



DISTINCTION BETWEEN VECTOR AND COORDINATE SYSTEM INVERSIONS BASED ON MECHANICAL EXAMPLES

Yukio Kobayashi

Department of Information Systems Science
Faculty of Science and Engineering
Soka University
1-236 Tangi-machi, Hachioji-shi
Tokyo 192-8577, Japan
e-mail: koba@t.soka.ac.jp

Abstract

Polar vectors, such as displacement, velocity, and force vectors, and axial vectors, including area, angular momentum, and angular velocity vectors, indicate different conversion rules for their components regarding inversion. The distinction between the inversion of a vector and that of a coordinate system is important; however, they are related through the expressions of the component transformation laws. In both inversion types, the signs of the components of inversed vectors depend on the definition of the vector product. Inversion of the

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angular momentum vector of an axial vector represents another approach for expressing the law of conservation of angular momentum for constant-velocity circular motion.

1. Introduction

Vector analysis is an effective mathematical method used in the field of physics. The vector quantities frequently encountered in this field are position, velocity, momentum, and angular momentum. These physical quantities are divided into polar and axial vectors according to their properties with respect to coordinate transformations, although they share the common characteristics of magnitude, direction, and sense.

In the inversion of the coordinate system, the vector should remain the same, regardless of whether it is axial or polar. However, Ando [1] presented a diagram in which the axial vector is also inverted. Some learners may wonder why both the coordinate systems and vectors are inverted.

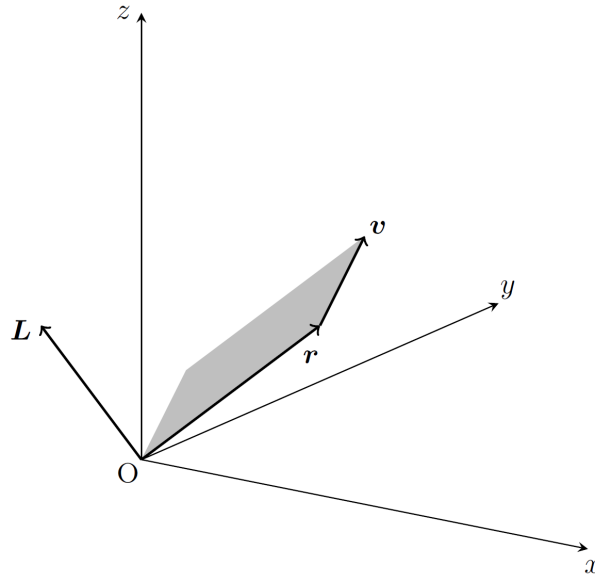


Figure 1. r , v , and L are position, velocity, and angular momentum vectors, respectively. $L = r \times mv$, where m denotes the mass.

In this study, the difference between the inversion of the vector and the inversion of the coordinate system was considered to address this question from the perspective of physics and mathematics education. In coordinate-system inversion, the sign of inversed components depends on the definition of the vector product. In addition to this issue, note that, in contrast to the pole vectors, the inversion of the axis vectors is closely related to the conservation laws of physical quantities.

2. Difference between Vector Inversion and Coordinate System Inversion

2.1. Representation of vector inversion

When mirroring with respect to a plane mirror, the polar and axial vectors change the sign of their components in different ways. Polar vectors change signs, such as line vectors, whereas axial vectors change signs, such as area vectors. The area vector is the vector in which the direction along the circumference of the plane piece is determined, and the magnitude of the area vector is the area of the plane piece. Because the vector product is related to the area vector, it is important to define its direction clearly. The vector product that multiplies \mathbf{a} by \mathbf{b} to the right of \mathbf{a} , as $\mathbf{a} \times \mathbf{b}$, has the same magnitude as the area of the parallelogram formed by \mathbf{a} and \mathbf{b} , and is defined as the vector in the direction in which the screw moves when turning it from \mathbf{a} to \mathbf{b} . In Section 2.4, we consider another definition of the vector product based on the explanation in reference [1].

