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LARGE DEVIATION FOR SEVERAL FRACTIONAL BROWNIAN MOTIONS AND DIFFUSION PROCESS

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Abstract

We examine the asymptotic behavior of a solution of a differential equation derived by several independent fractional Brownian motions with Hurst index $H \in (0; 1)$. We show the large deviations first for the linear combination of several fractional Brownian motions. Through this first step, the contraction principle allows us to study via the large deviations the behavior of the considered solution.

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0. Introduction

The theory of large deviations is concerned with the exponential decay of probabilities of large fluctuations in random systems. These probabilities are important in many fields including statistics, finance, and engineering, as they often yield valuable information about the large fluctuations of a random system around its most probable state or trajectory. In the context of equilibrium statistical mechanics, the theory of large deviations provides exponential-order estimates of probabilities that refine and generalize Einstein's theory of fluctuations. The behavior of such large fluctuations can be undertaken by stochastic differential equation modeling. In this paper, we consider the following SDE modeling any such system of large fluctuations:

$$Z_t = a_1 B_t^{H_1} + a_2 B_t^{H_2} + \cdots + a_N B_t^{H_N}, \tag{1}$$

a linear combination of several independent fractional Brownian motions B_t^H (fBm):

$$X_t^\varepsilon = x_0 + \int_0^t b(r, X_r^\varepsilon) dr + \varepsilon \int_0^t \sigma(r, X_r^\varepsilon) dZ_r \tag{2}$$

defined on a white noise probability space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}(\mathcal{S}'(\mathbb{R})), \mathbb{P})$, where $\mathcal{S}'(\mathbb{R})$ is a tempered distribution space called *dual of Schwartz space* on which $\mathcal{B}(\mathcal{S}'(\mathbb{R}))$ is a Borel algebra, and

* $x_0 \in \mathcal{S}'(\mathbb{R})$ is a measurable random variable;

* b and $\sigma : [0; T] \times \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ are measurable functions such that their integrals, respectively, with respect to dr and dZ_r are defined as white noise integral [4, 13].

These two functions satisfy the following assumptions:

For g and $h \in \mathcal{S}'(\mathbb{R})$, there exist constants M and L such that

- $| B(h) | \leq M$ and $| \sigma(h) | \leq M$;
- $| b(h) - b(g) | \leq L | h - g |$ and $| \sigma(h) - \sigma(g) | \leq L | h - g |$.

The existence and uniqueness of the solution of these kinds of equations have been proved by several authors among which we have [12] for a stochastic differential equation controlled by a fractional Brownian motion and an independent Brownian motion. However, our goal in this paper is to examine the asymptotic behavior of the solution of equation (2) via the Freidlin-Wentzell's large deviation [10] when ε tends to 0.

Moreover, in the articles [7] and [8], we have established the large deviations principle (LDP) for a solution of a stochastic differential equation directed simultaneously by a fractional Brownian motion and a standard Brownian motion. This paper is therefore a kind of generalization of this last article, by taking another equation derived by several independent fractional Brownian motions of different Hurst index. Like our work [7, 8], we will first treat the case of a combination of several fBm, i.e., when the drift *b* of the SDE is zero and then by using the contraction principle show the LDP of (2). The rest of this paper is organized as follows: Section 2 provides some preliminaries related to the fractional Brownian motion (fBm) and the LDP. Section 3 contains our results divided in two phases, the first one is the LDP for the finite combination of mBf and the second phase is the one for the solution (2).

1. Preliminaries

In this section, we recall some notions and results related to fractional Brownian motion [1, 3, 6, 9, 14].

