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ON A RECURRENCE RELATION FOR THE SUMS OF POWERS OF INTEGERS

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Abstract

Recently, Thomas and Namboothiri [1] derived a recurrence identity expressing an exponential power sum with negative powers in terms of another exponential power sum with positive powers. From this result, the authors obtained a corresponding recurrence relation for the ordinary power sums $S_k(n) = 1^k + 2^k + \cdots + n^k$. In this short note, we provide an alternative simple proof of the latter recurrence. Our proof is based on the following two ingredients: (i) an expression for $S_k(n+m)$, and (ii) the symmetry property of the power sum polynomials $S_k(n)$.

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1. Introduction

In a remarkable paper [1], Thomas and Namboothiri derived a recurrence identity expressing the exponential power sum $\sum_{s=1}^{k-1} s^p e^{\frac{-2\pi ims}{k}}$ in terms of another exponential power sum with positive powers (see Proposition 2.1 of [1]). As a consequence of this result, the authors obtained a corresponding recurrence relation for the ordinary power sums $S_k(n) = 1^k + 2^k + \cdots + n^k$. Specifically, they showed that (see, equation (8) of [1])

$$\sum_{s=1}^{k-1} s^p = k^p(k-1) + \sum_{a=0}^{p-1} (-1)^{p-a} \binom{p}{a} k^a \sum_{s=1}^{k-1} s^{p-a}.$$

It is easy to see that the above equation is equivalent to the recurrence relation given in Proposition 1.1 below. In Section 2, we provide a simple novel proof of the recurrence in question. In Section 3, we derive a variant of Proposition 1.1. We conclude in Section 4 with some final remarks.

Proposition 1.1. *For integers* $k, n \geq 1$, *let* $S_k(n)$ *denote the power sum* $1^k + 2^k + \cdots + n^k$. *Then,*

$$\sum_{j=0}^{k-1} (-1)^j \binom{k}{j} n^{k-j} S_j(n-1) = \begin{cases} 2S_k(n-1), & \text{if } k \text{ is odd;} \\ 0, & \text{if } k \text{ is even,} \end{cases} \tag{1}$$

where $S_0(n) = n$, *and where it is assumed that* $S_j(0) = 0$ *for all* j.

2. Proof of Proposition 1.1

To prove (1), let us first observe that, as shown in [2], $S_k(n+m)$ can be expressed as

$$S_k(n+m) = S_k(n) + S_k(m) + \sum_{j=0}^{k-1} \binom{k}{j} n^{k-j} S_j(m), \tag{2}$$

because

$$S_k(n+m) = 1^k + 2^k + \cdots + n^k + (n+1)^k + \cdots + (n+m)^k$$

$$= S_k(n) + \sum_{j=0}^{k} \binom{k}{j} n^{k-j} 1^j + \sum_{j=0}^{k} \binom{k}{j} n^{k-j} 2^j$$

$$+ \cdots + \sum_{j=0}^{k} \binom{k}{j} n^{k-j} m^j$$

$$= S_k(n) + \sum_{j=0}^{k} \binom{k}{j} n^{k-j} (1^j + 2^j + \cdots + m^j).$$

Therefore, making the transformations $n \to -n$ and $m \to n-1$, it follows that

$$S_k(-1) = S_k(-n) + S_k(n-1) + \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} n^{k-j} S_j(n-1). \tag{3}$$

On the other hand, by invoking the symmetry property of the power sum polynomials $S_k(n)$ (cf., e.g., Theorem 3 of [3])

$$S_k(-n-1) = -\delta_{k,0} + (-1)^{k+1} S_k(n), \tag{4}$$

we immediately deduce that, for all $k \geq 1$,

$$S_k(-n) = (-1)^{k+1} S_k(n-1). \tag{5}$$

In particular, from (5) we find that $S_k(-1) = 0$ provided $k \geq 1$. Hence, substituting (5) into (3) and taking into account that $S_k(-1) = 0$ for all $k \geq 1$ yields

$$(1 + (-1)^{k+1}) S_k(n-1) = \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} n^{k-j} S_j(n-1), \;\; k \geq 1,$$

which is just the recurrence relation (1).