The image of the line vector perpendicular to the plane mirror has a direction opposite to that of the original line vector. In the case of a plane parallel to the plane mirror, the direction determined along the circumference is not changed by mirror reflection; thus, turning the right-handed screw in this direction does not change the direction of the area vector. The image of the line vector parallel to the mirror has the same direction as the original line vector. In the case of a plane perpendicular to the mirror, the direction of the area vector is opposite, because the direction determined along the circumference is opposite to that determined by the mirror reflection.

The inversion of the vector itself is the mirror reflection for all the respective coordinate planes (xy -, yz -, and zx -planes). The position vector \mathbf{r} and velocity vector \mathbf{v} are polar vectors, and the angular momentum vector \mathbf{L} is an axial vector. Inverting \mathbf{r} , \mathbf{v} , and \mathbf{L} in Figure 1 yields \mathbf{r}' , \mathbf{v}' , and \mathbf{L}' in Figure 2. A comparison of these figures illustrates that the expressions

$$\mathbf{r} = -\mathbf{r}', \quad (1)$$

$$\mathbf{v} = -\mathbf{v}', \quad (2)$$

and

$$\mathbf{L} = \mathbf{L}' \quad (3)$$

holds, where $\mathbf{L} = \mathbf{r} \times m\mathbf{v}$, and $\mathbf{L}' = \mathbf{r}' \times m\mathbf{v}'$. The polarity vectors are opposite in the inversion as shown in equations (1) and (2), whereas the direction of the axial vector is invariant as shown in equation (3), which can also be derived from $\mathbf{r} \times m\mathbf{v} = (-\mathbf{r}') \times (-m\mathbf{v}')$.

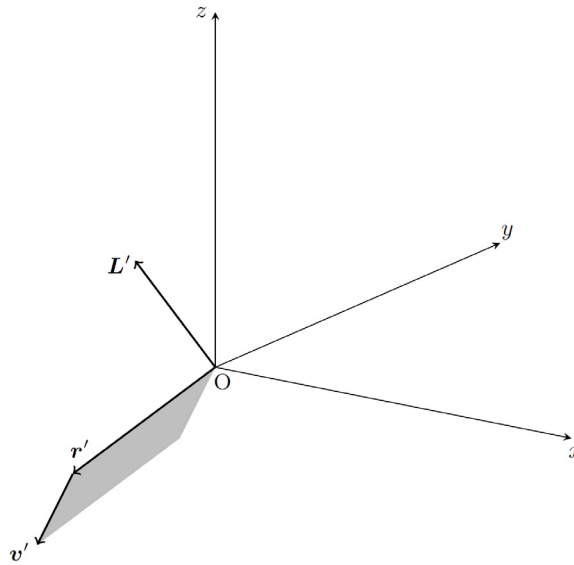


Figure 2. Inverted position, velocity, and angular momentum vectors.

Comparing Figures 1 and 2, equation (3) indicates that the angular momentum is invariant when an object moving in constant-velocity circular

motion passes through positions satisfying relations (1) and (2). Because equation (3) holds at any two positions of relation (1) on a circular orbit, equation (3) represents the law of conservation of angular momentum under equations (1) and (2). However, $m\mathbf{v} \neq m\mathbf{v}'$, that is, the momentum conservation law does not hold. The difference between polar vectors and axial vectors appears in the situation where the direction of motion changes in a constant velocity circular motion, whereas the direction of the axis of rotation does not. The reversal of the polar vectors (position and velocity vectors) is related to the conservation law of the axial vector (angular momentum) in the model of a constant-velocity circular motion of the earth around the sun in celestial mechanics.

2.2. Transformation of the components of a vector by inversion of the coordinate system

If a right-handed coordinate system is inverted, instead of vectors themselves, then it changes to a left-handed coordinate system. The components of the vectors measured in the right-handed coordinate system, (x, y, z) and (v_x, v_y, v_z) , shown in Figure 1, turn into $(\tilde{x}', \tilde{y}', \tilde{z}')$ and $(\tilde{v}'_x, \tilde{v}'_y, \tilde{v}'_z)$, respectively, when measured in the left-handed coordinate system shown in Figure 3. Let the fundamental vectors in the direction of each coordinate axis be $\mathbf{e}_x, \mathbf{e}_y,$ and $\mathbf{e}_z,$ and the fundamental vectors in the direction of each coordinate system after the inversion be $\mathbf{e}'_x, \mathbf{e}'_y,$ and $\mathbf{e}'_z,$ then according to the definition of the vector product determined in Section 2.1,

