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ON THE NUMERICAL APPROXIMATIONS OF BLOW-UP TIME IN SEMILINEAR PARABOLIC EQUATIONS

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Abstract

This paper investigates the numerical estimation of the blow-up time for solutions of semilinear parabolic problems defined on a bounded

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domain. We study the behavior of semidiscrete approximations applied to reaction-diffusion equations and establish both necessary and sufficient conditions for blow-up to occur in the discrete setting. For the numerical computations, the problem is transformed into a more tractable form using the arc-length transformation technique, which enables to generate a linearly convergent sequence to the blow-up time. This sequence is then accelerated using the Aitken Δ^2 method. Several numerical experiments are presented to illustrate the proposed approach.

1. Introduction

In recent years, there is a large number of nonlinear partial differential equations of parabolic type whose solution cannot be extended globally in time and becomes unbounded in finite time. Such a phenomenon is called *blow-up*, this can occur in nonlinear equations if the heat source is strong enough. One of the subareas of applied mathematics that has undergone a major expansion in the last century is the study of blow-up phenomenon. There has been an explosion of interest in blow-up results. Several papers contain numerous references on blow-up results [6, 7, 13-15]. We investigate the initial-boundary value problem

$$\begin{cases} u_t = F(t, x, u, \nabla u, \Delta u), & (x, t) \in \Omega \times (0, T), \\ u(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$. This problem belongs to a class of equations that are used to describe physical models arising in many fields of sciences. In the study of blow-up phenomenon, for instance for equation (1), we encounter some basic questions that must be considered, namely: does blow-up occur? where does it occur? when does it occur? and how does it occur? Our interest in this work is to answer to the first three questions, as well as to present some techniques used over the years by the main authors. In particular, we study existence of blow-up to a semilinear parabolic equation and also estimate the numerical blow-up time. The theoretical study of the blow-up of solutions to nonlinear partial differential equations has been the subject of investigations of many authors (see [2, 5, 10, 12]). As for the numerical approach, the simulation of blow-up phenomena remains a challenging task. For one thing, one has to deal with numerical data that grow indefinitely as the blow-up time approaches. This presents significant numerical challenges, particularly near the blow-up time. Most of the standard error estimates become useless as t approaches the blow-up time. However, there have been some attempts to establish numerical methods to capture blow-up phenomena, see for instance [1, 3, 4, 9, 11]).

2. Preliminaries

In this section, we establish some general and simple sufficient conditions under which blow-up occurs, as well as conditions that guarantee the convergence of the numerical blow-up time. We also present a numerical method to estimate this blow-up time. For convenience, we make the following assumptions, which will be used throughout the paper. We assume that problem 1 admits a unique local solution $u(\cdot, t)$ in a function space X, and that there exists a finite time T such that this solution cannot be extended in X beyond T. In other words, we assume:

(H_0) The solution $u(\cdot, t)$ of 1 blows up at T.

Moreover, we assume the existence of a functional $J : X \to \mathbb{R}$ such that, for the blow-up solution u of 1, the following holds:

(H_1) $\lim\limits_{t \to T} J[u](t) = \infty.$

To approximate problem (1), we consider a family of equations parameterized by h:

$$\begin{cases} u_t^h = F_h(t, x_h, u^h, \nabla_h u^h, \Delta_h u^h), & (x_h, t) \in \Omega_h \times (0, T), \\ u^h(x_h, t) = 0, & (x_h, t) \in \partial\Omega_h \times (0, T), \\ u^h(x_h, 0) = u_0^h(x_h), & x_h \in \Omega_h. \end{cases} \tag{2}$$

The parameter h represents the accuracy of approximation, which becomes better as h tends to zero. The approximations of $F, \nabla, \Delta, \Omega, x,$ and u_0 are denoted by $F_h, \nabla_h, \Delta_h, x_h$ and $u_0^h,$ respectively. We assume that problem (2) admits a unique local solution $u^h(\cdot, t)$ in a function space X_h. Furthermore, there exists a functional $J_h : X_h \to \mathbb{R}$ which approximates the continuous functional J. We introduce additional assumptions on the functionals J and J_h. Suppose:

(H_2) There exists a differentiable functional $I(t)$ satisfying the differential inequality:

$$\frac{d}{dt} I(t) \geq G(I(t)),$$

where $G : [0, \infty) \to \mathbb{R}$ is a continuous function such that

$$\begin{cases} G(s) > 0, & \text{for } s > R_0, \\ \displaystyle\int_{R_0}^{\infty} \frac{ds}{G(s)} < \infty. \end{cases} \tag{3}$$

Let $u(\cdot, t) \in X$ be a given solution of (1) that blows up at $t = T$, and let $u^h(\cdot, t) \in X_h$ be a solution of problem 2 for each h. We assume the existence of functionals $J : X \to \mathbb{R}$ and $J_h : X_h \to \mathbb{R}$ such that the functions $J(t) = J[u(\cdot, t)]$ and $J_h(t) = J_h[u^h(\cdot, t)]$ satisfy:

(H_3) The function $J(t)$ is continuous and satisfies assumption (H_1), while $J_h(t)$ is a C^1 function satisfying the differential inequality (H_2) for some function G.

(H_4) The family u^h approximates u in the following sense: for any $\varepsilon > 0$,

$$\lim_{h\to 0}\sup_{t\in[0, T-\varepsilon]} |J[u](t) - J_h[u^h](t)| = 0.$$

In particular, the following theorem was established in [15]:

Theorem 1. *Assume* $(H_0), (H_3)$ *and* (H_4) *hold. Then for sufficiently small h, the approximate solution* u^h *of problem* (2) *blows up in finite time* T_h *and* T_h *converges to T as h tends to zero.*

From a numerical computation of blow-up, the authors in [11] consider the following nonlinear ODE:

$$\begin{cases} \dfrac{dz}{dt} = g(z(t)), \quad 0 < t < T, \\ z(0) = z^0, \end{cases} \tag{4}$$

where

$$z = (z_1, ..., z_n)^T, \quad g(z) = (g_1(z), ..., g_n(z))^T \quad \text{and} \quad z^0 = (z_1^0, ..., z_1^0)^T,$$

we regard the variables t and z_i as functions of the arc length s. Since $ds^2 = dt^2 + dz_1^2 + \cdots + dz_n^2$, the variables $t(s)$ and $z_i(s)$ satisfy the differential equation

$$\begin{cases} \dfrac{d}{ds}\begin{pmatrix} t \\ z_1 \\ \vdots \\ z_n \end{pmatrix} = \dfrac{1}{\sqrt{1 + \sum_{i=1}^n g_i^2}}\begin{pmatrix} 1 \\ g_1 \\ \vdots \\ g_n \end{pmatrix}, \quad 0 < s < \infty, \\ t(0) = 0, \qquad z(0) = z^0. \end{cases} \tag{5}$$

This transformation is called the *arc length transformation*. Observe that in the transformed equation (5), the solution z never blows up for finite s. Moreover, when t approaches T (s approaches $+\infty$) and each component of the right-hand side of (5) satisfies

$$\lim_{t \to T} \frac{g_k}{\sqrt{1 + \sum_{i=1}^{n} g_i^2}} = \begin{cases} const(\neq 0) & \text{if } k \in \mathscr{B}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathscr{B} = \{k \mid y_k$ blows up with highest order$\}$. Therefore, the original equation is transformed into a numerically tractable one.

Remark 2. The proposed algorithm generates a linearly convergent sequence to the blow-up time and accelerates the sequence by the Aitken Δ^2 method for the case that the component $z_\nu(t)\ (\nu \in \mathrm{B})$ blows up in polynomial order.

We make the following assumption:

(A_0) None of the components of $z(t)$ blows up anywhere in $[0, T),$ but at $t = T,$ at least one component blows up (for brevity, we call the components which blow up the *blow-up components*).

