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SBA PLUS METHOD TO FIND SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER IN THE SENSE OF CAPUTO

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Abstract

In this article, we focus on the solution of some nonlinear partial differential equations of fractional order in Caputo sense using the SBA plus method.

The SBA plus method is based on the combination of the Adomian decomposition method, Picard's principle and the method of successive approximations. This method uses a process that converges rapidly to the exact solution, when it exists, in the function space, where the problem is posed.

1. SBA Plus Method

Following the work of KABORE et al. [13, 14], we attempt to solve nonlinear partial differential equations of fractional order in the Caputo sense using the SBA plus method. We begin with the description of SBA plus method.

In a Banach space E, we consider the following nonlinear fractional *ODE* problem:

$$\begin{cases} {}^{C}D_t^{\alpha}u = R(u) + N(u), \\ u^{(m)}(0) = \beta_m, \ m = \{0; 1; 2; \ldots; j-1\}, \end{cases} \tag{1.1}$$

where R is a linear operator on E, N is a non-linear operator on E, $u \in E$, ${}^{C}\mathcal{D}_t^{\alpha}$ is the fractional derivative of order α in Caputo's sense, $t \geq 0$; $\alpha > 0$; $u = u(t)$; $u \in C^m([0; T])$, $m \in \mathbb{N}$, $j = [\alpha] + 1$, and $u^{(m)}$ is the derivative of order m of the function u.

Putting

$$\begin{cases} L(\cdot) = {}^{c}\mathcal{D}^{\alpha}(\cdot), \\ L^{-1}(\cdot) = I^{\alpha}(\cdot), \end{cases} \tag{1.2}$$

where L^{-1} is the inverse of L in the Adomian sense and I^{α} is the fractional integral in the Riemann-Liouville sense, we have

$$Lu = Ru + Nu. \tag{1.3}$$

Composing (1.3) with L^{-1}, we obtain

$$L^{-1}Lu = L^{-1}Ru + L^{-1}Nu \tag{1.4}$$

with $L^{-1}R$ as a contracting operator.

On the other hand, we have

$$L^{-1}Lu(t) = u(t) - \sum_{m=0}^{j-1} \frac{t^m}{m!} u^{(m)}(0). \tag{1.5}$$

(1.4), (1.5) and the method of successive approximations give

$$u^k = \gamma^k(t) + L^{-1}Ru^k + L^{-1}Nu^{k-1}, \tag{1.6}$$

with $\gamma^k(t) = \sum_{m=0}^{j-1} \beta_m \frac{t^m}{m!}$.

Solving (1.6) by the method of approximations consists in determining the series of approximate solutions u^k's through iterations.

But this requires a judicious choice of the initial condition u^0 before hand.

Putting

$$u^k = \sum_{n=0}^{+\infty} u_n^k, \tag{1.7}$$

we deduce the following SBA algorithm:

$$\begin{cases} u_0^k = \gamma^k(t) + L^{-1}(N(u^{k-1})), \; k \geq 1, \\ u_{n+1}^k = L^{-1}(R(u_n^k)), \; n \geq 0. \end{cases} \tag{1.8}$$

Explicitly, the development of the algorithm (1.8) consists in first calculating the terms of the sequence $(u_n^k)_n$, and in deducing u^k if the series $u^k = \sum_{n=0}^{+\infty} u_n^k$ converges.

So for the first iteration, $k = 1$, we choose u^0 such that $Nu^0 = 0$, and calculate the terms of the sequence $(u_n^1)_n$. We deduce

$$u^1 = \sum_{n=0}^{+\infty} u_n^1.$$

Next, we evaluate Nu^1. If $Nu^1 = 0$, then u^1 is the general solution of the problem (1.1). Otherwise, if possible, we replace the initial problem by an equivalent transformation, with new non-linear term $\overline{N}$ so that by repeating the algorithm, we can obtain $\overline{N}u^1 = 0$.

1.1. Convergence of the SBA plus algorithm

For convergence, we refer [6, 13].