$$\mathbf{e}'_x = -\mathbf{e}_x, \quad (4)$$

$$\mathbf{e}'_y = -\mathbf{e}_y, \quad (5)$$

and

$$\mathbf{e}'_z = -\mathbf{e}_z \quad (6)$$

hold. Using equations (4), (5), and (6), the vector itself does not change. Thus, the components of the polar vector are:

$$\begin{aligned}
\mathbf{r} &= x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z \\
&= \tilde{x}'\mathbf{e}'_x + \tilde{y}'\mathbf{e}'_y + \tilde{z}'\mathbf{e}'_z \\
&= -\tilde{x}'\mathbf{e}_x - \tilde{y}'\mathbf{e}_y - \tilde{z}'\mathbf{e}_z
\end{aligned} \tag{7}$$

noting $\tilde{x}' = -x$, $\tilde{y}' = -y$, and $\tilde{z}' = -z$. The transformation formula for \mathbf{v} is similar to that in equation (7). For axial vectors, from

$$xv_y - yv_x = (-\tilde{x}') \cdot (-\tilde{v}'_y) - (-\tilde{y}') \cdot (-\tilde{v}'_x), \tag{8}$$

$L_z = x \cdot mv_y - y \cdot mv_x$ and $\tilde{L}'_z = -(\tilde{x}' \cdot m\tilde{v}'_y - \tilde{y}' \cdot m\tilde{v}'_x)$ satisfy $L_z = -\tilde{L}'_z$. Relationships similar to those in equation (8) are also true for other components. In the right-handed coordinate system,

$$\begin{aligned}
\mathbf{L} &= (x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z) \times m(v_x\mathbf{e}_x + v_y\mathbf{e}_y + v_z\mathbf{e}_z) \\
&= L_x\mathbf{e}_x + L_y\mathbf{e}_y + L_z\mathbf{e}_z,
\end{aligned} \tag{9}$$

whereas in the left-handed coordinate system,

$$\begin{aligned}
\mathbf{L} &= (\tilde{x}'\mathbf{e}'_x + \tilde{y}'\mathbf{e}'_y + \tilde{z}'\mathbf{e}'_z) \\
&\quad \times m(\tilde{v}'_x\mathbf{e}'_x + \tilde{v}'_y\mathbf{e}'_y + \tilde{v}'_z\mathbf{e}'_z) \\
&= \tilde{L}'_x\mathbf{e}'_x + \tilde{L}'_y\mathbf{e}'_y + \tilde{L}'_z\mathbf{e}'_z
\end{aligned} \tag{10}$$

using $\mathbf{e}'_x \times \mathbf{e}'_y = -\mathbf{e}'_z$. A comparison of equations (9) and (10) yields the following transformation formula for an axial vector:

$$\begin{aligned}
\mathbf{L} &= L_x\mathbf{e}_x + L_y\mathbf{e}_y + L_z\mathbf{e}_z \\
&= \tilde{L}'_x\mathbf{e}'_x + \tilde{L}'_y\mathbf{e}'_y + \tilde{L}'_z\mathbf{e}'_z \\
&= -\tilde{L}'_x\mathbf{e}_x - \tilde{L}'_y\mathbf{e}_y - \tilde{L}'_z\mathbf{e}_z.
\end{aligned} \tag{11}$$

The positive and negative values of L_z and \tilde{L}'_z differ, because $L_z = -\tilde{L}'_z$. This was also true for the other components.

Even without calculating the vector product, we can write equations (8) and (9) based on Figure 3. Note that the sign of the definition expressions for the components of the axial vectors differs between the right- and left-handed coordinate systems, as shown in $L_z = x \cdot mv_y - y \cdot mv_x$ and $\tilde{L}'_z = -(\tilde{x}' \cdot m\tilde{v}'_y - \tilde{y}' \cdot m\tilde{v}'_x)$. Comparing the components of vectors expressed in equations (1), (2), and (3) yields the relation between the inversion of the vector itself and the inversion of the coordinate system $x' = \tilde{x}'$, $v'_x = \tilde{v}'_x$, and $L'_x = -\tilde{L}'_x$. The same expressions also exists for the y - and z -components. Using these expressions and rewriting the transformation formulas (7) and (10) in the right-handed coordinate system, we can express the difference between the polar and axial vectors.