Definition 1.0.1. Let $N \in \mathbb{N},\ a_1, a_2, ..., a_N \in \mathbb{R}$ and $H_1, H_2, ..., H_N \in (0; 1)$. Then we define

$$Z_t = \sum_{k=1}^{N} a_k B_t^{H_k}, \quad t \in [0; T] \tag{3}$$

with it covariance

$$R(t, s) = \sum_{k=1}^{N} a_k^2 R_{H_k} = \sum_{k=1}^{N} a_k^2 \int_0^t \int_0^s \phi_k(u, r) du dr,$$

where $B_t^{H_k}$'s are independent fractional Brownian motions with hurt parameters H_k and covariances

$$R_{H_k} = \frac{1}{2}(t^{2H_k} + s^{2H_k} - |t-s|^{2H_k}) = \int_0^t \int_0^s \phi_k(u, r)dudr$$

and $\phi_k = \dfrac{\partial^2 R_{H_k}}{\partial t \partial s} = H_k(2H_k - 1)|t-s|^{2H_k - 2}$.

Now, consider $B_t^{H_k}$'s on probability space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}(\mathcal{S}'(\mathbb{R})), \mathbb{P})$, where $\mathcal{S}'(\mathbb{R})$ is the distribution tempered space of Schwartz space $\mathcal{S}(\mathbb{R})$ with the linear product norm $\langle ., . \rangle$.

In $\mathcal{S}'(\mathbb{R})$, $B_t^{H_k}$ can be defined as $B_t^{H_k} = \langle \omega_k, 1_{[0,t]} \rangle$ for all $\omega_k \in \mathcal{S}'(\mathbb{R})$ and $1_{\varepsilon[0,t]} \in \mathcal{S}(\mathbb{R}) \to 1_{[0,t]} \in \mathcal{S}'(\mathbb{R})$, so

$$Z_t = \sum_{k=1}^{N} a_k \langle \omega_k, 1_{[0,t]} \rangle.$$

Definition 1.0.2. Let $B_t^{H_k}$ be a fractional Brownian motion, for $\omega_k \in \mathcal{S}'(\mathbb{R})$ and $f_k : [0, T] \to \mathbb{R}$, a Gaussian process $\langle \omega_k, f_{[0,t]} \rangle = \int_0^t f_k(r) dB_t^{H_k}$ with the covariance:

$$|f_k|_{\phi_k}^2 = \langle f_{k[0,t]}, f_{k[0,s]} \rangle = \int_0^t \int_0^s f_k(u) f_k(r) \phi_k(u, r) dudr.$$

We know that $\mathcal{S}'(\mathbb{R})$ contains the Hilbert's space, so we define

$$L_{\phi_k}^2 = L_{\phi_k}^2(\mathbb{R}) = \{|f_k|_{\phi_k}^2 = \langle f_k, f_k \rangle_{\phi_k} \leq +\infty\} \tag{4}$$

which is the space of continuous functions $f_k : [0, T] \to \mathbb{R}$ of integrable squares.

In $\mathcal{S}'(\mathbb{R})$, Bochner-Minlos establishes the following theorem [3, 9]:

Theorem 1.1. *Let* $\mathbb{P}^{H_k}$ *be the probability measure of* $B_t^{H_k}$ *on* $\mathcal{S}'(\mathbb{R})$. *Then for* $f_k \in \mathcal{S}(\mathbb{R})$ *and* $\omega_k \in \mathcal{S}'(\mathbb{R})$,

$$\mathbb{E}(e^{i(\omega_k, f_k)}) = e^{-\frac{1}{2}| f_k |_{\phi_k}^2}$$

and it follows that $\mathbb{E}(\langle \omega_k, f_k \rangle) = 0$ *and* $\mathbb{E}(\langle \omega_k, f_k \rangle^2) = | f_k |_{\phi_k}^2$.

Moreover, in view of the fractional Brownian motion B^{H_k} irregularities in other spaces, some authors like Bender [4] present some results in $\mathcal{S}'(\mathbb{R})$, where there are no such irregularities. Among these theorems, we have the fractional formulas of Girsanov.

