



FUZZY FEEBLY F_{σ} -COMPLEMENTED SPACES

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Abstract

In this paper, the notion of fuzzy feebly F_{σ} -complemented space is introduced and several characterizations of fuzzy feebly F_{σ} -complemented spaces are established. The inter-relations between weakly F_{σ} -complemented spaces and fuzzy feebly F_{σ} -complemented spaces are established. The conditions, under which fuzzy feebly F_{σ} -complemented spaces become fuzzy F -spaces and fuzzy F' -spaces become fuzzy normal spaces, are also obtained in this paper.

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1. Introduction

The notion of fuzzy sets as a new approach to a mathematical representation of vagueness in everyday language was introduced by Zadeh [23] in his classical paper in 1965. The potential of fuzzy notion was realized by the researchers and has successfully been applied in all branches of mathematics. In 1968, Chang [4] introduced the concept of fuzzy topological spaces. The paper of Chang paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts.

In the recent years, there has been a growing trend to introduce and study different forms of fuzzy spaces in fuzzy topology. The notion of F -spaces in classical topology was introduced by Gillman and Henriksen [7] in which disjoint cozero subsets of X are contained in disjoint zero sets. Motivated on these lines, the notion of fuzzy F -spaces was introduced and studied by Thangaraj and Muruganatham in [19]. In 2004, Henriksen and Woods [9] introduced the notion of cozero complemented spaces and several characterizations of these spaces are established. Levy and Shapiro [11] studied cozero complemented spaces under the name “ z -good spaces” and Azarpanah and Karavan [1] studied cozero complemented spaces in the name “ m -spaces”. The concept of cozero complemented spaces is related to the structure of the space of minimal prime ideals of $C(X)$ and its compactness. The concept of F_{σ} -complemented spaces in fuzzy setting was introduced and studied by Thangaraj and Vikraman in [20]. Motivated by the works of Knox et al. [10] on weakly cozero complemented spaces in classical topology, the notion of fuzzy weakly F_{σ} -complemented spaces was introduced and studied in [21]. The term “*feebly cozero complemented*” arises in the study of f -rings and their lattices of z -ideals. The notion of feebly F_{σ} -complemented spaces was introduced and studied by Dube and Ighedo [5].

In this paper, the notion of fuzzy feebly F_{σ} -complemented spaces is introduced and several characterizations of fuzzy feebly F_{σ} -complemented

spaces are obtained. The inter-relations between fuzzy weakly F_σ -complemented spaces and fuzzy feebly F_σ -complemented spaces are studied. The conditions, under which fuzzy feebly F_σ -complemented spaces become fuzzy F -spaces and fuzzy F_σ -complemented spaces become feebly F_σ -complemented spaces, are explored. Fuzzy feebly F_σ -complemented spaces are considered for verifying fuzzy hyperconnectedness property. Fuzzy F' -spaces, which are also fuzzy P -spaces having fuzzy F_σ -complemented property, are also considered for fuzzy normality.

2. Preliminaries

Some basic notions and results used in the sequel are given in order to make the exposition self-contained. In this work, by (X, T) or simply by X , we denote a fuzzy topological space due to Chang [4]. Let X be a non-empty set and I the unit interval $[0, 1]$. A fuzzy set λ in X is a mapping from X into I . The *fuzzy set* 0_X is defined as $0_X(x) = 0$, for all $x \in X$ and the *fuzzy set* 1_X is defined as $1_X(x) = 1$, for all $x \in X$.

Definition 2.1 [4]. Let (X, T) be a fuzzy topological space and λ be any fuzzy set in (X, T) . Then the *interior*, the *closure* and the *complement* of λ are defined, respectively, as follows:

- (i) $\text{int}(\lambda) = \vee\{\mu/\mu \leq \lambda, \mu \in T\}$;
- (ii) $\text{cl}(\lambda) = \wedge\{\mu/\lambda \leq \mu, 1 - \mu \in T\}$;
- (iii) $\lambda'(x) = 1 - \lambda(x)$, for all $x \in X$.

For a family $\{\lambda_i/i \in I\}$ of fuzzy sets in (X, T) , the union $\psi = \vee_i(\lambda_i)$ and intersection $\delta = \wedge_i(\lambda_i)$, are defined, respectively, as

- (iv) $\psi(x) = \sup_i\{\lambda_i(x)/x \in X\}$.
- (v) $\delta(x) = \inf_i\{\lambda_i(x)/x \in X\}$.

Lemma 2.1 [2]. For a fuzzy set λ of a fuzzy topological space X ,

(i) $1 - \text{int}(\lambda) = \text{cl}(1 - \lambda)$ and (ii) $1 - \text{cl}(\lambda) = \text{int}(1 - \lambda)$.

Definition 2.2. A fuzzy set λ in a fuzzy topological space (X, T) is called

(i) *fuzzy G_δ -set* in (X, T) if $\lambda = \bigwedge_{i=1}^{\infty}(\lambda_i)$, where $\lambda_i \in T$ for $i \in J$;

fuzzy F_σ -set in (X, T) if $\lambda = \bigvee_{i=1}^{\infty}(\lambda_i)$, where $1 - \lambda_i \in T$ for $i \in J$ [3];

(ii) *fuzzy regular-open set* in (X, T) if $\lambda = \text{int cl}(\lambda)$;

fuzzy regular-closed set in (X, T) if $\lambda = \text{cl int}(\lambda)$ [2];

(iii) *fuzzy dense set* in (X, T) if there exists no fuzzy closed set μ in (X, T) such that $\lambda < \mu < 1$. That is, $\text{cl}(\lambda) = 1$, in (X, T) [15];

(iv) *fuzzy nowhere dense set* in (X, T) if there exists no non-zero fuzzy open set μ in (X, T) such that $\mu < \text{cl}(\lambda)$. That is, $\text{int cl}(\lambda) = 0$ in (X, T) [15];

(v) *fuzzy first category set* in (X, T) if $\lambda = \bigvee_{i=1}^{\infty}(\lambda_i)$, where (λ_i) 's are fuzzy nowhere dense sets in (X, T) . Any other fuzzy set in (X, T) is said to be of *fuzzy second category* [15];

(vi) *fuzzy somewhere dense set* if $\text{int cl}(\lambda) \neq 0$, for a fuzzy set in (X, T) [13].