(A_1) There may be many blow-up components, and they may tend to $+\infty$ or $-\infty$ with various orders. But the blow-up component with highest order tends necessarily to $+\infty$ and is an increasing function on $t \in [0, T).$ That is, for all $\nu \in \mathscr{B},$

$$z_\nu(t) \to +\infty, \quad t \uparrow T \quad \text{and} \quad g_\nu(z(t)) > 0, \quad t \in [0, T).$$

Theorem 3. *Suppose that the solution of* (4) *satisfies assumptions* (A_0) *and* (A_1). *Then* $\lim_{s \to +\infty} t(s) = T.$

(A_3) For $\nu \in \mathscr{B},$ the blow-up component z_ν satisfies

$$z_\nu(t) \sim \frac{1}{(T - t)^p}, \quad t \uparrow T, \quad p > 0.$$

From this and former assumptions, we have the following theorem:

Theorem 4. *Assume that the solution of* (4) *satisfies assumption* (A_0), (A_1) *and* (A_3) *and* $\{s_l\}$ *be the geometric sequence given by*

$$s_l = s_0 + \gamma^{\ell}, \quad s_0 > 0, \quad \gamma > 1, \quad \ell = 0, 1, 2, \ldots.$$

Using the sequence, if we define the sequence $\{t_\ell\}$ *by*

$$t_\ell \overset{def}{=} t(s_\ell) = \int_0^{s_\ell} \frac{ds}{\sqrt{1 + \sum_{i=1}^{n} g_i^2}}, \quad \ell = 0, 1, 2, \ldots,$$

then t_ℓ *converges to T linearly and the rate of convergence is* $\gamma^{-1/p}$.

Proof. For a proof, we refer the reader to [11]. □

Algorithm

(1) Let $s_0 > 0$ and $\gamma > 1$, and define the geometric sequence $\{s_\ell\}$ by

$$s_\ell = s_0 \cdot \gamma^{\ell}, \quad \ell = 0, 1, 2, \ldots.$$

(2) Integrate (5) from $s = 0$ to s_ℓ and put $t_\ell = t(s_\ell)$.

(3) Let $t_\ell^{(0)} = t_\ell$ $(\ell = 0, 1, 2, \ldots)$ and apply the Aitken Δ^2 method to the sequence recursively:

$$t_{\ell+2}^{(k+1)} = t_{\ell+2}^{(k)} - \frac{(t_{\ell+2}^{(k)} - t_{\ell+1}^{(k)})^2}{t_{\ell+2}^{(k)} - 2t_{\ell+1}^{(k)} + t_\ell^{(k)}}, \quad \ell \geq 2k, \quad k = 0, 1, 2, \ldots.$$

Using this algorithm, we can also estimate the blow-up rate p,

$$p_\ell \overset{def}{=} -1/\log_\gamma \left| \frac{t_\ell^{(0)} - t_L^{(K)}}{t_{\ell-1}^{(0)} - t_L^{(K)}} \right|, \quad k = 0, \ldots, K, \quad l = 2k, \ldots, L, \quad L \geq 2K.$$

3. Semilinear Parabolic PDEs

In this section, we first consider a semilinear partial differential equation whose solution blows up in finite time. Then, we investigate a semi-discrete numerical scheme and, by applying Theorem 1, we prove that the semi-discrete solution also blows up in finite time and that its blow-up time converges to the exact one. Finally, we provide an estimate of the blow-up time. We consider asymptotic behaviours of difference solutions for a semilinear parabolic equation

$$\begin{cases} u_t = \alpha\Delta u + f(u), & (x, t) \in \Omega \times (0, T), \\ u(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \tag{6}$$

where $f(u) = (u + \lambda)^p$, $p > 1$, α is a non-negative constant, $\lambda \geq 0$ and Ω is a bounded domain in $\mathbb{R}^N$. We assume that initial data $u_0(x)$ is a nonnegative smooth function and

$$u_0 \in \mathbf{C}^1([-1, 1]), \quad u_0(-1) = u_0(1) = 0,$$

$$u_0(x) \geq 0, \quad \Delta u_0 + f(u_0) \geq 0, \text{ if } -1 < x < 1. \tag{7}$$

This problem can be regarded as a heat conduction problem where u stands for the temperature, and the heat sources are prescribed on the boundaries. It is well established that problem (6) has a unique classical solution that blows up in finite time T (see [8, 16]). T is called the *blow-up time* of the solution u. The theoretical study of blow-up of solutions for this problem in particular case where $\alpha = 1$ and $\lambda = 0$ has been the subject of investigations of many authors (see [16] and the references cited therein). The authors proved that the blow-up occurs in a single point. Recently, Adou et al. [3] considered the numerical study of problem (6) where $\alpha = 1$ and estimated the numerical blow-up time by applying an efficient numerical algorithm to the semidiscrete problem. They also analyzed the convergence of the semidiscrete blow-up time. Our aim in this work is to use the semi-discretized equations of problem (6), instead of the fully discretized ones, to prove the convergence of the blow-up time and to compute an approximate value of it. Our result includes and extends those presented in [3, 11, 15].

