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A CONSTRUCTION OF FINITE PROJECTIVE PLANES

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Abstract

We propose a new method to construct a finite projective plane. Its incidence matrix is expressed in the special Paige-Wexler normal form whose lower right part is a circulant block matrix.

1. Introduction

A finite projective plane of order n is a set of $n^2 + n + 1$ lines and $n^2 + n + 1$ points satisfying the following conditions (cf. [6]):

(P1) every line contains $n + 1$ points;

(P2) every point is on $n + 1$ lines;

(P3) any two distinct lines intersect at exactly one point;

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(P4) any two distinct points lie on exactly one line.

In particular, a projective plane in which Desargues' theorem holds is called a *desarguesian projective plane*. A finite projective plane can also be represented by an incidence matrix $A = (A_{ij})$ of order $n^2 + n + 1$ such that $A_{ij} = 1$ if the point i is on the line j and $A_{ij} = 0$ otherwise. Then (P1)-(P4) are translated into the following:

(I1) $\sum_{i=1}^{n^2+n+1} A_{ij} = n + 1$ for $1 \le j \le n^2 + n + 1$;

(I2) $\sum_{j=1}^{n^2+n+1} A_{ij} = n + 1$ for $1 \le i \le n^2 + n + 1$;

(I3) $\sum_{i=1}^{n^2+n+1} A_{ij} A_{ij'} = 1$ for $1 \le j < j' \le n^2 + n + 1$;

(I4) $\sum_{j=1}^{n^2+n+1} A_{ij} A_{i'j} = 1$ for $1 \le i < i' \le n^2 + n + 1$.

The basic example of a finite projective plane is the 2-dimensional projective space $PG(2, q)$ over a finite field $\mathbb{F}_q$ of order q. It is known that $PG(2, q)$ is the only finite desarguesian projective plane (see [3]). The incidence matrices of finite desarguesian projective planes were given by Balbuena [1] via mutually orthogonal Latin squares.

Finite projective planes can be constructed from vector spaces (see [3]), planar ternary rings [4], and mutually orthogonal Latin squares (see [5]). Recently Crnković et al. [2] constructed $PG(2, q^2)$ from the unitary group $PSU(3, q)$. In this paper, we propose a new construction method of finite projective planes using special sequences of points.

2. Preliminaries

Let v_n and o_n be the $1 \times n$ matrices whose every entry is 1 and 0, respectively. Let O_n be the zero matrix of order n, I_n be the identity matrix

of order n, and $V_n(k) = (v_{ij})$ be the matrix of order n such that $v_{ij} = 1$ if $i = k$ and $v_{ij} = 0$ otherwise. We denote by $\mathrm{circ}(x_1, ..., x_n)$ the circulant matrix whose first row is $(x_1 \cdots x_n)$ and by M^t the transpose of a matrix M. Paige and Wexler [7] showed that the incidence matrix of any projective plane of order n can be expressed in the following form:

$$\begin{pmatrix} 1 & v_n & o_n & o_n & \cdots & o_n \\ v_n{}^t & O_n & V_n(1) & V_n(2) & \cdots & V_n(n) \\ o_n{}^t & V_n(1)^t & I_n & I_n & \cdots & I_n \\ o_n{}^t & V_n(2)^t & I_n & C_{11} & \cdots & C_{1n-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ o_n{}^t & V_n(n)^t & I_n & C_{n-11} & \cdots & C_{n-1\,n-1} \end{pmatrix} =: I(C),$$

where $C_{ij}(1 \le i, j \le n-1)$ are permutation matrices of order n and $C = (C_{ij})$.

Now we define

$$Q_m(k) = \mathrm{circ}(\delta_{k+1\,1}, \delta_{k+1\,2}, ..., \delta_{k+1\,m})$$

for $0 \le k \le m-1$ and

$$Q_{m_1, ..., m_r}(k_1, ..., k_r) = Q_{m_1}(k_1) \otimes \cdots \otimes Q_{m_r}(k_r)$$

for $r \ge 1$, where δ_{ij} is the Kronecker delta and $\otimes$ is the Kronecker product. Note that $Q_{m_1, ..., m_r}(0, ..., 0) = I_{m_1 \cdots m_r}$. We denote by $(y)_m$ the least integer x satisfying $x \equiv y \pmod m$ and $x > 0$.