Definition 2.3. A fuzzy topological space (X, T) is called a

(i) *fuzzy hyper connected space* if every non-null fuzzy open subset of (X, T) is a fuzzy dense set in (X, T) [12];

(ii) *fuzzy extremally disconnected space* if closure of every fuzzy open set is a fuzzy open set in (X, T) [8];

(iii) *fuzzy P -space* if each fuzzy G_δ -set in (X, T) is a fuzzy open set in (X, T) [14];

(iv) *fuzzy almost P -space* if for each non-zero fuzzy G_δ -set λ in (X, T) , $\text{int}(\lambda) \neq 0$ in (X, T) [16];

(v) *fuzzy F' -space* if $\lambda \leq 1 - \mu$, where λ and μ are fuzzy F_σ -sets in (X, T) , then $\text{cl}(\lambda) \leq 1 - \text{cl}(\mu)$ in (X, T) [18];

(vi) *fuzzy F -space* if for any two fuzzy F_σ -sets λ and μ in (X, T) with $\lambda \leq 1 - \mu$, there exist fuzzy G_δ -sets α and β in (X, T) such that $\lambda \leq \alpha$, $\mu \leq \beta$ and $\alpha \leq 1 - \beta$ in (X, T) [19];

(vii) *fuzzy F_σ -complemented space* if for each fuzzy F_σ -set λ in (X, T) , there exists a fuzzy F_σ -set μ in (X, T) such that $\lambda \leq 1 - \mu$ and $\text{cl}(\lambda \vee \mu) = 1$ [20];

(viii) *fuzzy weakly F_σ -complemented space* if for each pair of fuzzy F_σ -sets μ_1 and μ_2 with $\mu_1 \leq 1 - \mu_2$ in (X, T) , there exist fuzzy F_σ -sets λ_1 and λ_2 with $\lambda_1 \leq 1 - \lambda_2$ in (X, T) such that $\mu_1 \leq \lambda_1$, $\mu_2 \leq \lambda_2$ and $\text{cl}(\lambda_1 \vee \lambda_2) = 1$ [21];

(ix) *fuzzy normal space* if for every pair of disjoint fuzzy closed sets F and G of X , there exist disjoint fuzzy open sets U and V such that $F \leq U$ and $G \leq V$ [6];

(x) *fuzzy Oz -space* if each fuzzy regular closed set is a fuzzy G_δ -set in (X, T) [22].

3. Fuzzy Feebly F_σ -complemented Spaces

Motivated by the works of Dube and Ighedo [5] on feebly cozero complemented spaces in classical topology, the notion of fuzzy feebly F_σ -complemented space is defined as follows:

Definition 3.1. A fuzzy topological space (X, T) is called a *fuzzy feebly F_σ -complemented space* if for each pair of fuzzy F_σ -sets μ_1 and μ_2 with $\mu_1 \leq 1 - \mu_2$ in (X, T) , there exist fuzzy F_σ -sets λ_1 and λ_2 in (X, T) such that $\mu_1 \leq 1 - \lambda_1$, $\mu_2 \leq 1 - \lambda_2$ and $\text{cl}(\lambda_1 \vee \lambda_2) = 1$.

Example 3.1. Let $X = \{p, q, r\}$. Let $I = [0, 1]$. The fuzzy sets $\alpha, \beta, \gamma, \lambda$ and δ are defined on X as follows:

$$\alpha : X \rightarrow I \text{ is defined by } \alpha(p) = 0.8; \alpha(q) = 0.5; \alpha(r) = 0.7;$$

$$\beta : X \rightarrow I \text{ is defined by } \beta(p) = 0.6; \beta(q) = 0.9; \beta(r) = 0.4;$$

$$\gamma : X \rightarrow I \text{ is defined by } \gamma(p) = 0.4; \gamma(q) = 0.7; \gamma(r) = 0.8;$$

$$\lambda : X \rightarrow I \text{ is defined by } \lambda(a) = 0.6; \lambda(b) = 0.5; \lambda(c) = 0.6;$$

$$\delta : X \rightarrow I \text{ is defined by } \delta(a) = 0.4; \delta(b) = 0.3; \delta(c) = 0.3;$$

$$\theta : X \rightarrow I \text{ is defined by } \theta(a) = 0.2; \theta(b) = 0.5; \theta(c) = 0.3.$$

Then,

$$\begin{aligned} T = \{ & 0, \alpha, \beta, \gamma, \alpha \vee \beta, \alpha \vee \gamma, \beta \vee \gamma, \alpha \wedge \beta, \alpha \wedge \gamma, \beta \wedge \gamma, \alpha \vee [\beta \wedge \gamma], \\ & \beta \vee [\alpha \wedge \gamma], \gamma \vee [\alpha \wedge \beta], \alpha \wedge [\beta \vee \gamma], \beta \wedge [\alpha \vee \gamma], \gamma \wedge [\alpha \vee \beta], \\ & \alpha \vee \beta \vee \gamma, 1\} \end{aligned}$$

is a fuzzy topology on X . By computation, we find that

$$\lambda = \{1 - \alpha\} \vee \{1 - \beta\} \vee \{1 - \gamma\},$$

$$\delta = \{1 - [\alpha \vee \beta]\} \vee \{1 - [\alpha \vee \gamma]\} \vee \{1 - [\beta \vee \gamma]\},$$

$$\theta = \{1 - [\alpha \vee \beta]\} \vee \{1 - \alpha\} \vee \{1 - [\alpha \vee \gamma]\},$$

$$1 - (\alpha \wedge \beta) = \{1 - (\gamma \vee [\alpha \wedge \beta])\} \vee \{1 - (\alpha \wedge [\beta \vee \gamma])\} \vee \{1 - (\beta \wedge [\alpha \vee \gamma])\},$$

$$1 - (\beta \wedge \gamma) = \{1 - (\beta \wedge [\alpha \vee \gamma])\} \vee \{1 - (\gamma \wedge [\alpha \vee \beta])\} \vee \{1 - (\alpha \vee \beta \vee \gamma)\}$$

are fuzzy F_σ -sets in X .

Now, for the fuzzy F_σ -sets δ and $1 - (\alpha \wedge \beta)$ in (X, T) with $\delta \leq 1 - \{1 - (\alpha \wedge \beta)\}$, there exist fuzzy F_σ -sets λ and θ in (X, T) such that $\delta \leq 1 - \lambda$ and $1 - (\alpha \wedge \beta) \leq 1 - \theta$. Also, $\text{cl}(\lambda \vee \theta) = \text{cl}(\lambda) = 1$.

For the fuzzy F_σ -sets δ and $1 - (\beta \wedge \gamma)$ in (X, T) with $\delta \leq 1 - \{1 - (\beta \wedge \gamma)\}$, there exist fuzzy F_σ -sets λ and θ in (X, T) such that $\delta \leq 1 - \lambda$ and $1 - (\beta \wedge \gamma) \leq 1 - \theta$. Also, $\text{cl}(\lambda \vee \theta) = \text{cl}(\lambda) = 1$.

For the fuzzy F_σ -sets δ and θ in (X, T) with $\delta \leq 1 - \theta$, there exist fuzzy F_σ -sets λ and $1 - (\alpha \wedge \beta)$ in (X, T) such that $\delta \leq 1 - \lambda$ and $\theta \leq 1 - \{1 - (\alpha \wedge \beta)\}$. Also, $\text{cl}(\lambda \vee \{1 - (\alpha \wedge \beta)\}) = \text{cl}(\lambda) = 1$.

For the fuzzy F_σ -sets $1 - (\alpha \wedge \beta)$ and θ in (X, T) with $1 - (\alpha \wedge \beta) \leq 1 - \theta$, there exist fuzzy F_σ -sets δ and λ in (X, T) such that $1 - (\alpha \wedge \beta) \leq 1 - \delta$ and $\theta \leq 1 - \lambda$. Also, $\text{cl}(\delta \vee \lambda) = \text{cl}(\lambda) = 1$.

For the fuzzy F_σ -sets $1 - (\beta \wedge \gamma)$ and θ in (X, T) with $1 - (\beta \wedge \gamma) \leq 1 - \theta$, there exist fuzzy F_σ -sets δ and λ in (X, T) such that $1 - (\beta \wedge \gamma) \leq 1 - \delta$ and $\theta \leq 1 - \lambda$. Also, $\text{cl}(\delta \vee \lambda) = \text{cl}(\lambda) = 1$.