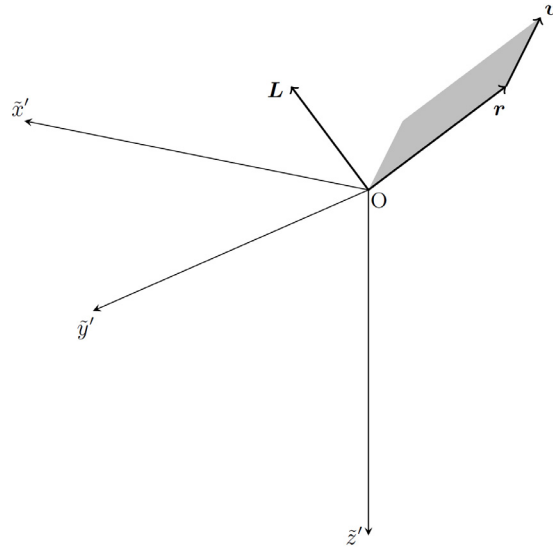


Figure 3. Inversion of the coordinate system.

2.3. Fundamental vectors representing the inversion of a vector

Using equations (1), (2), and (3), the components of inversed vectors can be expressed as

$$\begin{aligned}
 \mathbf{r}' &= x'\mathbf{e}_x + y'\mathbf{e}_y + z'\mathbf{e}_z \\
 &= -x\mathbf{e}_x - y\mathbf{e}_y - z\mathbf{e}_z
 \end{aligned} \tag{12}$$

and

$$\begin{aligned}\mathbf{L}' &= L'_x \mathbf{e}_x + L'_y \mathbf{e}_y + L'_z \mathbf{e}_z \\ &= L_x \mathbf{e}_x + L_y \mathbf{e}_y + L_z \mathbf{e}_z\end{aligned}\quad (13)$$

and \mathbf{v}' can be expressed as equation (12). During inversion, the sign of the polar vector components change, whereas that of the axial vector does not change.

If the fundamental vectors in the direction of each coordinate axis are also inverted, then the transformation formulae expressed by equations (4), (5), and (6) hold true. Using \mathbf{e}'_x , \mathbf{e}'_y , and \mathbf{e}'_z , equations (12) and (13) are rewritten as:

$$\mathbf{r}' = x\mathbf{e}'_x + y\mathbf{e}'_y + z\mathbf{e}'_z, \quad (14)$$

and

$$\mathbf{L}' = -L_x \mathbf{e}'_x - L_y \mathbf{e}'_y - L_z \mathbf{e}'_z, \quad (15)$$

respectively. On inversion, the signs between the polarity and axial vectors are different, as shown in equations (14) and (15). In equations (7) and (10), the vector itself remains unchanged and the coordinate system is inverted, whereas in equations (14) and (15), the vector itself and the coordinate system are inverted in the same manner.

2.4. Differences in inversion depending on the definition of the vector product

When only the coordinate system is inverted, the vector remains unchanged. Thus, $\mathbf{L}' = -\mathbf{L}$ presented in reference [1] does not appear to match those in Figures 1 and 2. Learners may be confused if the angular momentum vectors after the inversion of the coordinate system are expressed by $\mathbf{L}' = -\mathbf{L}$, even though the directions of \mathbf{L} and \mathbf{L}' are the same, as shown in Figures 1 and 2. This indicates that the sign of the transformation formula for the inversion differs depending on the definition of the vector product.