Remark 2.1. By letting $n = 2$ in (1), we obtain the following identity for $k \geq 1$:

$$\sum_{j=0}^{k-1} \left(-\frac{1}{2}\right)^j \binom{k}{j} = \begin{cases} 1/2^{k-1}, & \text{if } k \text{ is odd;} \\ 0, & \text{if } k \text{ is even.} \end{cases}$$

3. A Variant of Proposition 1.1

It is worthwhile to point out that, by making the transformation $n \to -n$ and using the symmetry property (4), we can express (1) in the alternate form

$$n^k + \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} n^{k-j} S_j(n) = \begin{cases} 2S_k(n), & \text{if } k \text{ is odd;} \\ 0, & \text{if } k \text{ is even.} \end{cases} \tag{6}$$

Let us note, incidentally, that letting $n = 1$ in the preceding equation leads to the well-known binomial identity

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} = 0, \text{ for all } k \geq 1.$$

On the other hand, putting $n = -1$ in (2) and recalling that $S_k(-1) = 0$ for all $k \geq 1$, we obtain

$$m^k = (-1)^{k-1} \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} S_j(m), \ k \geq 1.$$

Thus, renaming m as n in the last equation and substituting the resulting expression for n^k into (6) gives rise to the following variant of (1):

$$\sum_{j=0}^{k-1} (-1)^j \binom{k}{j} (n^{k-j} - (-1)^k) S_j(n) = \begin{cases} 2S_k(n), & \text{if } k \text{ is odd;} \\ 0, & \text{if } k \text{ is even.} \end{cases} \tag{7}$$

For the case where $k \geq 1$ is odd, from (7) we have

$$S_k(n) = \frac{1}{2} \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} (n^{k-j} + 1) S_j(n), \text{ odd } k \geq 1,$$

which should be compared with the formula for $S_k(n)$ that is obtained from (1) when $k \geq 1$ is odd, namely

$$S_k(n) = \frac{1}{2}\sum_{j=0}^{k-1}(-1)^j \binom{k}{j}(n+1)^{k-j} S_j(n), \text{ odd } k \geq 1. \tag{8}$$

Likewise, for the case in which $k \geq 2$ is even, from (7) we obtain the identity

$$\sum_{j=0}^{k-1}(-1)^j \binom{k}{j}(n^{k-j} - 1) S_j(n) = 0, \text{ even } k \geq 2. \tag{9}$$

Notice that (9) allows one to obtain the odd-indexed power sum $S_{k-1}(n)$ in terms of $S_0(n)$, $S_1(n)$, ..., $S_{k-2}(n)$.

4. Concluding Remarks

We end this note with the following remarks.

Remark 4.1. The recurrence relation (1) was already proved by El-Mikkawy and Atlan in Theorem 3.1 (viii) of [4] by manipulating the exponential generating function of $S_k(n)$.

Remark 4.2. In Proposition 8.6.1 of [5], Treeby obtained the following recursive formula

$$S_k(n) = \frac{1}{k-1}\sum_{j=1}^{k-1}(-1)^j \binom{k}{j-1}(n+1)^{k-j} S_j(n), \tag{10}$$

which holds for any odd integer $k \geq 3$.

Remark 4.3. By combining (8) and (10), we obtain in turn the recursive formula

$$S_k(n) = \frac{1}{k+1}\sum_{j=0}^{k-1}(-1)^j \binom{k+1}{j}(n+1)^{k-j} S_j(n), \text{ odd } k \geq 1, \tag{11}$$

which applies to any odd integer $k \geq 1$. Note that, for each $j = 0, 1, \ldots, k-1$, the product $(n+1)^{k-j} S_j(n)$ is a polynomial in n of degree $k+1$.

Example 4.4. When $k = 3$, from (11) it follows that

$$S_3(n) = \frac{1}{4}[(n+1)^3 S_0(n) - 4(n+1)^2 S_1(n) + 6(n+1) S_2(n)]$$

$$= \frac{1}{4}[n(n+1)^3 - 2n(n+1)^3 + n(n+1)^2(2n+1)],$$

which, after simplifying, gives us the well-known result

$$S_3(n) = \frac{1}{4} n^2 (n+1)^2 = (S_1(n))^2.$$

A beautiful proof without words of the identity $S_3(n) = (S_1(n))^2$ can be found in the recent paper [6]. Finally, it should be noted that the formula (11) can also be obtained directly starting from (9). We leave it as an easy exercise for the interested reader to check this fact.

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