For the fuzzy F_σ -sets λ and δ in (X, T) with $\lambda \leq 1 - \delta$, there exist fuzzy F_σ -sets θ and $1 - (\beta \wedge \gamma)$ in (X, T) such that $\lambda \leq 1 - \theta$ and $\delta \leq 1 - \{1 - (\beta \wedge \gamma)\}$. Also, $\text{cl}(\theta \vee 1 - (\beta \wedge \gamma)) = \text{cl}(\lambda) = 1$.

For the fuzzy F_σ -sets λ and θ in (X, T) with $\lambda \leq 1 - \theta$, there exist fuzzy F_σ -sets θ and $1 - (\beta \wedge \gamma)$ in (X, T) such that $\lambda \leq 1 - \theta$ and $\theta \leq 1 - \{1 - (\beta \wedge \gamma)\}$. Also, $\text{cl}(\theta \vee [1 - (\beta \wedge \gamma)]) = \text{cl}(\lambda) = 1$. Hence it follows that (X, T) is a fuzzy feebly F_σ -complemented space. It is to be noted that $\lambda \not\leq 1 - \{1 - (\alpha \wedge \beta)\}$, $\lambda \not\leq 1 - \{1 - (\beta \wedge \gamma)\}$ and

$$[1 - (\alpha \wedge \beta)] \not\leq 1 - \{1 - (\beta \wedge \gamma)\}$$

in (X, T) .

Example 3.2. Let $X = \{a, b, c\}$. Define fuzzy sets α, β, γ on X as follows:

$$\alpha : X \rightarrow I \text{ is defined by } \alpha(a) = 0.5; \alpha(b) = 0.7; \alpha(c) = 0.4;$$

$$\beta : X \rightarrow I \text{ is defined by } \beta(a) = 0.8; \beta(b) = 0.5; \beta(c) = 0.6;$$

$$\gamma : X \rightarrow I \text{ is defined by } \gamma(a) = 0.6; \gamma(b) = 0.6; \gamma(c) = 0.7.$$

Then,

$$T = \{0, \alpha, \beta, \gamma, \alpha \vee \beta, \alpha \vee \gamma, \beta \vee \gamma, \alpha \wedge \beta, \alpha \wedge \gamma, \beta \wedge \gamma, \alpha \vee [\beta \wedge \gamma], \\ \beta \vee [\alpha \wedge \gamma], \gamma \vee [\alpha \wedge \beta], \gamma \wedge [\alpha \vee \beta], \alpha \vee \beta \vee \gamma, 1\}$$

is a fuzzy topology on X . By computation, we find that

$$1 - (\alpha \wedge \beta) = \{1 - \alpha\} \vee \{1 - \beta\} \vee \{1 - \gamma\};$$

$$1 - (\gamma \wedge [\alpha \vee \beta]) = \{1 - [\alpha \vee \beta]\} \vee \{1 - [\beta \vee \gamma]\} \vee \{1 - [\alpha \vee \gamma]\};$$

$$1 - ([\beta \wedge \gamma]) = \{1 - \beta\} \vee \{1 - \gamma\} \vee \{1 - (\gamma \wedge [\alpha \vee \beta])\}$$

and

$$1 - \beta = \{1 - \beta\} \vee \{1 - [\alpha \vee \beta \vee \gamma]\} \vee \{1 - [\beta \vee [\alpha \wedge \gamma]]\}$$

are fuzzy F_σ -sets in X .

Now for the fuzzy F_σ -sets $1 - (\alpha \wedge \beta)$ and $1 - (\gamma \wedge [\alpha \vee \beta])$, there exist fuzzy F_σ -sets $1 - \beta$ and $1 - (\beta \wedge \gamma)$ in X such that $\{1 - [\alpha \wedge \beta]\} \leq 1 - \{1 - \beta\}$ and $\{1 - (\gamma \wedge [\alpha \vee \beta])\} \leq 1 - \{1 - [\beta \wedge \gamma]\}$ and on computation, we find that $\text{cl}(\{1 - \beta\} \vee \{1 - [\beta \wedge \gamma]\}) = 1 - [\beta \wedge \gamma] \neq 1$.

Also, for the fuzzy F_σ -sets $1 - (\alpha \wedge \beta)$ and $1 - \beta$, there exist fuzzy F_σ -sets $1 - (\gamma \wedge [\alpha \vee \beta])$ and $1 - [\beta \wedge \gamma]$ in X such that $\{1 - (\alpha \wedge \beta)\} \leq 1 - \{1 - (\gamma \wedge [\alpha \vee \beta])\}$ and $\{1 - \beta\} \leq 1 - \{1 - [\beta \wedge \gamma]\}$ whereas on computation, $\text{cl}(\{1 - (\gamma \wedge [\alpha \vee \beta])\} \vee \{1 - [\beta \wedge \gamma]\}) = 1 - [\beta \wedge \gamma] \neq 1$. Hence (X, T) is not a fuzzy feebly F_σ -complemented space.

Proposition 3.1. *If a fuzzy topological space (X, T) is a fuzzy feebly F_σ -complemented space, then for each pair of fuzzy F_σ -sets μ_1 and μ_2 with $\mu_1 \leq 1 - \mu_2$ in (X, T) , there exist fuzzy G_δ -sets δ_1 and δ_2 in (X, T) such that $\mu_1 \leq \delta_1$, $\mu_2 \leq \delta_2$ and $\text{int}(\delta_1) \leq 1 - \text{int}(\delta_2)$.*

Proof. Suppose that μ_1 and μ_2 are fuzzy F_σ -sets with $\mu_1 \leq 1 - \mu_2$ in (X, T) . Since (X, T) is a fuzzy feebly F_σ -complemented space, there exist fuzzy F_σ -sets λ_1 and λ_2 in (X, T) such that $\mu_1 \leq 1 - \lambda_1$, $\mu_2 \leq 1 - \lambda_2$ and $\text{cl}(\lambda_1 \vee \lambda_2) = 1$. Let $\delta_1 = 1 - \lambda_1$ and $\delta_2 = 1 - \lambda_2$. Then, δ_1 and δ_2 are fuzzy G_δ -sets in (X, T) . Now $\delta_1 \wedge \delta_2 = (1 - \lambda_1) \wedge (1 - \lambda_2) = 1 - (\lambda_1 \vee \lambda_2)$, then $\text{int}(\delta_1 \wedge \delta_2) = \text{int}(1 - (\lambda_1 \vee \lambda_2)) = 1 - \text{cl}(\lambda_1 \vee \lambda_2) = 1 - 1 = 0$. This implies that $\text{int}(\delta_1) \wedge \text{int}(\delta_2) = 0$ and hence $\text{int}(\delta_1) \leq 1 - \text{int}(\delta_2)$. Thus, for the fuzzy F_σ -sets μ_1 and μ_2 with $\mu_1 \leq 1 - \mu_2$ in (X, T) , there exist fuzzy G_δ -sets δ_1 and δ_2 in (X, T) such that $\mu_1 \leq \delta_1$, $\mu_2 \leq \delta_2$ and $\text{int}(\delta_1) \leq 1 - \text{int}(\delta_2)$.