4. Semidiscrete Problem

Let I be a positive integer. Set $h = \frac{2}{I}$ and define the grid $x_i = ih - 1$, for $i = 0, \ldots, I$. Let δ^2 denote the standard second order difference operator. We approximate the solution u of the problem (6) by the solution $U_h(t)$ of the semidiscrete equations, where $U_h(t) = (U_0(t), \ldots, U_I(t))^T$ of the semidiscrete equations:

$$\begin{cases} \frac{d}{dt}U_i(t) = \alpha\delta^2 U_i(t) + (U_i(t) + \lambda)^p, & i = 1, \ldots, I-1, \ t \in [0, T], \\ U_I(t) = U_0(t) = 0, & t \geq 0, \\ U_i(0) = U_i^0, & i = 0, \ldots, I, \end{cases} \tag{8}$$

where

$$\delta^2 U_i(t) = \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{h^2}, \quad i = 1, \ldots, I-1.$$

Definition 5. Let be a function $V_h \in \mathbf{C}^1([0, T], \mathbb{R}^{I+1})$. Then V_h is an *upper solution* of (8) if

$$\frac{d}{dt}V_i(t) - \alpha\delta^2 V_i(t) \geq f(V_i(t)), \quad i = 1, \ldots, I-1, \quad t \in [0, T],$$

$$V_0(t) \geq 0, \ \ V_I(t) \geq 0, \ \ t \in [0, T],$$

$$V_i(0) \geq U_i^0, \ \ i = 0, \ldots, I.$$

On the other hand, we say that $V_h \in \mathbf{C}^1([0, T], \mathbb{R}^{I+1})$ is a *lower solution* of (8) if these inequalities above are reversed.

Theorem 6. *Let* $f \in \mathbf{C}^0(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ *and* $\alpha > 0$. *If*

$$V_h, W_h \in \mathbf{C}^1([0, T], \mathbb{R}^{I+1})$$

are such that, for each $t \in [0, T]$,

$$\frac{d}{dt}V_i - \alpha\delta^2 V_i - f(V_i) < \frac{d}{dt}W_i - \alpha\delta^2 W_i - f(W_i), \quad 1 \le i \le I - 1, \tag{9}$$

$$V_0(t) < W_0(t), \quad V_I(t) < W_I(t), \tag{10}$$

$$V_i(0) < W_i(0), \quad 0 \le i \le I, \tag{11}$$

then

$$V_i(t) < W_i(t), \quad t \in [0, T]. \tag{12}$$

Proof. We define the vector $Z_h(t) = W_h(t) - V_h(t)$. Let t_0 be the first $t \in [0, T]$ such that $Z_i(t) > 0$ for $t \in (0, t_0)$, $0 \le i \le I$, but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, ..., I\}$. We can see that

if $i_0 = 0$, respectively (if $i_0 = I$), then $Z_0(t_0) = 0$, respectively $(Z_I(t_0) = 0)$; it is a contradiction because of (10).

If $i_0 \in \{1, ..., I - 1\}$, then

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \le 0,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \ge 0,$$

which implies that

$$\frac{dZ_{i_0}(t_0)}{dt} - \alpha\delta^2 Z_{i_0}(t_0) + f(W_{i_0}(t_0)) - f(V_{i_0}(t_0)) \le 0.$$

But this inequality contradicts (9), and the proof is complete. □

Theorem 7. *Let* $f \in \mathbf{C}^0(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, $i = 1, ..., I - 1$ *be locally Lipschitz continuous functions second argument and* $\alpha > 0$. *If*

$$V_h, W_h \in \mathbf{C}^1([0, T], \mathbb{R}^{I+1})$$

are such that, for each $t \in [0, T]$,

$$\frac{d}{dt}V_i - \alpha\delta^2 V_i - f(V_i) \le \frac{d}{dt}W_i - \alpha\delta^2 W_i - f(W_i), \ \ 1 \le i \le I-1,$$

$$V_0(t) \le W_0(t), \ \ V_I(t) \le W_I(t),$$

$$V_i(0) \le W_i(0), \ \ 0 \le i \le I,$$

then

$$V_i(t) \le W_i(t), \ \ t \in [0, T].$$

Lemma 8. *Let* U_h *be the solution of* (8). *Then* $U_i(t) > 0, \ 1 \le i \le I$.