In Section 2.1, a vector product is defined based on its geometric meaning. Therefore, as shown in Section 2.2, the sign of the defining formula for the components of the axial vector differs between the right- and left-handed coordinate systems. However, the sense of a vector itself is not the opposite. The rule of the fundamental vectors is $\mathbf{e}_x \times \mathbf{e}_y = \mathbf{e}_z$ in the right-handed coordinate system, whereas $\mathbf{e}'_x \times \mathbf{e}'_y = -\mathbf{e}'_z$ in the left-handed coordinate system. In contrast, the vector product $\mathbf{a} \times \mathbf{b}$ seems to have been defined as $(a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x)$ in the right-handed coordinate system, and as $(\tilde{a}'_y \tilde{b}'_z - \tilde{a}'_z \tilde{b}'_y, \tilde{a}'_z \tilde{b}'_x - \tilde{a}'_x \tilde{b}'_z, \tilde{a}'_x \tilde{b}'_y - \tilde{a}'_y \tilde{b}'_x)$ in the left-handed coordinate system, although these definitions are not specified in reference [1]. Using these definitions, the signs of the components are identical in both coordinate systems.

Defining vector products in this way, in the case of axial vectors, $L_z = x \cdot mv_y - y \cdot mv_x$, $\tilde{L}'_z = \tilde{x}' \cdot m\tilde{v}'_y - \tilde{y}' \cdot m\tilde{v}'_x$, then $xv_y - yv_x = (-\tilde{x}') \cdot (-\tilde{v}'_y) - (-\tilde{y}') \cdot (-\tilde{v}'_x)$. Thus, $L_z = \tilde{L}'_z$, which indicates that the positive and negative components are the same in the right- and left-handed coordinate systems. Equation (11) does not hold, in contrast to the diagram in Figure 3, because the rightmost sides of equations (9) and (10) represent vectors of opposite senses. The angular momenta \mathbf{L}' in $\mathbf{L}' = -\mathbf{L}$ in reference [1] and \mathbf{L}' in Figures 1 and 2 represent the vectors of opposite senses, although they have the same symbols.

3. Issues Related to the Introduction of Vector Quantities from the Perspective of the Measurement of Vector Quantities

Polar and axial vectors have different properties when the vectors themselves are inversed and when the coordinate system is inversed. Furthermore, differences exist in the measurements when the physical quantities are defined in terms of vectors. Examples are provided from mechanics and electromagnetism.

The directions and senses of polar vectors, such as displacement, velocity, and force, depend on the physical phenomena. The motion of an object and the direction and sensation of the forces acting on it can be measured; thus, they are usually expressed in terms of the directed line segments in these directions and senses. In contrast to these polar vectors, the directions and senses of the axial vectors, such as area, angular momentum, and angular velocity, are human-defined. Angular momentum was introduced to describe the intensity of rotation and was defined as the vector product of position and momentum. The direction of angular momentum is defined as perpendicular to both the position and velocity, although the object does not rotate in this direction.

In electromagnetism, the electric field is a polar vector, whereas the magnetic field and magnetic flux density are axial vectors. When the force acting on a stationary charge is measured, the electric field around that charge is determined as the force per unit charge. However, inconveniences arise in the method for introducing the magnetic flux density from the Lorentz force per unit charge $\mathbf{v} \times \mathbf{B}$, where \mathbf{v} denotes the velocity of the electric charge, and \mathbf{B} denotes the magnetic flux density around the electric charge [2]. Thus, using Bio-Savart's law, the magnetic flux density is expressed using the vector product of the current density and the observed position measured from the origin. Even if the force exerted on the electric charge can be calculated from the change in velocity (i.e., acceleration), the components parallel to the velocity of the electric charge of the magnetic flux density cannot be determined, although the components perpendicular to the velocity can be determined.

4. Problems Related to Angular Momentum

As an example of a physical application, we consider the problem of showing that the product of inertia of a slab is zero. In contrast to the previous sections, the cylindrical coordinate system shown in Figure 4 is convenient. According to a previous study [3], Figure 4 illustrates that the relationships between the cylindrical and Cartesian coordinates for the i -th

mass element are given by:

$$x_i = \rho_i \cos \varphi_i, \quad (16)$$

$$y_i = \rho_i \sin \varphi_i, \quad (17)$$

and

$$z_i = z_i, \quad (18)$$

where ρ_i and φ_i are the radial and azimuthal coordinates of the i -th mass element, respectively. The relationship of the fundamental vectors between the cylindrical and Cartesian fundamental vectors can be represented as:

$$\hat{\rho}_i = \mathbf{e}_x \cos \varphi_i + \mathbf{e}_y \sin \varphi_i, \quad (19)$$

$$\hat{\phi}_i = -\mathbf{e}_x \sin \varphi_i + \mathbf{e}_y \cos \varphi_i. \quad (20)$$

