles.

2. A Visual Proof of Sum of the Powers of a Natural Number

The geometric series is expressed as follows:

$$\sum_{k=0}^{n-1} r^k = \frac{r^n - 1}{r - 1}.$$

The meanings of denominators 3 and 7 of $\sum_{k=0}^{n-1} 4^k = (4^n - 1)/3$ and $\sum_{k=0}^{n-1} 8^k = (8^n - 1)/7$, respectively, can also be observed at a glance by comparing the two diagrams shown in Figures 1 and 2. All of the red rectangles are composed of the same number of rectangles, just as another visual proof [4] is based on self-similarity. Note that the focus is on the number of rectangles in the red rectangle, rather than on the area of the red rectangle. This means a one-dimensional modification of the two- and three-dimensional representations, in the sense that the sum is not expressed by the area of rectangles. The two different diagrams shown in Figures 1 and 2 indicate different ways of counting the number of rectangles. A similar idea can be applied to the visual proof [5] of $\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$.

The diagrams in Figure 1 indicate a pictorial expression,

$$\square + 4^3 = 1 + \square + \square + \square + \square \qquad (1)$$

and thus the relational expression

$$4^3 - 1 = \square + \square + \square \tag{2}$$

holds. As noted above, (1) and (2) do not indicate the relationship between the areas of the rectangles but instead the relationship between the numbers of rectangles. The red rectangles in these expressions represent $4^0 + 4^1 + 4^2$. The right-hand side of (2) indicates that the number of red rectangles is $4 - 1$ in the diagram shown in Figure 1. Therefore, we obtain

$$\sum_{k=0}^{n-1} 4^k = \frac{4^n - 1}{4 - 1}.$$

Similarly, the number 2

$$\sum_{k=0}^{n-1} 8^k = \frac{8^n - 1}{8 - 1}$$

can also be pictorially shown.

3. Concluding Remarks

Even at the high school mathematics level, it is important to cultivate insight into the geometric meaning of geometric series rather than memorizing the formula for the geometric series. When instructing on these pictorial proofs, instructors should not only show Figures 1 and 2, but ought also to confirm, with the expression of a geometric series equation (3) that the geometric series contains the same polynomial [6]:

$$(4^0 + 4^1 + 4^2 + \cdots + 4^{n-1}) + 4^n = 1 + 4(4^0 + 4^1 + 4^2 + \cdots + 4^{n-1}). \tag{3}$$

From a geometric standpoint, the same polynomial $4^0 + 4^1 + 4^2 + \cdots + 4^{n-1}$ is related to self-similarity. Even in understanding PWW, the correspondence between an equation and a diagram is essential, following the pictorial proof of the inequation developed by Khattri [7]. It is also important to have the resourcefulness to draw an array of rectangles instead of a tree for sum of the powers of a large natural number.