Corollary 3.1. *If μ_1 and μ_2 are disjoint fuzzy F_σ -sets in a fuzzy feebly F_σ -complemented space (X, T) , then*

(i) *there exist fuzzy G_δ -sets δ_1 and δ_2 in (X, T) such that $\mu_1 \leq \delta_1$, $\mu_2 \leq \delta_2$ and $\text{int}(\delta_1) \leq 1 - \text{int}(\delta_2)$;*

(ii) *there exist fuzzy G_δ -sets δ_1 and δ_2 in (X, T) such that*

$$\text{int}(\mu_1) \leq \text{int}(\delta_1) \leq 1 - \text{int}(\delta_2) \leq 1 - \text{int}(\mu_2).$$

Proof. (i) Suppose that μ_1 and μ_2 are disjoint fuzzy F_σ -sets in a fuzzy feebly F_σ -complemented space (X, T) . Then, $\mu_1 \wedge \mu_2 = 0$ implies that $\mu_1 \leq 1 - \mu_2$ and by Proposition 3.1, there exist fuzzy G_δ -sets δ_1 and δ_2 in (X, T) such that $\mu_1 \leq \delta_1$, $\mu_2 \leq \delta_2$ and $\text{int}(\delta_1) \leq 1 - \text{int}(\delta_2)$.

(ii) By (i), there exist fuzzy G_δ -sets δ_1 and δ_2 in (X, T) such that $\mu_1 \leq \delta_1$, $\mu_2 \leq \delta_2$ and $\text{int}(\delta_1) \leq 1 - \text{int}(\delta_2)$. Now, $\mu_1 \leq \delta_1$ and $\mu_2 \leq \delta_2$

imply that $\text{int}(\mu_1) \leq \text{int}(\delta_1)$ and $\text{int}(\mu_2) \leq \text{int}(\delta_2)$. Then, $\text{int}(\mu_1) \leq \text{int}(\delta_1) \leq 1 - \text{int}(\delta_2) \leq 1 - \text{int}(\mu_2)$.

Proposition 3.2. *If $\mu_1 \leq 1 - \mu_2$, for a pair of fuzzy F_σ -sets μ_1 and μ_2 in a fuzzy feebly F_σ -complemented space (X, T) , then $\text{int}(\mu_1) \leq 1 - \text{int}(\mu_2)$ in (X, T) .*

Proof. Suppose that μ_1 and μ_2 are fuzzy F_σ -sets with $\mu_1 \leq 1 - \mu_2$ in (X, T) . Since (X, T) is a fuzzy feebly F_σ -complemented space, by Proposition 3.1, there exist fuzzy G_δ -sets δ_1 and δ_2 in (X, T) such that $\mu_1 \leq \delta_1$, $\mu_2 \leq \delta_2$ and $\text{int}(\delta_1) \leq 1 - \text{int}(\delta_2)$. Now, $\mu_1 \leq \delta_1$ and $\mu_2 \leq \delta_2$ imply that $\text{int}(\mu_1) \leq \text{int}(\delta_1)$ and $\text{int}(\mu_2) \leq \text{int}(\delta_2)$. Then, $\text{int}(\mu_1) \leq \text{int}(\delta_1) \leq 1 - \text{int}(\delta_2) \leq 1 - \text{int}(\mu_2)$ and thus $\text{int}(\mu_1) \leq 1 - \text{int}(\mu_2)$ in (X, T) .

Proposition 3.3. *If μ_1 and μ_2 are disjoint fuzzy F_σ -sets in a fuzzy feebly F_σ -complemented space (X, T) , then there exist a fuzzy open set γ and a fuzzy closed set β in (X, T) such that $\text{int}(\mu_1) \leq \gamma \leq \beta \leq 1 - \text{int}(\mu_2)$.*

Proof. Suppose that μ_1 and μ_2 are disjoint fuzzy F_σ -sets in a fuzzy feebly F_σ -complemented space (X, T) . By Corollary 3.1(ii), there exist fuzzy G_δ -sets δ_1 and δ_2 in (X, T) such that $\text{int}(\mu_1) \leq \text{int}(\delta_1) \leq 1 - \text{int}(\delta_2) \leq 1 - \text{int}(\mu_2)$. Let $\gamma = \text{int}(\delta_1)$ and $\beta = 1 - \text{int}(\delta_2)$. Then, γ is a fuzzy open set and β is a fuzzy closed set in (X, T) . Hence, for the fuzzy F_σ -sets μ_1 and μ_2 with $\mu_1 \leq 1 - \mu_2$, there exist a fuzzy open set γ and a fuzzy closed set β in (X, T) such that $\text{int}(\mu_1) \leq \gamma \leq \beta \leq 1 - \text{int}(\mu_2)$.

Proposition 3.4. *If a fuzzy topological space (X, T) is a fuzzy feebly F_σ -complemented space, then for each pair of fuzzy G_δ -sets δ_1 and δ_2 in (X, T) with $1 - \delta_1 \leq \delta_2$, there exist fuzzy F_σ -sets λ_1 and λ_2 in (X, T) such that $\lambda_1 \leq \delta_1$, $\lambda_2 \leq \delta_2$ and $\text{cl}(\lambda_1 \vee \lambda_2) = 1$ in (X, T) .*

Proof. Suppose that δ_1 and δ_2 are fuzzy G_δ -sets with $1 - \delta_1 \leq \delta_2$ in (X, T) . Then, $(1 - \delta_1)$ and $(1 - \delta_2)$ are fuzzy F_σ -sets with $(1 - \delta_1) \leq 1 - (1 - \delta_2)$ in (X, T) . Since (X, T) is a fuzzy feebly F_σ -complemented space, there exist fuzzy F_σ -sets λ_1 and λ_2 in (X, T) such that $(1 - \delta_1) \leq 1 - \lambda_1$, $(1 - \delta_2) \leq 1 - \lambda_2$ and $\text{cl}(\lambda_1 \vee \lambda_2) = 1$. Then, $\lambda_1 \leq \delta_1$, $\lambda_2 \leq \delta_2$ and $\text{cl}(\lambda_1 \vee \lambda_2) = 1$. Since $\lambda_1 \vee \lambda_2 \leq \delta_1 \vee \delta_2$, $\text{cl}(\lambda_1 \vee \lambda_2) \leq \text{cl}(\delta_1 \vee \delta_2)$ and hence $1 \leq \text{cl}(\delta_1 \vee \delta_2)$. That is, $\text{cl}(\delta_1 \vee \delta_2) = 1$ in (X, T) . Hence, for the fuzzy G_δ -sets δ_1 and δ_2 in (X, T) with $1 - \delta_1 \leq \delta_2$, there exist fuzzy F_σ -sets λ_1 and λ_2 in (X, T) such that $\lambda_1 \leq \delta_1$, $\lambda_2 \leq \delta_2$ and $\text{cl}(\lambda_1 \vee \lambda_2) = 1$.

Proposition 3.5. *If δ_1 and δ_2 are fuzzy G_δ -sets with $1 - \delta_1 \leq \delta_2$ in a fuzzy feebly F_σ -complemented space (X, T) , then $\text{cl}(\delta_1 \vee \delta_2) = 1$ in (X, T) .*

Proof. Suppose that δ_1 and δ_2 are fuzzy G_δ -sets with $1 - \delta_1 \leq \delta_2$ in (X, T) . Then, by Proposition 3.4, there exist fuzzy F_σ -sets λ_1 and λ_2 in (X, T) such that $\lambda_1 \leq \delta_1$, $\lambda_2 \leq \delta_2$ and $\text{cl}(\lambda_1 \vee \lambda_2) = 1$ in (X, T) . Now, $\lambda_1 \vee \lambda_2 \leq \delta_1 \vee \delta_2$ in (X, T) implies that $\text{cl}(\lambda_1 \vee \lambda_2) \leq \text{cl}(\delta_1 \vee \delta_2)$. Thus $1 \leq \text{cl}(\delta_1 \vee \delta_2)$. That is, $\text{cl}(\delta_1 \vee \delta_2) = 1$ in (X, T) .