Proof. Let $\beta = \min_{1 \le i \le I-1} U_i^0$. We introduce the vector V_h defined by $V_i = \beta e^{-\gamma_h t}\sin\left(\frac{\pi}{2}ih\right), \ \ 0 \le i \le I$, where $\gamma_h = \frac{2 - 2\cos\left(\frac{\pi}{2}h\right)}{h^2}$. It is not hard to see that

$$\frac{dU_i}{dt} - \alpha\delta^2 U_i \ge \frac{dV_i}{dt} - \alpha\delta^2 V_i = 0, \ 1 \le i \le I-1, \ \ t \in [0, T],$$

$$U_0(t) = V_0(t) = 0, \ \ U_I(t) = V_I(t) = 0, \ \ t \in [0, T],$$

$$U_i(0) \ge V_i(0), \ \ 0 \le i \le I.$$

We deduce from Theorem 7 that

$$U_i(t) \ge \beta e^{-\gamma_h t}\cos\left(\frac{\pi}{2}(ih - 1)\right), \ \ 0 \le i \le I.$$

This implies that $U_i(t) > 0, 1 \le i \le I$. This completes the proof. □

The next theorem establishes that, for each fixed time interval $[0, T]$, where u is defined, the solution of semidiscrete problem (8) approximates u, as $h \to 0$.

Theorem 9. *Assume*

(i) *The reaction function* $f \in \mathbf{C}^1([0, \infty))$, *and the problem* (6) *has a solution* $u \in \mathbf{C}^{4,1}([-1, 1] \times [0, T])$.

(ii) *The initial condition* U_h^0 *satisfies*:

$$\| U_h^0 - u_h(0) \|_\infty = o(1), \quad h \to 0, \tag{13}$$

where $u_h(t) = (u(x_0, t), \ldots, u(x_{I+1}))^T$, $t \in [0, T]$. *Then*, *for* h *sufficiently small*, *problem* (8) *has a unique solution*

$$U_h \in \mathbf{C}^1([0, T], \mathbb{R}^{I+1})$$

such that

$$\max_{t \in [0, T]} \| U_h(t) - u_h(t) \|_\infty = O(\| U_h^0 - u_h(0) \|_\infty + h^2), \quad h \to 0. \tag{14}$$

Proof. Since the proof is a standard one, we omit it. We refer the readers to [1, 3, 6]. □

5. Convergence of the Semidiscrete Blow-up Time

In this section, we prove the convergence of the semidiscrete blow-up time to that of the original problem as the mesh size goes to zero. This proof will be done in three steps.

Step 1 (Blow-up of *u*)**.** For sufficiently large initial data, the solution of problem (6) blows up in finite time. See, for instance [8, 16], which satisfies condition H_0.

Step 2 (Blow-up of U_h and convergence of blow-up time)**.** Define the energy *I* as follows:

$$I[u](t) = \frac{1}{2}\alpha \int_{-1}^{1} u_x^2 dx - \frac{1}{p+1} \int_{-1}^{1} (u + \lambda)^{p+1} dx. \tag{15}$$

We can easily check that

$$\frac{d}{dt} I[u] = -\int_{-1}^{1} u_t^2 dx \leq 0. \tag{16}$$

Therefore, I is a monotone non-increasing function of t.