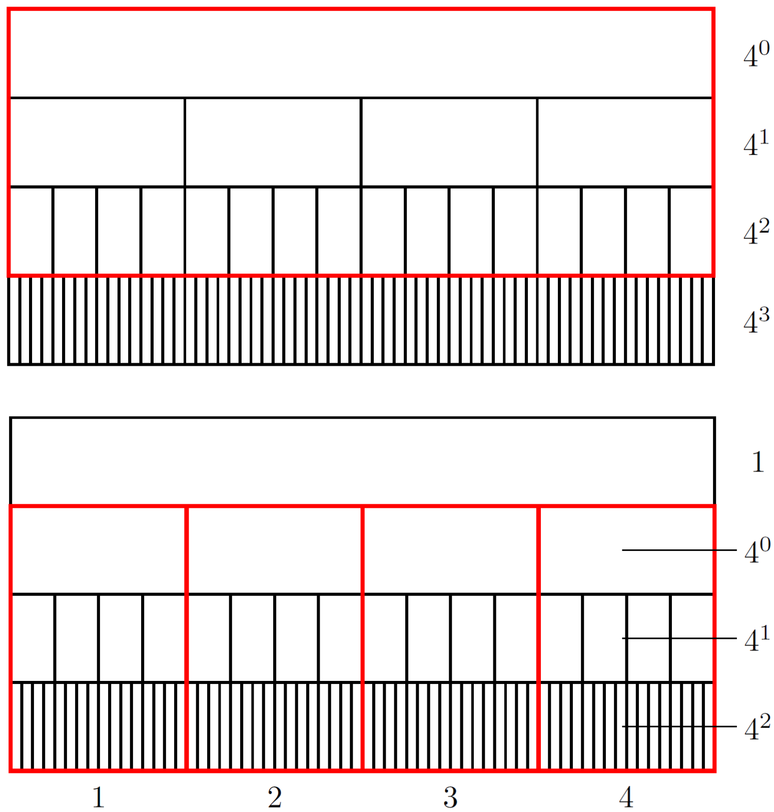
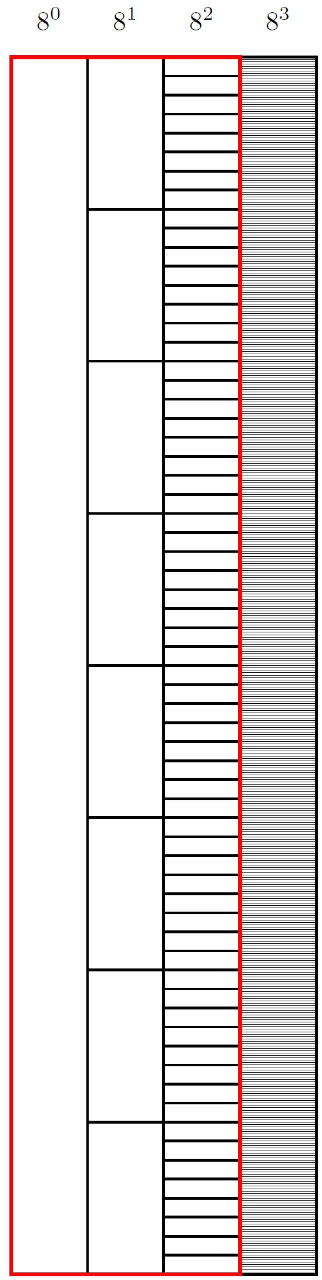


Figure 1. Visual proof of sum of the powers of 4. All the red rectangles are composed of the same number of rectangles. The diagram above is compared with the diagram below. Focus on the internal structure of the red rectangle, rather than on the area of the red rectangle. In the diagram below, 1, 2, 3, and 4 indicate four red rectangles.


8^0 8^1 8^2 8^3

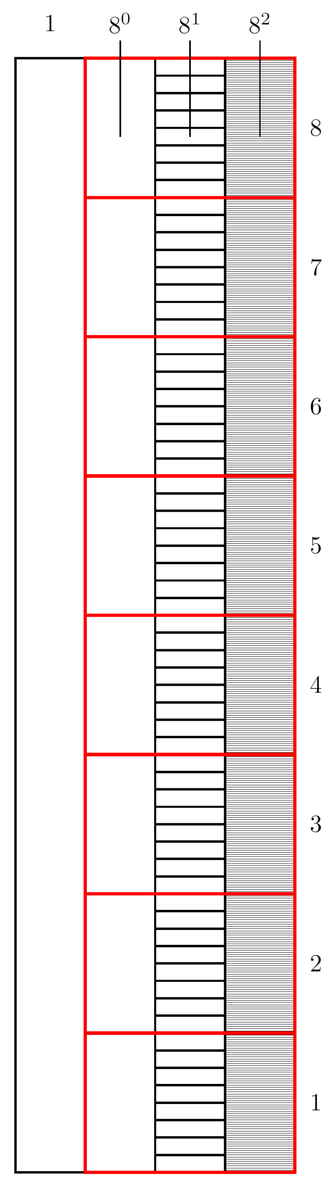


Figure 2. Visual proof of sum of the powers of 8. All of the red rectangles are composed of the same number of rectangles. The diagram above is compared with the diagram below. Focus on the internal structure of the red rectangle, rather than on the area of the red rectangle. In the diagram below, 1, 2, ..., and 8 indicate that there are eight red rectangles.

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