Definition 2.1. Let $S_{m_i} = \{0, 1, ..., m_i - 1\}$, $m_1 \cdots m_r > 2$, and

$$\Omega(m_1, ..., m_r) = \{d_i = (d_{i1}, ..., d_{ir}) \mid d_{i1} \in S_{m_1}, ..., d_{ir} \in S_{m_r}\} \backslash (0, ..., 0).$$

Let $\{d_1, ..., d_{m_1 \cdots m_r - 1}\}$ be an arrangement of $\Omega(m_1, ..., m_r)$. If there exists exactly one pair $(d_i, d_{(i+j)_{m_1 \cdots m_r - 1}})$ such that

$$(d_{(i+j)_{m_1\cdots m_r-1}1} - d_{i\,1} \pmod{m_1}, \ldots, d_{(i+j)_{m_1\cdots m_r-1}r} - d_{i\,r} \pmod{m_r}) = d_k$$

for any $j \in \{1, 2, \ldots, m_1 \cdots m_r - 2\}$ and any $d_k \in \Omega(m_1, \ldots, m_r)$, then we call $\{d_1, \ldots, d_{m_1\cdots m_r-1}\}$ a *complete sequence* of length $m_1 \cdots m_r - 1$.

For $r = 1$ and $m_1 = 2$, we define a complete sequence by $\{d_1 = (1)\}$.

For a complete sequence $\{d_1, \ldots, d_{m_1\cdots m_r-1}\}$, we write

$$I(d_1, \ldots, d_{m_1\cdots m_r-1})$$
$$= I(\mathrm{circ}(Q_{m_1,\ldots,m_r}(d_{11}, \ldots, d_{1r}), \ldots, Q_{m_1,\ldots,m_r}(d_{m_1\cdots m_r-1\,1}, \ldots, d_{m_1\cdots m_r-1\,r}))).$$

We first show the following simple lemma.

Lemma 2.2. *Let j be an integer less than $m_1 \cdots m_r$. Then j is uniquely expressed as*

$$j = a_1 m_2 \cdots m_r + a_2 m_3 \cdots m_r + \cdots + a_{r-1} m_r + a_r, \tag{1}$$

where $0 \le a_i < m_i$ for $1 \le j \le r$.

Proof. Let $m_{r+1} = 1$. Taking $a_1 = \lfloor j/(m_2 \cdots m_r m_{r+1}) \rfloor$ and

$$a_i = \lfloor (j - (a_1 m_2 \cdots m_r m_{r+1} + \cdots + a_{i-1} m_i \cdots m_r m_{r+1}))/(m_{i+1} \cdots m_r m_{r+1}) \rfloor$$

recursively, we obtain (1).

Next, suppose that

$$a_1 m_2 \cdots m_r + \cdots + a_{r-1} m_r + a_r = a_1' m_2 \cdots m_r + \cdots + a_{r-1}' m_r + a_r'.$$

Then we have $a_r \equiv a_r' \pmod{m_r}$, so that $a_r = a_r'$ and

$$a_1 m_2 \cdots m_{r-1} + \cdots + a_{r-2} m_{r-1} + a_{r-1}$$
$$= a_1' m_2 \cdots m_{r-1} + \cdots + a_{r-2}' m_{r-1} + a_{r-1}'.$$

Similarly, we have $a_i = a_i'$ for $1 \le i \le r - 1$. Hence the uniqueness holds. □

The following result is our main tool.

Theorem 2.3. *Let* $n = m_1 \cdots m_r$ *and let* $\{d_1, ..., d_{n-1}\}$ *be a complete sequence. Then* $I(d_1, ..., d_{n-1})$ *is an incidence matrix of a finite projective plane of order* n.