2. Examples

Example 2.1. Consider the following model:

$$\begin{cases} {}^c\mathcal{D}_t^{\alpha} u = \dfrac{\partial u}{\partial x} + u \dfrac{\partial u}{\partial x} + u - u^2, \\ u(x, 0) = e^x, \\ u'(x, 0) = 0, \end{cases} \tag{2.1}$$

where $t \geq 0$; $1 < \alpha \leq 2$; $u = u(x, t)$; $u \in C^2([0; T])$; ${}^c\mathcal{D}^\alpha(\cdot)$ is the derivative in Caputo sense; $\mathcal{I}^\alpha(\cdot)$ is the integral in the Riemann-Liouville sense.

Putting

$$\begin{cases} L(\cdot) = {}^c\mathcal{D}^\alpha(\cdot), \\ L^{-1}(\cdot) = I^\alpha(\cdot), \\ Ru = \dfrac{\partial u}{\partial x} + u, \\ Nu = u\dfrac{\partial u}{\partial x} - u^2, \end{cases} \tag{2.2}$$

we obtain the algorithm SBA:

$$\begin{cases} u_0^k = u^k(0) + L^{-1}Nu^{k-1}, \\ u_{n+1}^k = L^{-1}R(u_n^k), \end{cases} \quad n \geq 0;\ k \geq 1. \tag{2.3}$$

Using the algorithm (2.3), we obtain the following for the first iteration $k = 1$:

$$\begin{cases} u_0^1 = e^x \\ u_1^1 = (2e^x)\dfrac{t^\alpha}{\Gamma(\alpha+1)} \\ u_2^1 = (4e^x)\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ u_3^1 = (8e^x)\dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ \quad\vdots \\ u_n^1 = (2)^n e^x \dfrac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \end{cases} \tag{2.4}$$

So the approximate solution to the problem (2.1) at the first iteration is given by

$$u^1 = \sum\nolimits_{n=0}^{+\infty} u_n^1 = e^x\mathbb{E}(2t^\alpha). \tag{2.5}$$

For the second iteration $k = 2$, we have

$$\begin{cases} u_0^2 = u^2(0) + L^{-1}Nu^1, \\ u_{n+1}^2 = L^{-1}R(u_n^2), \end{cases} \quad n \geq 0. \tag{2.6}$$

Now,

$$Nu^1 = u^1 \frac{\partial u^1}{\partial x} - (u^1)^2 = e^{2x}\mathbb{E}^2(2t^\alpha) - e^{2x}\mathbb{E}^2(2t^\alpha) = 0.$$

So,

$$\begin{cases} u_0^2 = e^x \\ u_1^2 = (2e^x)\frac{t^\alpha}{\Gamma(\alpha+1)} \\ u_2^2 = (4e^x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ u_3^2 = (8e^x)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ \quad\vdots \\ u_n^2 = (2)^n e^x \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \end{cases} \tag{2.7}$$

Hence the approximate solution to the problem (2.1) at the second iteration is given by

$$u^2 = e^x\mathbb{E}(2t^{\frac{3}{2}}).$$

Recursively, for $k \geq 2$,

$$u^k(x, t) = \sum_{n=0}^{+\infty} u_n^k(x, t) = e^x\mathbb{E}(2t^{\frac{3}{2}}).$$

Consequently, the solution to the problem (2.1) is given by

$$u = \lim_{k\to+\infty} u^k = e^x\mathbb{E}(2t^\alpha).$$

For $\alpha = \frac{4}{3}$, we have $u = e^x\mathbb{E}(2t^{\frac{4}{3}})$.

For $\alpha = \frac{3}{2}$, we have $u = e^x\mathbb{E}(2t^{\frac{3}{2}})$.