Proposition 3.6. *If δ_1 and δ_2 are fuzzy G_δ -sets with $1 - \delta_1 \leq \delta_2$ in a fuzzy feebly F_σ -complemented space (X, T) , then $1 - \text{cl}(\delta_1) \leq \text{cl}(\delta_2)$ in (X, T) .*

Proof. Suppose that δ_1 and δ_2 are fuzzy G_δ -sets with $1 - \delta_1 \leq \delta_2$ in (X, T) . By Proposition 3.5, $\text{cl}(\delta_1 \vee \delta_2) = 1$ in (X, T) . Then, $1 - \text{cl}(\delta_1 \vee \delta_2) = 0$ and by Lemma 2.1, $\text{int}[1 - (\delta_1 \vee \delta_2)] = 1 - \text{cl}(\delta_1 \vee \delta_2) = 0$ in (X, T) . Then, $\text{int}[(1 - \delta_1) \wedge (1 - \delta_2)] = 0$ and $\text{int}(1 - \delta_1) \wedge \text{int}(1 - \delta_2) = 0$. This implies that $\text{int}(1 - \delta_1) \leq 1 - [\text{int}(1 - \delta_2)]$ and thus $1 - \text{cl}(\delta_1) \leq 1 - [1 - \text{cl}(\delta_2)]$. Hence, $1 - \text{cl}(\delta_1) \leq \text{cl}(\delta_2)$ in (X, T) .

Proposition 3.7. *If δ_1 and δ_2 are fuzzy G_δ -sets with $1 - \delta_1 \leq \delta_2$ in a fuzzy feebly F_σ -complemented space (X, T) , then δ_1 and δ_2 are fuzzy somewhere dense sets in (X, T) .*

Proof. Suppose that δ_1 and δ_2 are fuzzy G_δ -sets with $1 - \delta_1 \leq \delta_2$ in (X, T) . By Proposition 3.6, $1 - \text{cl}(\delta_1) \leq \text{cl}(\delta_2)$ in (X, T) . Since $1 - \text{cl}(\delta_1)$ is a fuzzy open set in (X, T) , $\text{int} \text{cl}(\delta_2) \neq 0$ and thus δ_2 is a fuzzy somewhere dense set in (X, T) . Now, $1 - \delta_1 \leq \delta_2$ implies that $1 - \delta_2 \leq \delta_1$ and by Proposition 3.6, $1 - \text{cl}(\delta_2) \leq \text{cl}(\delta_1)$ in (X, T) . Then, $\text{int} \text{cl}(\delta_1) \neq 0$ and thus δ_1 is a fuzzy somewhere dense set in (X, T) .

Proposition 3.8. *If $\mu_1 \leq 1 - \mu_2$, for a pair of fuzzy F_σ -sets μ_1 and μ_2 in a fuzzy feebly F_σ -complemented space (X, T) , then $\text{cl} \text{int}(\mu_2) \neq 1$, and $\text{cl} \text{int}(\mu_1) \neq 1$ in (X, T) .*

Proof. Suppose that $\mu_1 \leq 1 - \mu_2$, for a pair of fuzzy F_σ -sets μ_1 and μ_2 in (X, T) . Since (X, T) is a fuzzy feebly F_σ -complemented space, by Proposition 3.1, there exist fuzzy G_δ -sets δ_1 and δ_2 in (X, T) such that $\mu_1 \leq \delta_1$, $\mu_2 \leq \delta_2$ and $\text{int}(\delta_1) \leq 1 - \text{int}(\delta_2)$. Then, $\text{int}(\mu_1) \leq \text{int}(\delta_1) \leq 1 - \text{int}(\delta_2) \leq 1 - \text{int}(\mu_2)$. Let $\gamma = \text{int}(\delta_1)$ and $\beta = 1 - \text{int}(\delta_2)$. Then, γ is a fuzzy open set and β is a fuzzy closed set in (X, T) and thus $\text{int}(\mu_1) \leq \gamma \leq \beta \leq 1 - \text{int}(\mu_2)$. Now $\gamma \leq 1 - \text{int}(\mu_2)$ implies that $\text{int}[1 - \text{int}(\mu_2)] \neq 0$. Then, by Lemma 2.1, $1 - \text{cl} \text{int}(\mu_2) \neq 0$ and thus $\text{cl} \text{int}(\mu_2) \neq 1$ in (X, T) . Also, $\text{int}(\mu_1) \leq \beta$ implies that $\text{cl} \text{int}(\mu_1) \leq \text{cl}(\beta) = \beta \leq 1$ and thus $\text{cl} \text{int}(\mu_1) \neq 1$ in (X, T) .

Proposition 3.9. *If a fuzzy topological space (X, T) is a fuzzy feebly F_σ -complemented space, then for each pair of fuzzy F_σ -sets μ_1 and μ_2 with $\mu_1 \leq 1 - \mu_2$ in (X, T) , there exist fuzzy F_σ -sets λ_1 and λ_2 in (X, T) such that $\mu_1 \leq 1 - (\lambda_1 \wedge \mu_2)$, $\mu_2 \leq 1 - (\lambda_2 \wedge \mu_1)$ and $\text{cl}(\lambda_1 \vee \lambda_2) = 1$.*

Proof. Suppose that μ_1 and μ_2 are fuzzy F_σ -sets with $\mu_1 \leq 1 - \mu_2$ in (X, T) . Since (X, T) is a fuzzy feebly F_σ -complemented space, there exist fuzzy F_σ -sets λ_1 and λ_2 in (X, T) such that $\mu_1 \leq 1 - \lambda_1$, $\mu_2 \leq 1 - \lambda_2$ and $\text{cl}(\lambda_1 \vee \lambda_2) = 1$. Now, $\mu_1 \leq 1 - \lambda_1$, $\mu_1 \leq 1 - \mu_2$ imply that $\mu_1 \leq (1 - \lambda_1) \vee (1 - \mu_2)$, and $\mu_2 \leq 1 - \lambda_2$, $\mu_2 \leq 1 - \mu_1$ imply that $\mu_2 \leq (1 - \lambda_2) \vee (1 - \mu_1)$. Then, $\mu_1 \leq 1 - (\lambda_1 \wedge \mu_2)$ and $\mu_2 \leq 1 - (\lambda_2 \wedge \mu_1)$, and $\text{cl}(\lambda_1 \vee \lambda_2) = 1$ in (X, T) .

4. Fuzzy Feebly F_σ -complemented Spaces and Other Fuzzy Topological Spaces

In this section, we relate fuzzy feebly F_σ -complemented spaces to some other well known fuzzy topological spaces.

The following proposition shows that fuzzy weakly F_σ -complemented spaces are fuzzy feebly F_σ -complemented spaces.