Theorem 1.2 (Fractional Girsanov formula I [14])**.** *Let* $\xi \in L^p(\mathbb{P}^{H_k})$ *for* $p > 1$ *and* $f_k \in L^2_{\phi_k}(\mathbb{R})$. *Then for all* $\gamma_k(t) = \int_0^t f_k(r)\phi_k(r, s)dr$, *the map* $w_k \mapsto \xi(\omega_k + \gamma_k)$ *for all* $q < p$ *and*

$$\int_{\mathcal{S}'(\mathbb{R})} \xi(\omega_k + \gamma_k) d\mathbb{P}^{H_k} = \int_{\mathcal{S}'(\mathbb{R})} \xi(w_k) \exp\left\{\langle \omega_k, f_k \rangle - \frac{1}{2}| f_k |_{\phi_k}^2\right\} d\mathbb{P}^{H_k}.$$

Theorem 1.3 (Fractional Girsanov formula II [9, 14])**.** *Let* γ_k *and g be continuous functions with* $supp\, \gamma_k(t) \subset [0, T]$ *and* $supp\, g \subset [0, T]$ *such that* $\langle g, f_k \rangle = (\gamma_k, l)_{L^2(\mathbb{R})}$ *for all* $f \in \mathcal{S}'(\mathbb{R})$, *i.e.*,

$$\gamma_k(t) = \int_0^t g(r)\phi_k(r, s)dr, \quad 0 \le s \le t \le T.$$

Define a probability measure $\widetilde{\mathbb{P}}^{H_k}$ *on* $\mathcal{S}'(\mathbb{R})$ *by*

$$\frac{d\widetilde{\mathbb{P}}^{H_k}}{d\mathbb{P}^{H_k}} = \exp\left\{-\langle \omega_k, f_k \rangle - \frac{1}{2}| f_k |_{\phi_k}^2\right\}.$$

Then the process defined by $\tilde{\omega}_k = \omega_k + \gamma_k$ *is a fractional Brownian motion under* $\widetilde{\mathbb{P}}^{H_k}$.

In addition to these statements, we rely on the following definition and theorem of large deviations to show, respectively, that the process (3) and the EDS (2) obey the large deviations principle (LDP).

Definition 1.4 (Large deviations principle (LDP) [6]). The family $(X_t^{\varepsilon})_{\varepsilon>0}$ of probabilities $\mathbb{P}^{\varepsilon}$ is said to satisfy *large deviations principle* if there exist a rate function I defined on $\mathbb{L}^2(\mathbb{R})$ and a speed ε tending to 0 such that:

(i) $0 \leq I(h) \leq +\infty$, for all $h \in \mathbb{L}^2(\mathbb{R})$;

(ii) I is lower semi-continuous that is, for all $a < +\infty$, $\{h : I(h) \leq a\}$ is a closed set of $\mathbb{L}^2(\mathbb{R})$;

(iii) for all $a < +\infty$, $\{h : I(h) \leq a\}$ is a compact set of $\mathbb{L}^2(\mathbb{R})$, in which case I is a good rate function;

(iv) for a closed set $C \subset \mathbb{L}^2(\mathbb{R})$,

$$\lim_{\varepsilon \to 0} \sup \varepsilon \log \mathbb{P}^{\varepsilon}(X_t^{\varepsilon} \in C) \leq -\inf_{h \in C} I(h); \tag{5}$$

(v) for an open set $O \subset \mathbb{L}^2(\mathbb{R})$,

$$\lim_{\varepsilon \to 0} \inf \varepsilon \log \mathbb{P}^{\varepsilon}(X_t^{\varepsilon} \in O) \geq -\inf_{h \in O} I(h). \tag{6}$$

Theorem 1.5 (Contraction principle [6]). *Let E_1 and $E_2 \subset \mathbb{L}^2(\mathbb{R})$ and $g : E_1 \to E_2$ be a continuous function. If the family $(X_t^{\varepsilon})_{\varepsilon>0}$ satisfies the large deviations principle of a rate function I, then the family $g((X_t^{\varepsilon})_{\varepsilon>0})$ satisfies the LDP on E_2 of the rate function J defined by*

$$J(z) = \inf\{I(h) : h \in E_1,\ z = g(h)\},$$

for each $z \in E_2$.