Example 2.2. Consider the following fractional order partial differential equation:

$$\begin{cases} {}^c\mathcal{D}_t^{\alpha}u = \frac{\partial}{\partial x}\left(u^2\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) - \frac{\partial u^2}{\partial x} + 2u(1 - 2u - 6u^2), \\ u(x, 0) = e^{2x}, \\ u'(x, 0) = 0, \end{cases} \tag{2.8}$$

where $t \geq 0$; $1 < \alpha \leq 2$; $u = u(x, t)$; $u \in C^2([0; T])$; ${}^c\mathcal{D}^{\alpha}(\cdot)$ is the derivative in Caputo sense; and $\mathcal{I}^{\alpha}(\cdot)$ is the integral in the Riemann-Liouville sense.

Putting

$$\begin{cases} L(\cdot) = {}^c\mathcal{D}^{\alpha}(\cdot), \\ L^{-1}(\cdot) = I^{\alpha}(\cdot), \\ Ru = 2u, \\ Nu = \frac{\partial}{\partial x}\left(u^2\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) - \frac{\partial u^2}{\partial x} - 4u^2 - 12u^3, \end{cases}$$

we obtain

$$\begin{cases} u_0^k = u^k(0) + L^{-1}Nu^{k-1}, \\ u_{n+1}^k = L^{-1}R(u_n^k), \end{cases} \quad n \geq 0;\ k \geq 1. \tag{2.9}$$

Using the algorithm, we obtain the following for the first iteration $k = 1$:

$$\begin{cases} u_0^1 = e^{2x} \\ u_1^1 = 2e^{2x} \dfrac{t^{\alpha}}{\Gamma(\alpha+1)} \\ u_2^1 = 4e^{2x} \dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ \quad \vdots \\ u_n^1 = (2)^n e^{2x} \dfrac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \end{cases} \tag{2.10}$$

Consequently, the approximate solution of the problem (2.8) at the first iteration is given by

$$u^1 = \sum_{n=0}^{+\infty} u_n^1 = e^{2x}\mathbb{E}(2t^{\alpha}). \tag{2.11}$$

For the second iteration $k = 2,$ we have

$$\begin{cases} u_0^2 = u^2(0) + L^{-1}Nu^1, \\ u_{n+1}^2 = L^{-1}R(u_n^2), \end{cases} \quad n \geq 0. \tag{2.12}$$

Now,

$$Nu^1 = \frac{\partial}{\partial x}\left((u^1)^2 \frac{\partial u^1}{\partial x}\right) + \frac{\partial}{\partial x}\left(u^1 \frac{\partial u^1}{\partial x}\right) - \frac{\partial (u^1)^2}{\partial x} - 4(u^1)^2 - 12(u^1)^3$$

$$= 12e^{6x}\mathbb{E}^3(2t^{\alpha}) + 8e^{4x}\mathbb{E}^2(2t^{\alpha}) - 4e^{4x}\mathbb{E}^2(2t^{\alpha})$$

$$- 4e^{4x}\mathbb{E}^2(2t^{\alpha}) - 12e^{6x}\mathbb{E}^3(2t^{\alpha}) = 0.$$

So,

$$\begin{cases} u_0^2 = e^{2x} \\ u_1^2 = 2e^{2x} \dfrac{t^{\alpha}}{\Gamma(\alpha+1)} \\ u_2^2 = 4e^{2x} \dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ \quad \vdots \\ u_n^1 = (2)^n e^{2x} \dfrac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \end{cases} \tag{2.13}$$

Hence the approximate solution of the problem (2.8) at the second iteration is: $u^2 = e^{2x}\mathbb{E}(2t^{\alpha})$.