Proof. (I1) and (I2) can be easily verified. Write

$$(c_{ab}) = \mathrm{circ}(Q_{m_1,...,m_r}(d_{11}, ..., d_{1r}), ..., Q_{m_1,...,m_r}(d_{n-11}, ..., d_{n-1\,r})),$$

$$(q(d_k)_{ij}) = Q_{m_1,...,m_r}(d_{k1}, ..., d_{kr}).$$

In order to prove (I3), it suffices to show that

$$\sum_{a=1}^{n^2-n} c_{ab}c_{ab'} = \begin{cases} 0 & \text{if } \lceil b/n \rceil = \lceil b'/n \rceil \text{ or } b \equiv b' \pmod n, \\ 1 & \text{otherwise.} \end{cases}$$

Case 1. $\lceil b/n \rceil = \lceil b'/n \rceil$. Since $\sum_{i=1}^{n} q(d_k)_{ij} q(d_k)_{ij'} = 0$ for $j \neq j'$, we have

$$\sum_{a=1}^{n^2-n} c_{ab}c_{ab'} = \sum_{l=0}^{n-2} \sum_{i=1}^{n} q(d_{(\lceil b/n \rceil - l)_{n-1}})_{ij}\, q(d_{(\lceil b/n \rceil - l)_{n-1}})_{ij'} = 0,$$

where $j = b - n(\lceil b/n \rceil - 1)$ and $j' = b' - n(\lceil b/n \rceil - 1)$.

Case 2. $\lceil b/n \rceil \neq \lceil b'/n \rceil$ and $b \equiv b' \pmod n$. Since

$$\sum_{i=1}^{n} q(d_k)_{ij} q(d_{k'})_{ij} = 0$$

for $k \neq k'$, we have

$$\sum_{a=1}^{n^2-n} c_{ab}c_{ab'} = \sum_{l=0}^{n-2} \sum_{i=1}^{n} q(d_{(\lceil b/n \rceil - l)_{n-1}})_{ij}\, q(d_{(\lceil b'/n \rceil - l)_{n-1}})_{ij} = 0,$$

where $j = b - n(\lceil b/n \rceil - 1) = b' - n(\lceil b'/n \rceil - 1)$.

Case 3. $\lceil b/n \rceil \neq \lceil b'/n \rceil$ and $b \not\equiv b' \pmod n$. Write

$$k = \lceil b'/n \rceil - \lceil b/n \rceil, \quad j = b - n(\lceil b/n \rceil - 1), \quad j' = b' - n(\lceil b'/n \rceil - 1).$$

Without loss of generality, we may assume that $j' > j$. By Lemma 2.2, we can express

$$j' - j = a_1 m_2 \cdots m_r + a_2 m_3 \cdots m_r + \cdots + a_{r-1} m_r + a_r,$$

where $0 \le a_i < m_i$ for $1 \le j \le r$. By Definition 2.1, there exists only one pair $(d_s, d_{(s+k)_{n-1}})$ such that

$$(d_{(s+k)_{n-1}1} - d_{s1} \,(\mathrm{mod}\, m_1), \ldots, d_{(s+k)_{n-1}r} - d_{sr} \,(\mathrm{mod}\, m_r))$$

$$\equiv (a_1 \,(\mathrm{mod}\, m_1), \ldots, a_r \,(\mathrm{mod}\, m_r)).$$

Here we denote by $[y]_m$ the least integer x satisfying $x \equiv y \,(\mathrm{mod}\, m)$ and $x \ge 0$. Writing

$$d = [d_{(s+k)_{n-1}1} - d_{s1}]_{m_1} m_2 \cdots m_r + \cdots + [d_{(s+k)_{n-1}r-1} - d_{s\,r-1}]_{m_{r-1}} m_r + [d_{(s+k)_{n-1}r} - d_{sr}]_{m_r},$$

we have

$$\sum_{i=1}^{n} q(d_s)_{ij}\, q(d_{(s+k)_{n-1}})_{ij'} = \sum_{i=1}^{n} q(d_s)_{ij}\, q(d_s)_{i\ j+(j'-j)-d} = \sum_{i=1}^{n} q(d_s)_{ij}\, q(d_s)_{ij} = 1. \tag{2}$$