2. Main Results

2.1. Large deviation for a linear combination of factional Brownian motion

Consider the family $Z_t^{\varepsilon} = \varepsilon Z_t = \sum_{k=1}^{N} \varepsilon a_k B_t^{H_k}$ of the probability measures $\mathbb{P}$, where Z_t is defined in (3) and $f_k : [0, T] \to \mathbb{R}$ is a bounded and continuous function. We denote by $\mathbb{P}^{H_k}$ the probability measure of $B_t^{H_k}$ and $\mathbb{L}_{\phi}^2 = L_{\phi_k}^2 \times \cdots \times L_{\phi_N}^2$.

Theorem 2.1. *The family* $(Z_t^{\varepsilon})_{\varepsilon > 0}$ *satisfies the LDP with the good rate function* $I : \mathbb{L}_{\phi_k}^2 \to [0; +\infty]$ *given by*

$$I(f_k) = \begin{cases} \dfrac{1}{2} \displaystyle\sum_{k=1}^{N} a_k^2 | f_k |_{\phi_k}^2, & \text{if } f_k \in L_{\phi_k}^2, \\ +\infty, & \text{otherwise.} \end{cases} \tag{7}$$

In other words:

$*$ *I is lower semi-continuous on* $\mathbb{L}_{\phi_k}^2$.

$*$ $\{f_k \in L_{\phi_k}^2,\ a \in \mathbb{R}_+^*,\ I(f_k) \le a\}$ *is a compact subset of* $\mathbb{L}_{\phi_k}^2$.

$*$ *For a closed set* $C \subset \mathbb{L}_{\phi_k}^2$ *and an open set* $O \subset \mathbb{L}_{\phi_k}^2$,

$$\lim_{\varepsilon \to 0} \sup \varepsilon^2 \log \mathbb{P}^{\varepsilon}(Z_t^{\varepsilon} \in C) \le -I(f_k) \le \lim_{\varepsilon \to 0} \inf \varepsilon^2 \log \mathbb{P}^{\varepsilon} (Z_t^{\varepsilon} \in O).$$

Proof. We determine the lower and upper bounds of the large deviations. Since $B_t^{H_k}$'s are independent, we have $\dfrac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \dfrac{d\tilde{\mathbb{P}}^{H_1, \varepsilon}}{d\mathbb{P}^{H_1, \varepsilon}} \times \cdots \times \dfrac{d\tilde{\mathbb{P}}^{H_N, \varepsilon}}{d\mathbb{P}^{H_N, \varepsilon}}$ [11]. By Theorem 1.3, it follows that

$$\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}} = \exp\left\{\frac{a_1}{\varepsilon}\langle\omega_1,\, f_1\rangle + \frac{a_1^2}{2\varepsilon^2}| f_1 |_{\phi_1}^2\right\}$$

$$\times\cdots\times\exp\left\{\frac{a_N}{\varepsilon}\langle\omega_N,\, f_N\rangle + \frac{a_N^2}{2\varepsilon^2}| f_N |_{\phi_N}^2\right\}.$$

Lower bound. Let O be an open subset of $L_{\phi_k}^2$. Then for $\varepsilon > 0$ and $\varepsilon \to 0,$ we have for $f_k^\varepsilon \in \mathcal{S}(\mathbb{R})$ and $\omega_k \in \mathcal{S}'(\mathbb{R}),$

$$\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}} = \exp\left\{\frac{a_1}{\varepsilon}\langle\omega_1,\, f_1^\varepsilon\rangle + \frac{a_1^2}{2\varepsilon^2}| f_1^\varepsilon |_{\phi_1}^2\right\}$$

$$\times\cdots\times\exp\left\{-\frac{a_N}{\varepsilon}\langle\omega_N,\, f_N^\varepsilon\rangle + \frac{a_N^2}{2\varepsilon^2}| f_N^\varepsilon |_{\phi_N}^2\right\}.$$