Recursively, for $k \geq 2$,

$$u^k(x, t) = \sum_{n=0}^{+\infty} u_n^k(x, t) = e^{2x}\mathbb{E}(2t^{\alpha}).$$

Hence the solution to the problem (2.8) is given by

$$u = \lim_{k \to +\infty} u^k = e^{2x}\mathbb{E}(2t^{\alpha}).$$

Example 2.3. Consider the following system:

$$\begin{cases} {}^{c}\mathcal{D}_t^{\beta} u = -\dfrac{\partial^4 u}{\partial x^4} + u^4 \dfrac{\partial^2 u}{\partial x^2} - u^2\left(\dfrac{\partial^2 u}{\partial x^2}\right)^3 - 2\dfrac{\partial^2 u}{\partial x^2}, \\ u(x, 0) = \cos x, \end{cases} \tag{2.14}$$

where $t \geq 0$; $0 < \beta \leq 1$; $u = u(x, t)$; $u \in C^1([0; T])$; ${}^{c}\mathcal{D}^{\beta}(\cdot)$ is the derivative in Caputo sense; and $\mathcal{I}^{\beta}(\cdot)$ is the integral in the Riemann-Liouville sense.

Putting

$$\begin{cases} L(\cdot) = {}^{c}\mathcal{D}^{\beta}(\cdot), \\ L^{-1}(\cdot) = I^{\beta}(\cdot), \\ Ru = -\dfrac{\partial^4 u}{\partial x^4} - 2\dfrac{\partial^2 u}{\partial x^2}, \\ Nu = u^4 \dfrac{\partial^2 u}{\partial x^2} - u^2\left(\dfrac{\partial^2 u}{\partial x^2}\right)^3, \end{cases}$$

we obtain

$$\begin{cases} u_0^k = u^k(0) + L^{-1}Nu^{k-1}, \\ u_{n+1}^k = L^{-1}R(u_n^k), \end{cases} \quad n \geq 0;\ k \geq 1. \tag{2.15}$$

Using the algorithm (2.15), we obtain for the first iteration $k = 1$,

$$\begin{cases} u_0^1 = \cos x \\ u_1^1 = \cos x \dfrac{t^\beta}{\Gamma(\beta+1)} \\ u_2^1 = \cos x \dfrac{t^{2\beta}}{\Gamma(2\beta+1)} \\ \quad \vdots \\ u_n^1 = \cos x \dfrac{t^{n\beta}}{\Gamma(n\beta+1)}. \end{cases} \tag{2.16}$$

Therefore, the approximate solution of the problem (2.14) at the first iteration is given by

$$\begin{aligned} u^1 &= \sum_{n=0}^{+\infty} u_n^1 \\ &= \sum_{n=0}^{+\infty} \cos x \frac{t^{n\beta}}{\Gamma(n\beta+1)} \\ &= \cos x \mathbb{E}(t^\beta). \end{aligned}$$

For the second iteration $k = 2$, we have

$$\begin{cases} u_0^2 = u^2(0) + L^{-1}Nu^1, \\ u_{n+1}^2 = L^{-1}R(u_n^2), \end{cases} \quad n \geq 0. \tag{2.17}$$

Now,

$$\begin{aligned} Nu^1 &= (u^1)^4 \frac{\partial^2 u^1}{\partial x^2} - (u^1)^2 \left(\frac{\partial^2 u^1}{\partial x^2}\right)^3 \\ &= -\cos^5 x\mathbb{E}(t^\beta) + \cos^5 x\mathbb{E}(t^\beta) = 0. \end{aligned}$$

So,

$$\begin{cases} u_0^2 = \cos x \\ u_1^2 = \cos x \dfrac{t^\beta}{\Gamma(\beta+1)} \\ u_2^2 = \cos x \dfrac{t^{2\beta}}{\Gamma(2\beta+1)} \\ \quad \vdots \\ u_n^2 = \cos x \dfrac{t^{n\beta}}{\Gamma(n\beta+1)}. \end{cases} \tag{2.18}$$

Hence the approximate solution to the problem (2.14) at the second iteration is given by

$$u^2 = \cos x \mathbb{E}(t^\beta).$$

Recursively, for $k \geq 2$,

$$u^k(x, t) = \sum_{n=0}^{+\infty} u_n^k(x, t) = \cos x \mathbb{E}(t^\beta).$$

So the solution to the problem (2.14) is given by

$$u = \lim_{k \to +\infty} u^k(x, t), \text{ or}$$

$$u(x, t) = \cos x \mathbb{E}(t^\beta).$$

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