Next, let $t \ne s$ and let l be the least integer satisfying

$$d_{(t+k)_{n-1}l} - d_{tl} \not\equiv a_l \,(\mathrm{mod}\, m_l).$$

Write

$$j_0 = \lfloor j/(m_{l+1} \cdots m_r m_{r+1}) \rfloor, \quad j_0' = \lfloor j'/(m_{l+1} \cdots m_r m_{r+1}) \rfloor,$$

$$t_0 = \lfloor t/(m_{l+1} \cdots m_r m_{r+1}) \rfloor, \quad t_0' = \lfloor (t+k)_{n-1}/(m_{l+1} \cdots m_r m_{r+1}) \rfloor,$$

$$d_0 = [d_{t_0'1} - d_{t_0 1}]_{m_1} m_2 \cdots m_l + \cdots + [d_{t_0' l-1} - d_{t_0 l-1}]_{m_{l-1}} m_l + [d_{t_0' l} - d_{t_0 l}]_{m_l},$$

where $m_{r+1} = 1$. Since

$$\begin{aligned}
&\sum_{i=1}^{m_1 \cdots m_l} Q_{m_1, \ldots, m_l}(d_{t_0 1}, \ldots, d_{t_0 l})_{ij_0} Q_{m_1, \ldots, m_l}(d_{t_0' 1}, \ldots, d_{t_0' l})_{ij_0'} \\
&= \sum_{i=1}^{m_1 \cdots m_l} Q_{m_1, \ldots, m_l}(d_{t_0 1}, \ldots, d_{t_0 l})_{ij_0} \\
&\quad \times Q_{m_1, \ldots, m_l}(d_{t_0 1}, \ldots, d_{t_0 l})_{i(j_0 + (j_0' - j_0) - d_0)_{m_1 \cdots m_l - 1}} \\
&= 0,
\end{aligned}$$

we have

$$\sum_{i=1}^{n} q(d_t)_{ij}\, q(d_{(t+k)_{n-1}})_{ij'} = 0. \tag{3}$$

From (2) and (3), it follows that

$$\sum_{a=1}^{n^2-n} c_{ab} c_{ab'} = \sum_{l=0}^{n-2} \sum_{i=1}^{n} q(d_{(\lceil b/n \rceil - l)_{n-1}})_{ij}\, q(d_{(\lceil b/n \rceil + k - l)_{n-1}})_{ij'} = 1.$$

(I4) can be proved in the same way as (I3). □

3. Construction

Throughout this section, p denotes a prime number. It has been conjectured that any finite projective plane has prime power order. If the conjecture is true, then the length of any complete sequence must be of the form $p^r - 1$. Here we give an example of a complete sequence of length $p^r - 1$ for $r \geq 1$.

Lemma 3.1. *Let* $\{\beta, \beta^p, \ldots, \beta^{p^{r-1}}\}$ *be a normal basis of* $\mathbb{F}_{p^r}$ *and* μ *be a generator of* $\mathbb{F}_{p^r} \setminus 0$. *Let* $\mu^i = \sum_{j=0}^{r-1} b_{i\, r-j} \beta^{p^j}$ *and* $b_i = (b_{i1}, \ldots, b_{ir})$ *for*

$1 \le i \le p^r - 1$. *Regard* b_{ij} *as an integer. Then* $\{b_1, ..., b_{p^r-1}\}$ *is a complete sequence*.

Proof. It is obvious for $p = 2$ and $r = 1$. Suppose that $(p, r) \ne (2, 1)$. From

$$\{\mu^{i+j} - \mu^i \mid 1 \le i \le p^r - 1\} = \{\mu, ..., \mu^{p^r-1}\}$$

for $1 \le j \le p^r - 2$, the lemma follows. □

Theorem 2.3 and Lemma 3.1 immediately imply the following result.

Theorem 3.2. $I(b_1, ..., b_{p^r-1})$ *is an incidence matrix of a finite projective plane of order* $p^r - 1$.

We conjecture that the above finite projective plane is desarguesian.

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