We define a functional J by

$$J[u](t) = \int_{-1}^{1} (u(x, t) + \lambda)^2 dx. \tag{17}$$

For this J, we obtain

$$\begin{aligned}
\frac{d}{dt} J &= 2\int_{-1}^{1} (u + \lambda) u_t dx \\
&= 2\left(\alpha\lambda(u_x(1) - u_x(-1)) - \alpha\int_{-1}^{1} u_x^2 dx + \int_{-1}^{1} (u + \lambda)^{p+1} dx \right) \\
&= 2\alpha\lambda(u_x(1) - u_x(-1)) - 4I[u] + \frac{2(p-1)}{p+1} \int_{-1}^{1} (u + \lambda)^{p+1} dx \\
&\geq 2\alpha\lambda(u_x(1) - u_x(-1)) - 4I[u_0] + \frac{2(p-1)}{p+1} \int_{-1}^{1} (u + \lambda)^{p+1} dx.
\end{aligned}$$

By Jensen's inequality, we have

$$\frac{d}{dt} J \geq \xi - 4I[u_0] + c(J[u])^{\frac{p+1}{2}},$$

where $\xi = 2\alpha\lambda(u_x(1) - u_x(-1))$ and $c > 0$.

It is clear that

$$\lim_{t \to T} J[u](t) = \infty$$

since

$$\lim_{t \to T} \| u(., t) \|_2 = \infty$$

(see Weissler [16] and Friedman and Giga [7]).

We introduce functionals I_h and J_h as follows:

$$I_h[U_h](t) = \frac{1}{2}\alpha\left(\frac{1}{h}\sum_{i=0}^{I}(U_{i+1}(t) - U_i(t))^2\right) - \frac{1}{p+1}\left(h\sum_{i=0}^{I}(U_i(t) + \lambda)^{p+1}\right)$$

and

$$J_h[U_h](t) = h\sum_{i=0}^{I}(U_i(t) + \lambda)^2.$$

It is not hard to see that

$$\frac{d}{dt}J_h[U_h] \geq \beta - 4I_h[U_h^0] + c_1(J_h[U_h])^{\frac{p+1}{2}},$$

where

$$\beta = -2\alpha\lambda\frac{U_I + U_1}{h}, \ c_1 > 0,$$

since

$$\frac{d}{dt}I_h[U_h](t) = -h\sum_{i=1}^{I}(U_i)_t^2(t) \leq 0.$$

Setting $G(J_h) = \gamma + c_1 J_h^{\frac{p+1}{2}}$, with $\gamma = \beta - 4I_h[U_h^0]$, we obtain

$$\frac{d}{dt}J_h[U_h] \geq G(J_h).$$

Lemma 10. *Let* $G : [0, +\infty) \to \mathbb{R}$ *be a functional such that* $G(s) = \gamma + cs^{\frac{p+1}{2}}$ *with* $\gamma = -4I_h[U_h^0]$ *and* $c > 0$. *Then G satisfies*

$$\begin{cases} G(s) > 0 \quad \text{for } s > R_0, \\ \int_{R_0}^{\infty}\frac{ds}{G(s)} < \infty. \end{cases}$$

Proof. For any $\gamma \in \mathbb{R}$, there exists a number R_0 such that for any $s > R_0$, $G(s) > 0$.

Using the criterion for improper integrals, we obtain

$$\int_{R_0}^{\infty} \frac{ds}{G(s)} < \infty, \text{ for } p > 1,$$

which satisfies (H_1) and (H_2). □

Step 3 (Convergence of J_h)**.** From Theorem 9 and continuity of J and J_h, we deduce that for any $\varepsilon > 0$,

$$\lim_{h \to 0} \sup_{t \in [0, T-\varepsilon]} |J[u](t) - J_h[U_h](t)| = 0.$$

This completes the three conditions of Theorem 1. Hence, we conclude that the approximate solution U_h of problem (8) blows up in finite time T_h for sufficiently small h and T_h converges to T as h tends to zero.

6. Numerical Experiments

We now present numerical approximations of the blow-up time for problem (6) using the initial condition $\varphi(x_i) = 20\cos\left(\frac{\pi}{2} x_i\right)$ for $i = 0, ..., I$. These approximations are obtained by applying the standard central difference method to problem (6), which leads to the following system ODEs:

$$\frac{d}{dt}\begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_{n-2} \\ U_{n-1} \end{pmatrix} = \frac{\alpha}{h^2}\begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & -2 & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_{n-2} \\ U_{n-1} \end{pmatrix} + \begin{pmatrix} (U_1+\lambda)^p \\ (U_2+\lambda)^p \\ \vdots \\ (U_{n-2}+\lambda)^p \\ (U_{n-1}+\lambda)^p \end{pmatrix},$$

$$U_i^0 = \varphi(x_i), \quad i = 0, ..., I.$$

We apply our algorithm by setting $s_\ell = 2^{15+\ell}$ $(\ell = 0, ..., 12)$. To be more confident in our algorithm, we perform the experiment while progressively refining the mesh size.