Thus,

$$\frac{d\hat{\rho}_i}{dt} = \hat{\phi}_i \frac{d\varphi_i}{dt}. \quad (21)$$

The position \mathbf{r}_i for the i -th mass element can be expressed as:

$$\mathbf{r}_i = \hat{\rho}_i \rho_i + \mathbf{e}_z z_i \quad (22)$$

and the velocity of the i -th mass element is:

$$\begin{aligned} \mathbf{v}_i &= \frac{d\mathbf{r}_i}{dt} \\ &= \hat{\phi}_i \rho_i \frac{d\varphi_i}{dt} \end{aligned} \quad (23)$$

using equation (21). Using $\hat{\rho}_i \times \hat{\phi}_i = \mathbf{e}_z$ and $\mathbf{e}_z \times \hat{\phi}_i = -\hat{\rho}_i$ obtained from equations (19) and (20), the angular momentum of the i -th element with mass of m_i is

$$\begin{aligned} \mathbf{L}_i &= \mathbf{r}_i \times m_i \mathbf{v}_i \\ &= \mathbf{e}_z m_i \rho_i^2 \omega - \hat{\phi}_i m_i \rho_i z_i \omega \end{aligned} \quad (24)$$

by calculating the vector product of the two vectors expressed by equations (22) and (23), where ω denotes the angular velocity. The total angular momentum of the slab is the sum of the angular momentum of all elements, as represented by equation (24):

$$L = e_z \left(\sum_i m_i \rho_i^2 \right) \omega - \left(\sum_i \hat{\phi}_i m_i \rho_i z_i \right) \omega, \quad (25)$$

where the second term on the right-hand side is zero because of symmetry with respect to the z -axis, as shown in Figure 5, and symmetry with respect to the xy -plane, as shown in Figure 6.

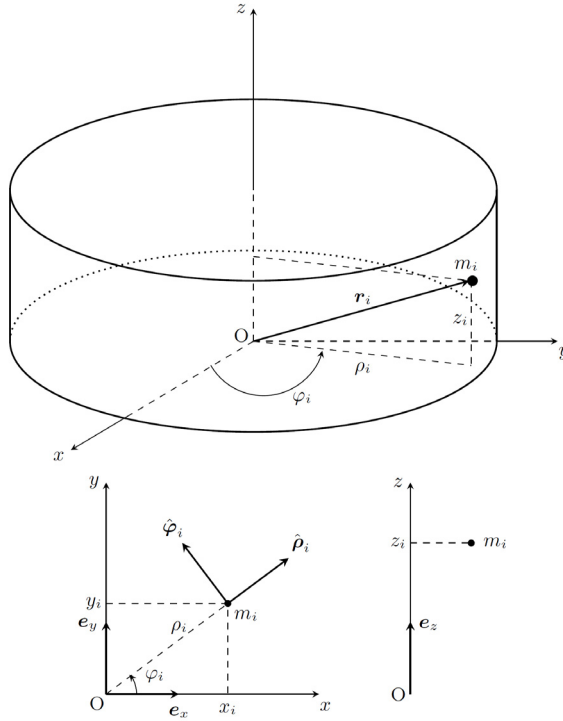


Figure 4. Cylindrical coordinate system. $\hat{\rho}_i$ and $\hat{\phi}_i$ are the cylindrical fundamental vectors, whereas e_x and e_y are the Cartesian fundamental vectors. e_z is the fundamental vector in these coordinate systems. The diagrams below illustrate equations (16), (17), and (18).