Proposition 4.1. *If a fuzzy topological space (X, T) is a fuzzy weakly F_σ -complemented space, then (X, T) is a fuzzy feebly F_σ -complemented space.*

Proof. Suppose that μ_1 and μ_2 are fuzzy F_σ -sets with $\mu_1 \leq 1 - \mu_2$ in (X, T) . Since (X, T) is a fuzzy weakly F_σ -complemented space, there exist fuzzy F_σ -sets λ_1 and λ_2 with $\lambda_1 \leq 1 - \lambda_2$ in (X, T) such that $\mu_1 \leq \lambda_1$, $\mu_2 \leq \lambda_2$ and $\text{cl}(\lambda_1 \vee \lambda_2) = 1$. Then, $\mu_1 \leq \lambda_1 \leq 1 - \lambda_2$ and $\mu_2 \leq \lambda_2 \leq 1 - \lambda_1$ in (X, T) . Thus, for the fuzzy F_σ -sets μ_1 and μ_2 with $\mu_1 \leq 1 - \mu_2$, there exist fuzzy F_σ -sets λ_1 and λ_2 in (X, T) such that $\mu_1 \leq 1 - \lambda_2$, $\mu_2 \leq 1 - \lambda_1$ and $\text{cl}(\lambda_1 \vee \lambda_2) = 1$. Hence (X, T) is a fuzzy feebly F_σ -complemented space.

Remark 4.1. The converse of Proposition 4.1 need not be true. That is, fuzzy feebly F_σ -complemented spaces need not be fuzzy weakly F_σ -

complemented spaces. In Example 3.1, (X, T) is a fuzzy feebly F_σ -complemented space but not a fuzzy weakly F_σ -complemented space. It is because for the fuzzy F_σ -sets $\{1 - (\beta \wedge \gamma)\}$ and θ in (X, T) , there exist fuzzy F_σ -sets λ and $1 - (\alpha \wedge \beta)$ such that $[1 - (\beta \wedge \gamma)] \leq \lambda$ and $\theta \leq 1 - (\alpha \wedge \beta)$, but $\lambda \not\leq 1 - \{1 - (\alpha \wedge \beta)\}$ even though $\text{cl}(\lambda \vee \{1 - [\alpha \wedge \beta]\}) = \text{cl}(\lambda) = 1$ in (X, T) .

It is to be noted that fuzzy F -spaces in which fuzzy G_δ -sets are fuzzy open, are fuzzy normal spaces and fuzzy F -spaces in which fuzzy G_δ -sets are fuzzy closed, are fuzzy F' -spaces [19]. Now one will be expected to study those fuzzy F -spaces in which fuzzy G_δ -sets are fuzzy dense sets.

The following proposition gives a condition under which fuzzy F -spaces become fuzzy feebly F_σ -complemented spaces.

Proposition 4.2. *If a fuzzy topological space (X, T) is a fuzzy F -space in which fuzzy G_δ -sets are fuzzy dense, then (X, T) is a fuzzy feebly F_σ -complemented space.*

Proof. Suppose that μ_1 and μ_2 are fuzzy F_σ -sets with $\mu_1 \leq 1 - \mu_2$ in (X, T) . Since (X, T) is a fuzzy F -space, for the fuzzy F_σ -sets μ_1 and μ_2 , there exist fuzzy G_δ -sets α and β in (X, T) such that $\mu_1 \leq \alpha$, $\mu_2 \leq \beta$ and $\alpha \leq 1 - \beta$ in (X, T) . Now $\mu_1 \leq 1 - [1 - \alpha]$ and $\mu_2 \leq 1 - [1 - \beta]$. Let $\lambda_1 = 1 - \alpha$ and $\lambda_2 = 1 - \beta$. Then, λ_1 and λ_2 are fuzzy F_σ -sets in (X, T) . Thus, for the fuzzy F_σ -sets μ_1 and μ_2 , there exist fuzzy F_σ -sets λ_1 and λ_2 in (X, T) such that $\mu_1 \leq 1 - \lambda_1$, $\mu_2 \leq 1 - \lambda_2$.

Since $\alpha \leq 1 - \beta$ in (X, T) , $\text{int}(\alpha) \leq \text{int}(1 - \beta) = 1 - \text{cl}(\beta)$. By hypothesis, the fuzzy G_δ -set β is fuzzy dense and thus $\text{cl}(\beta) = 1$ in (X, T) . Then, $\text{int}(\alpha) \leq 1 - \text{cl}(\beta) = 1 - 1 = 0$ and thus $\text{int}(\alpha) = 0$ in (X, T) . Also, $\alpha \leq 1 - \beta$ implies that $\text{cl}(\alpha) \leq \text{cl}(1 - \beta) = 1 - \text{int}(\beta)$. By hypothesis, $\text{cl}(\alpha) = 1$ in (X, T) . Then, $1 \leq 1 - \text{int}(\beta)$ and it follows that $\text{int}(\beta) = 0$.

Now

$$\begin{aligned} \text{cl}(\lambda_1 \vee \lambda_2) &= \text{cl}([1 - \alpha] \vee [1 - \beta]) = \text{cl}(1 - [\alpha \wedge \beta]) = 1 - \text{int}[\alpha \wedge \beta] \\ &= 1 - \{\text{int}(\alpha) \wedge \text{int}(\beta)\} = 1 - \{0 \wedge 0\} = 1 \end{aligned}$$

and thus for the fuzzy F_σ -sets μ_1 and μ_2 , the existence of fuzzy F_σ -sets λ_1 and λ_2 in (X, T) with $\mu_1 \leq 1 - \lambda_1$, $\mu_2 \leq 1 - \lambda_2$ and $\text{cl}(\lambda_1 \vee \lambda_2) = 1$, shows that (X, T) is a fuzzy feebly F_σ -complemented space.

Proposition 4.3. *If a fuzzy topological space (X, T) is a fuzzy F -space and fuzzy P -space, then (X, T) is not a fuzzy feebly F_σ -complemented space.*

Proof. Suppose that μ_1 and μ_2 are fuzzy F_σ -sets with $\mu_1 \leq 1 - \mu_2$ in (X, T) . Since (X, T) is a fuzzy F -space, for the fuzzy F_σ -sets μ_1 and μ_2 , there exist fuzzy G_δ -sets α and β in (X, T) such that $\mu_1 \leq \alpha$, $\mu_2 \leq \beta$ and $\alpha \leq 1 - \beta$. Now $\mu_1 \leq 1 - [1 - \alpha]$ and $\mu_2 \leq 1 - [1 - \beta]$. Let $\lambda_1 = 1 - \alpha$ and $\lambda_2 = 1 - \beta$. Then, λ_1 and λ_2 are fuzzy F_σ -sets in (X, T) . Thus, for the fuzzy F_σ -sets μ_1 and μ_2 , there exist fuzzy F_σ -sets λ_1 and λ_2 in (X, T) such that $\mu_1 \leq 1 - \lambda_1$, $\mu_2 \leq 1 - \lambda_2$. Now

$$\begin{aligned} \text{cl}(\lambda_1 \vee \lambda_2) &= \text{cl}([1 - \alpha] \vee [1 - \beta]) = \text{cl}(1 - [\alpha \wedge \beta]) \\ &= 1 - \text{int}[\alpha \wedge \beta] = 1 - \{\text{int}(\alpha) \wedge \text{int}(\beta)\}. \end{aligned}$$

Since (X, T) is a fuzzy P -space, the fuzzy G_δ -sets α and β are fuzzy open sets in (X, T) and thus $\text{int}(\alpha) \wedge \text{int}(\beta) = \alpha \wedge \beta \neq 0$. Therefore, $\text{cl}(\lambda_1 \vee \lambda_2) = 1 - [\alpha \wedge \beta] \neq 1$. Thus, it follows that (X, T) is not a fuzzy feebly F_σ -complemented space.