The following tables present the numerical blow-up times for different values of α, λ and p when $h \to 0$.

Table 1. Numerical blow-up times obtained for $\lambda = 0$ and $p = 2$

	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$
I	T_h	T_h	T_h
32	0.056153585726042	0.063039830254959	0.071396287717730
64	0.056184237790581	0.063074177577534	0.071420088854879
128	0.056192225348881	0.063082524783378	0.071423146565352
254	0.056194274559320	0.063084394296039	0.071422317477852
512	0.056194812452228	0.063084753014666	0.07142126851443
1024	0.056194945631782	0.063084777235026	0.071420592315325

Table 2. Numerical blow-up times obtained for $\alpha = 2$ and $p = 3$

	$\lambda = 0$	$\lambda = 1$	$\lambda = 2$
I	T_h	T_h	T_h
32	0.001265044707703	0.001145576673927	0.001042403386883
64	0.001265311727703	0.001145781382974	0.001042562066021
128	0.001265398202745	0.001145847919998	0.001042613785775
254	0.001265424400689	0.001145868041527	0.001042629569605
512	0.001265432009786	0.001145874060747	0.001042634187341
1024	0.001265434157714	0.001145875732904	0.001042635499912

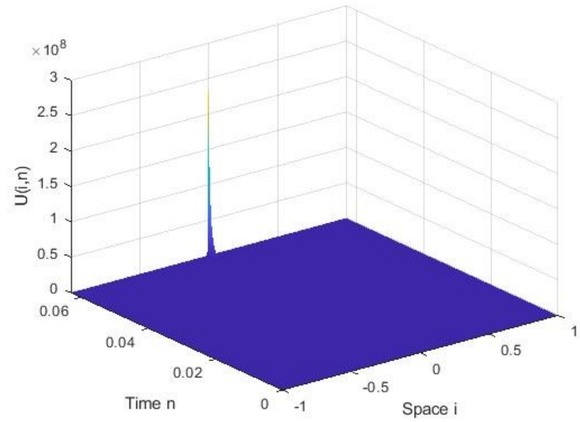

Figure 1. The behaviour of the blow-up solution in (x, t, u)-space for $\alpha = 2,\ \lambda = 0$ and $p = 2$ when $I = 512$.

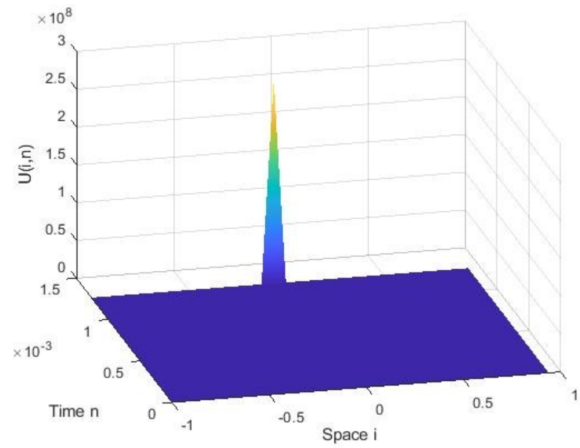

Figure 2. The behaviour of the blow-up solution in (x, t, u)-space for $\alpha = 2,\ \lambda = 0$ and $p = 3$ when $I = 64$.

7. Conclusion

In this work, we proposed a numerical method to estimate the blow-up time of semilinear parabolic equations using an arc-length transformation and the Aitken Δ^2 method. We established sufficient conditions ensuring the convergence of the numerical blow-up time to the theoretical one. Numerical experiments confirmed the efficiency and robustness of the proposed method. Future work will consider the extension of this approach to systems of equations, higher-dimensional domains, and more complex boundary conditions.

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