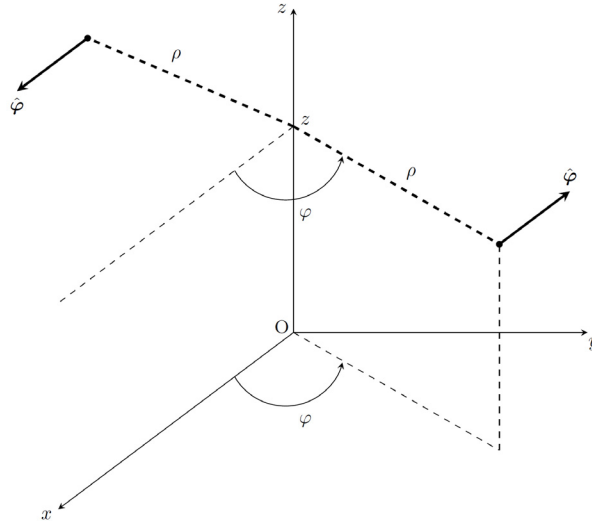


Figure 5. Slab symmetric with respect to the z -axis.

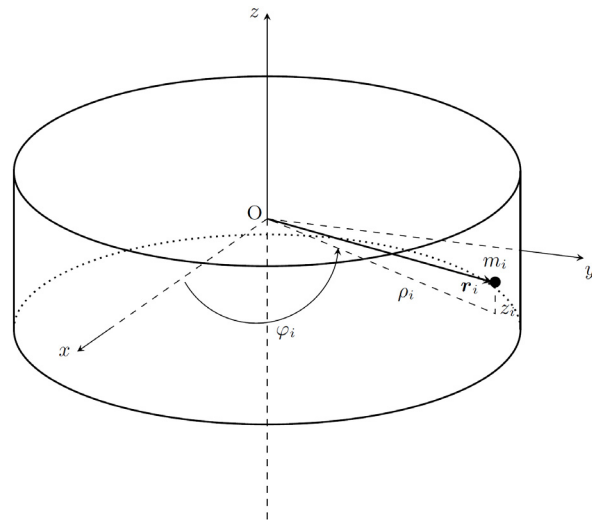


Figure 6. Cylindrical coordinate system with origin at the center of the slab symmetric with respect to the xy -plane. This slab contains the element at $z = -z_i$ for the i -th element at $z = z_i$.

Because the only fundamental vector that represents equation (25) is \mathbf{e}_z , the effect of the inversion can be easily observed. The perpendicular distance

from the z -axis, ρ_i^2 , is invariant under the inversion of \mathbf{r}_i and \mathbf{v}_i . The angular momentum vector under the inversion of the coordinate system is

$$\mathbf{L}' = \mathbf{e}'_z \left(\sum_i m_i \rho_i^2 \right) \omega. \quad (26)$$

Equation (26) indicates $\mathbf{L}' = -\mathbf{L}$ because $\mathbf{e}'_z = -\mathbf{e}_z$.

5. Conclusions

In the transformation laws of the vectors, confusion arises if it is not clear what is inverted. The components of the vectors are inverted and thus differ according to the coordinate system, although the vector itself does not depend on the choice of the coordinate system. If the vector itself is inverted without changing the coordinate system, equations (1), (2), and (3) hold between the inversed vector and the other vectors. If the coordinate system is reversed without changing the vector, equations (7) and (10) hold true for the old and new coordinate systems, respectively. If not only the vector itself but also the coordinate system is inverted in the same manner, equations (13) and (14) hold true. In all these relationships, a difference between the polar and axial vectors can be found.

When a physical quantity related to the extent, such as an area, is represented by a directed line segment, the sign of the components of inversed vectors depends on how the direction is determined. The vector product $\mathbf{r} \times m\mathbf{v}$ is expressed by a directed line segment with the same magnitude as the area of the parallelogram spanned by \mathbf{r} and $m\mathbf{v}$. The axial vectors represented by (polar vector) \times (polar vector) are the vector products of the fundamental vectors. The sign of the components of $\mathbf{r} \times m\mathbf{v}$ depends on how the vector products of fundamental vectors such as $\mathbf{e}_x \times \mathbf{e}_y$ are defined. However, the polar and axial vectors can be distinguished by the difference in the inversion rule, independent of how the vector product is defined.

Some undergraduates are unable to connect the algebraic and geometric aspects of vectors, whereas others lack an adequate understanding of one- and two-dimensional vectors [4, 5]. In such situations, the vector transformation rules discussed in this article are likely to lead to misunderstanding and confusion. It is essential to provide mathematics education for physics that does not impede the acquisition of subjects, such as mechanics and electromagnetism, which require an understanding of physical laws based on the characteristics of physical quantities.

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