Now, $\omega_k = \tilde{\omega}_k - \frac{1}{\varepsilon}\gamma_k^\varepsilon$ with $\gamma_k^\varepsilon = a_k\int_0^t f_k^\varepsilon(r)\phi_k(r,\, s)dr,$ and

$$\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}} = \exp\left\{\frac{a_1}{\varepsilon}\langle\tilde{\omega}_1,\, f_1^\varepsilon\rangle - \frac{a_1}{\varepsilon^2}\langle\gamma_1^\varepsilon,\, f_1^\varepsilon\rangle + \frac{a_1^2}{2\varepsilon^2}| f_1^\varepsilon |_{\phi_1}^2\right\}$$

$$\times\cdots\times\exp\left\{\frac{a_N}{\varepsilon}\langle\tilde{\omega}_N,\, f_N^\varepsilon\rangle - \frac{a_N}{\varepsilon^2}\langle\gamma_N^\varepsilon,\, f_N^\varepsilon\rangle + \frac{a_N^2}{2\varepsilon^2}| f_N^\varepsilon |_{\phi_N}^2\right\}$$

$$= \exp\left\{\frac{a_1}{\varepsilon}\langle\tilde{\omega}_1,\, f_1^\varepsilon\rangle - \frac{a_1^2}{\varepsilon^2}| f_1^\varepsilon |_{\phi_1}^2 + \frac{a_1^2}{2\varepsilon^2}| f_1^\varepsilon |_{\phi_1}^2\right\}$$

$$\times\cdots\times\exp\left\{\frac{a_N}{\varepsilon}\langle\tilde{\omega}_N,\, f_N^\varepsilon\rangle - \frac{a_N^2}{\varepsilon^2}| f_N^\varepsilon |_{\phi_N}^2 + \frac{a_N^2}{2\varepsilon^2}| f_N^\varepsilon |_{\phi_N}^2\right\}$$

$$= \exp\left\{\frac{a_1}{\varepsilon}\langle\tilde{\omega}_1,\, f_1^\varepsilon\rangle - \frac{a_1^2}{2\varepsilon^2}| f_1^\varepsilon |_{\phi_1}^2\right\}$$

$$\times\cdots\times\exp\left\{\frac{a_N}{\varepsilon}\langle\tilde{\omega}_N,\, f_N^\varepsilon\rangle - \frac{a_N^2}{2\varepsilon^2}| f_N^\varepsilon |_{\phi_N}^2\right\}$$

$$= \exp\left\{\frac{1}{\varepsilon}\left\langle\sum_{k=1}^{N} a_k\tilde{\omega}_k,\ f_k^{\varepsilon}\right\rangle - \frac{1}{2\varepsilon^2}\sum_{k=1}^{N} a_k^2\ |f_k^{\varepsilon}|_{\phi_k}^2\right\},$$

$$\mathbb{P}(Z_t^{\varepsilon} \in O) = \exp\left\{-\frac{1}{2\varepsilon^2}\sum_{k=1}^{N} a_k^2\ |f_k^{\varepsilon}|_{\phi_k}^2\right\}$$

$$\times\tilde{\mathbb{E}}\left(\chi_{Z_t^{\varepsilon}\in O}\exp\left\{\frac{1}{\varepsilon}\left\langle\sum_{k=1}^{N} a_k\tilde{\omega}_k,\ f_k^{\varepsilon}\right\rangle\right\}\right). \tag{8}$$