Proposition 4.4. *If a fuzzy topological space (X, T) is a fuzzy F -space but not a fuzzy almost P -space, then (X, T) is a fuzzy feebly F_σ -complemented space.*

Proof. Suppose that μ_1 and μ_2 are fuzzy F_σ -sets with $\mu_1 \leq 1 - \mu_2$ in (X, T) . Since (X, T) is a fuzzy F -space, for the fuzzy F_σ -sets μ_1 and μ_2 , there exist fuzzy G_δ -sets α and β in (X, T) such that $\mu_1 \leq \alpha$, $\mu_2 \leq \beta$ and $\alpha \leq 1 - \beta$ in (X, T) . Now $\mu_1 \leq 1 - [1 - \alpha]$ and $\mu_2 \leq 1 - [1 - \beta]$. Let $\lambda_1 = 1 - \alpha$ and $\lambda_2 = 1 - \beta$. Then, λ_1 and λ_2 are fuzzy F_σ -sets in (X, T) . Thus, for the fuzzy F_σ -sets μ_1 and μ_2 , there exist fuzzy F_σ -sets λ_1 and λ_2 in (X, T) such that $\mu_1 \leq 1 - \lambda_1$, $\mu_2 \leq 1 - \lambda_2$. Now

$$\begin{aligned} \text{cl}(\lambda_1 \vee \lambda_2) &= \text{cl}([1 - \alpha] \vee [1 - \beta]) = \text{cl}(1 - [\alpha \wedge \beta]) \\ &= 1 - \text{int}[\alpha \wedge \beta] = 1 - \{\text{int}(\alpha) \wedge \text{int}(\beta)\}. \end{aligned}$$

Since (X, T) is not a fuzzy almost P -space, for the fuzzy G_δ -sets α and β in (X, T) , either $\text{int}(\alpha) = 0$ or $\text{int}(\beta) = 0$ and hence $\text{int}(\alpha) \wedge \text{int}(\beta) = 0$ in (X, T) . This implies that $\text{cl}(\lambda_1 \vee \lambda_2) = 1 - \{0\} = 1$ and thus for the fuzzy F_σ -sets μ_1 and μ_2 , the existence of fuzzy F_σ -sets λ_1 and λ_2 in (X, T) with $\mu_1 \leq 1 - \lambda_1$, $\mu_2 \leq 1 - \lambda_2$ and $\text{cl}(\lambda_1 \vee \lambda_2) = 1$, shows that (X, T) is a fuzzy feebly F_σ -complemented space.

The following proposition gives a condition under which fuzzy feebly F_σ -complemented spaces become fuzzy F -spaces.

Proposition 4.5. *If a fuzzy topological space (X, T) is a fuzzy feebly F_σ -complemented and fuzzy P space, then (X, T) is a fuzzy F -space.*

Proof. Suppose that μ_1 and μ_2 are fuzzy F_σ -sets with $\mu_1 \leq 1 - \mu_2$ in (X, T) . Since (X, T) is a fuzzy feebly F_σ -complemented space, by Corollary 3.1(i), there exist fuzzy G_δ -sets δ_1 and δ_2 in (X, T) such that $\mu_1 \leq \delta_1$, $\mu_2 \leq \delta_2$ and $\text{int}(\delta_1) \leq 1 - \text{int}(\delta_2)$. Since (X, T) is a fuzzy P space, the fuzzy G_δ -sets δ_1 and δ_2 are fuzzy open sets and thus $\text{int}(\delta_1) = \delta_1$ and $\text{int}(\delta_2) = \delta_2$ in (X, T) . Thus, for the fuzzy F_σ -sets μ_1

and μ_2 , the existence of fuzzy G_δ -sets δ_1 and δ_2 in (X, T) such that $\mu_1 \leq \delta_1$, $\mu_2 \leq \delta_2$ and $\delta_1 \leq 1 - \delta_2$, shows that (X, T) is a fuzzy F -space.

Corollary 4.1. *If a fuzzy topological space (X, T) is a fuzzy weakly F_σ -complemented and fuzzy P space, then (X, T) is a fuzzy F -space.*

Proof. The proof follows from Propositions 4.1 and 4.5.

The following propositions give condition under which fuzzy F_σ -complemented spaces become feebly F_σ -complemented spaces.

Proposition 4.6. *If (X, T) is a fuzzy F_σ -complemented and fuzzy F' -space, then (X, T) is a fuzzy feebly F_σ -complemented space.*

Proof. Let μ_1 be a non-zero fuzzy F_σ -set in (X, T) . Since (X, T) is a fuzzy F_σ -complemented space, there exists a fuzzy F_σ -set μ_2 in (X, T) such that $\mu_1 \leq 1 - \mu_2$ and $\text{cl}(\mu_1 \vee \mu_2) = 1$ in (X, T) . Thus, there exists a pair of fuzzy F_σ -sets μ_1 and μ_2 with $\mu_1 \leq 1 - \mu_2$ in (X, T) . Clearly, $\text{cl}(\mu_1)$ and $\text{cl}(\mu_2)$ are fuzzy F_σ -sets in (X, T) . Let $\lambda_1 = \text{cl}(\mu_1)$ and $\lambda_2 = \text{cl}(\mu_2)$. Now

$$\text{cl}[\lambda_1 \vee \lambda_2] = \text{cl}[\text{cl}(\mu_1) \vee \text{cl}(\mu_2)] = \text{cl}[\text{cl}(\mu_1 \vee \mu_2)] = \text{cl}(\mu_1 \vee \mu_2) = 1.$$

Since (X, T) is a fuzzy F' -space, for the fuzzy F_σ -sets μ_1 and μ_2 with $\mu_1 \leq 1 - \mu_2$, $\text{cl}(\mu_1) \leq 1 - \text{cl}(\mu_2)$ and thus $\lambda_1 \leq 1 - \lambda_2$ in (X, T) . It follows that $\mu_1 \leq \lambda_1 \leq 1 - \lambda_2$ and $\mu_2 \leq \lambda_2 \leq 1 - \lambda_1$ in (X, T) . Thus, for the fuzzy F_σ -sets μ_1 and μ_2 with $\mu_1 \leq 1 - \mu_2$, there exist fuzzy F_σ -sets λ_1 and λ_2 in (X, T) such that $\mu_1 \leq 1 - \lambda_2$, $\mu_2 \leq 1 - \lambda_1$ and $\text{cl}(\lambda_1 \vee \lambda_2) = 1$ in (X, T) . Hence (X, T) is a fuzzy feebly F_σ -complemented space.

Corollary 4.2. *If a fuzzy topological space (X, T) is a fuzzy F' -space, fuzzy F_σ -complemented and fuzzy P -space, then (X, T) is a fuzzy F -space.*

Proof. The proof follows from Propositions 4.5 and 4.6.