By Markov's inequality, we obtain

$$\mathbb{P}(Z_t^{\varepsilon} \in O) \geq \exp\left\{-\frac{1}{2\varepsilon^2}\sum_{k=1}^{N} a_k^2\ |f_k^{\varepsilon}|_{\phi_k}^2\right\}\tilde{\mathbb{P}}(Z_t^{\varepsilon} \in O),$$

and hence

$$\liminf_{\varepsilon\to 0}\ \varepsilon^2\log\mathbb{P}(Z_t^{\varepsilon} \in O) \geq -\frac{1}{2}\sum_{k=1}^{N} a_k^2 |f_k|_{\phi_k}^2 = -I(f_k).$$

Upper bound. Let C be a closed subset of $L_{\phi_k}^2$. According to the fractional Girsanov formula *I*, for $\omega_k \in \mathcal{S}'(\mathbb{R})$ and $f_k^{\varepsilon} \in \mathcal{S}(\mathbb{R})$,

$$\mathbb{P}(Z_t^{\varepsilon} \in C) \leq \mathbb{E}\left(\chi_{\{Z_t^{\varepsilon}\in C\}}\exp\left\{\frac{1}{\varepsilon}\sum_{k=1}^{N} a_k\langle\omega_k,\ f_k^{\varepsilon}\rangle\right\}\right)$$

$$= \tilde{\mathbb{E}}\left(\chi_{\{Z_t^{\varepsilon}\in C\}}\exp\left\{\frac{1}{\varepsilon}\sum_{k=1}^{N} a_k\langle\omega_k + \gamma_k^{\varepsilon},\ f_k^{\varepsilon}\rangle\right\}\right),$$

according to [14].

Using Theorems 1.2 and 1.3, we obtain

$$\mathbb{P}(Z_t^{\varepsilon} \in C) \leq \tilde{\mathbb{E}}\left(\exp\left\{\frac{2}{\varepsilon}\sum_{k=1}^{N} a_k\langle\tilde{\omega}_k,\ f_k^{\varepsilon}\rangle - \frac{2}{\varepsilon^2}\sum_{k=1}^{N} a_k^2 |f_k^{\varepsilon}|_{\phi_k}^2\right\}\right)$$

$$\times \exp\left\{-\frac{1}{2\varepsilon^2}\sum_{k=1}^{N} a_k^2 | f_k^\varepsilon |_{\phi_k}^2\right\}$$

$$= 1 \times \exp\left\{-\frac{1}{2\varepsilon^2}\sum_{k=1}^{N} a_k^2 | f_k^\varepsilon |_{\phi_k}^2\right\},$$

so

$$\lim_{\varepsilon\to 0} \sup \varepsilon^2 \log \mathbb{P}^\varepsilon(Z_t^\varepsilon \in C) \leq -\frac{1}{2}\sum_{k=1}^{N} a_k^2 | f_k |_{\phi_k}^2 = -I(f_k).$$

It remains to show that I is a good rate function. We see that I is the combination of a_k and $\frac{1}{2}| f_k |_{\phi_k}$. Since $\frac{1}{2}| f_k |_{\phi_k}$ ’s are good rate functions, the result follows.

2.2. Large deviations for a solution of stochastic differential equation derived by several fractional Brownian motions

In this subsection, we study the diffusion process $X_t^\varepsilon(2)$ when the drift is not 0 and ε tends to 0. We denote the probability law of X_t^ε by $\mu^\varepsilon = \mathbb{P}^\varepsilon \circ F^{-1}$, where

* $\mathbb{P}^\varepsilon$ is the probability law of $Z_t^\varepsilon(1)$;
* $F(f_k) = \sum_{k=1}^{N} h_k$ with the solution h_k of ordinary differential equation:

$$h_k(t) = x_0 + \int_0^t b(h_k(r))dr + \int_0^t \sigma(h_k(r)) f_k(r)\phi_k(r, s)dr,$$

for which $f_k \in L_{\phi_k}^2$ induced by LDP of the process $B_t^{H_k}$.

Theorem 2.2. *The family* $(X_t^\varepsilon)_{\varepsilon>0}$ *satisfies a large deviation principle with the good rate function* $J : \mathbb{L}_{\phi_k^{-1}}^2 \to [0, +\infty]$ *given by*

$$J(h_k) = \sum_{k=1}^{N} a_k^2 J_{H_k}(h_k),$$

where

$$J_{H_k}(h_k) = \begin{cases} \frac{1}{2}\|\sigma^{-1}(h_k)[h_k - b(h_k)]\|^2_{\phi_k^{-1}}, & \text{for } h_k \in L^2_{\phi_k^{-1}}, \\ +\infty, & \text{otherwise.} \end{cases} \tag{9}$$

In other words:

∗ *J is a good rate function*;

∗ *for a closed set* $C \subset \mathbb{L}^2_{\phi_k}$ *and an open set* $O \subset \mathbb{L}^2_{\phi_k}$,

$$\limsup_{\varepsilon \to 0} \varepsilon^2 \log \mu^\varepsilon (X_t^\varepsilon \in C) \leq -J(h_k) \leq \liminf_{\varepsilon \to 0} \varepsilon^2 \log \mu^\varepsilon (X_t^\varepsilon \in O).$$

Proof. Since J_{H_k}'s are good rate functions, J is a good rate function. Therefore, h_k is continuous, and consequently F is continuous.

Since F is a continuous function and the process Z_t^ε of probability law $\mathbb{P}^\varepsilon$ has a LDP with the good rate function I (7), according to Theorem 1.5, the family $(X_t^\varepsilon)_{\varepsilon > 0}$ satisfies a large deviation principle. Indeed, for an open set O and a closed set C,

$$h_k(t) = x_0 + \int_0^t b(h_k(r))dr + \int_0^t \sigma_H(h_k(r)) f_k(r) \phi(r, s) dr$$

$$\Rightarrow f_k(t) = \frac{1}{\sigma(h_k(t))\phi(t, s)}[\dot{h}_k(t) - b(h_k(t))].$$

Thus

$$\begin{aligned} \liminf_{\varepsilon \to 0} \varepsilon^2 \log \mu^\varepsilon [X_t^\varepsilon \in O] &= \liminf_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}^\varepsilon \circ F^{-1}[X_t^\varepsilon \in O] \\ &= \liminf_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}^\varepsilon [F^{-1}(X_t^\varepsilon) \in F^{-1}[O]] \end{aligned}$$

$$= \liminf_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}^{\varepsilon}[Z_t^{\varepsilon} \in F^{-1}[O]]$$

$$\geq - \inf_{h_k \in L^2_{\phi_k}(\mathbb{R})} \left\{ I(f_k) = \frac{1}{2} \sum_{k=1}^{N} a_k^2 | f_k |^2_{\phi_k}, \, F = h_k \right\}$$

$$= -\frac{1}{2} \sum_{k=1}^{N} a_k^2 |\sigma^{-1}(h_k)[\dot{h}_k - b(h_k)]|^2_{\phi_k^{-1}}$$

$$= -J(h_k)$$

$$\geq \limsup_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}^{\varepsilon}[Z_t^{\varepsilon} \in F^{-1}(C)]$$

$$= \limsup_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}^{\varepsilon}[F^{-1}(X_t^{\varepsilon}) \in F^{-1}(C)]$$

$$= \limsup_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}^{\varepsilon} \circ F^{-1}[X_t^{\varepsilon} \in C]$$

$$= \limsup_{\varepsilon \to 0} \varepsilon^2 \log \mu^{\varepsilon}[X_t^{\varepsilon} \in C].$$

3. Conclusion

In this paper, we have studied the asymptotic behavior of the solution (2) by using Freidlin-Wentzell's [10] method. This construction is carried out in the tempered distribution space $\mathcal{S}'(\mathbb{R})$, where certain mathematical notions are applicable, for example, the Bochner-Minlos Theorem 1.1 and fractional white noise. It would be interesting to extend it in a space larger than that considered here.

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