Proposition 4.7. *If $\mu_1 \leq 1 - \mu_2$, where μ_1 and μ_2 are fuzzy open sets in a fuzzy extremally disconnected space (X, T) , then $\text{cl}(\mu_1) \leq 1 - \text{cl}(\mu_2)$ in (X, T) .*

Proof. Suppose that $\mu_1 \leq 1 - \mu_2$, where μ_1 and μ_2 are fuzzy open sets in (X, T) . Then, $\text{int}(\mu_1) \leq \text{int}(1 - \mu_2)$. By Lemma 2.1, $\text{int}(1 - \mu_2) = 1 - \text{cl}(\mu_2)$. Since μ_1 is a fuzzy open set in (X, T) , $\text{int}(\mu_1) = \mu_1$ and thus $\mu_1 \leq 1 - \text{cl}(\mu_2)$. Now, $\text{cl}(\mu_1) \leq \text{cl}[1 - \text{cl}(\mu_2)] = 1 - \text{int}[\text{cl}(\mu_2)]$. Since (X, T) is a fuzzy extremally disconnected space, for the fuzzy open set μ_2 in (X, T) , $\text{cl}(\mu_2)$ is a fuzzy open set in (X, T) and $\text{cl}(\mu_2) = \text{int}[\text{cl}(\mu_2)]$ and thus $\text{cl}(\mu_1) \leq 1 - \text{cl}(\mu_2)$ in (X, T) .

Proposition 4.8. *If (X, T) is a fuzzy F_σ -complemented and fuzzy extremally disconnected space in which fuzzy F_σ -sets are fuzzy open, then (X, T) is a fuzzy feebly F_σ -complemented space.*

Proof. Let μ_1 be a non-zero fuzzy F_σ -set in (X, T) . Since (X, T) is a fuzzy F_σ -complemented space, there exists a fuzzy F_σ -set μ_2 in (X, T) such that $\mu_1 \leq 1 - \mu_2$ and $\text{cl}(\mu_1 \vee \mu_2) = 1$ in (X, T) . Thus, there exists a pair of fuzzy F_σ -sets μ_1 and μ_2 with $\mu_1 \leq 1 - \mu_2$ in (X, T) . Now $\text{cl}(\mu_1)$ and $\text{cl}(\mu_2)$ are fuzzy F_σ -sets in (X, T) . Let $\lambda_1 = \text{cl}(\mu_1)$ and $\lambda_2 = \text{cl}(\mu_2)$. Now

$$\text{cl}[\lambda_1 \vee \lambda_2] = \text{cl}[\text{cl}(\mu_1) \vee \text{cl}(\mu_2)] = \text{cl}[\text{cl}(\mu_1 \vee \mu_2)] = \text{cl}(\mu_1 \vee \mu_2) = 1.$$

By hypothesis, the fuzzy F_σ -sets μ_1 and μ_2 are fuzzy open sets in (X, T) . Now (X, T) being a fuzzy extremally disconnected space, for the open sets μ_1 and μ_2 with $\mu_1 \leq 1 - \mu_2$, by Proposition 4.7, $\text{cl}(\mu_1) \leq 1 - \text{cl}(\mu_2)$ and

thus $\lambda_1 \leq 1 - \lambda_2$ in (X, T) . It follows that $\mu_1 \leq \lambda_1 \leq 1 - \lambda_2$ and $\mu_2 \leq \lambda_2 \leq 1 - \lambda_1$. Thus, for the fuzzy F_σ -sets μ_1 and μ_2 with $\mu_1 \leq 1 - \mu_2$, there exist fuzzy F_σ -sets λ_1 and λ_2 in (X, T) such that $\mu_1 \leq 1 - \lambda_2$, $\mu_2 \leq 1 - \lambda_1$ and $\text{cl}(\lambda_1 \vee \lambda_2) = 1$ in (X, T) . Hence, (X, T) is a fuzzy feebly F_σ -complemented space.

Proposition 4.9. *If (X, T) is a fuzzy F' -, F_σ -complemented P -space, then (X, T) is a fuzzy normal space.*

Proof. Let γ and δ be disjoint fuzzy closed sets in (X, T) . Then, $\gamma \wedge \delta = 0$ in (X, T) and this implies that $\gamma \leq 1 - \delta$. Since each fuzzy closed set is a fuzzy F_σ -set in (X, T) , γ and δ are fuzzy F_σ -sets in (X, T) . By hypothesis, (X, T) is a fuzzy F' -space and fuzzy F_σ -complemented space. Then, by Proposition 4.6, (X, T) is a fuzzy feebly F_σ -complemented space. By Proposition 4.5, the fuzzy feebly F_σ -complemented and fuzzy P space (X, T) is a fuzzy F -space. Thus, for the fuzzy F_σ -sets γ and δ , there exist fuzzy G_δ -sets α and β in (X, T) such that $\gamma \leq \alpha$, $\delta \leq \beta$ and $\alpha \leq 1 - \beta$ in (X, T) . Now (X, T) being a fuzzy P -space, the fuzzy G_δ -sets α and β are fuzzy open sets in (X, T) . Thus, for the fuzzy closed sets γ and δ such that $\gamma \leq 1 - \delta$ in (X, T) , there exist fuzzy open sets α and β in (X, T) such that $\gamma \leq \alpha$, $\delta \leq \beta$ and $\alpha \leq 1 - \beta$, in (X, T) and hence (X, T) is a fuzzy normal space.

Theorem 4.1 [17]. *If λ is a fuzzy first category set in a fuzzy globally disconnected space (X, T) , then λ is a fuzzy F_σ -set in (X, T) .*

Proposition 4.10. *If $\mu_1 \leq 1 - \mu_2$, where μ_1 and μ_2 are fuzzy first category sets in a fuzzy globally disconnected and fuzzy feebly F_σ -complemented space (X, T) , then there exist fuzzy G_δ -sets δ_1 and δ_2 in (X, T) such that $\mu_1 \leq \delta_1$, $\mu_2 \leq \delta_2$ and $\text{int}(\delta_1) \leq 1 - \text{int}(\delta_2)$.*

Proof. Suppose that $\mu_1 \leq 1 - \mu_2$, where μ_1 and μ_2 are fuzzy first category sets in (X, T) . Since (X, T) is a fuzzy globally disconnected space, by Theorem 4.1, μ_1 and μ_2 are fuzzy F_σ -sets in (X, T) . Now (X, T) being a fuzzy feebly F_σ -complemented space, for the fuzzy F_σ -sets μ_1 and μ_2 with $\mu_1 \leq 1 - \mu_2$ in (X, T) , by Proposition 3.1, there exist fuzzy G_δ -sets δ_1 and δ_2 in (X, T) such that $\mu_1 \leq \delta_1$, $\mu_2 \leq \delta_2$ and $\text{int}(\delta_1) \leq 1 - \text{int}(\delta_2)$.

Theorem 4.2 [22]. *If λ is a fuzzy regular open set in a fuzzy Oz-space (X, T) , then λ is a fuzzy F_σ -set in (X, T) .*

Proposition 4.11. *If $\mu_1 \leq 1 - \mu_2$, where μ_1 and μ_2 are fuzzy regular open sets in a fuzzy feebly F_σ -complemented and fuzzy Oz-space (X, T) , then there exist fuzzy G_δ -sets δ_1 and δ_2 in (X, T) such that $\mu_1 \leq \delta_1$, $\mu_2 \leq \delta_2$ and $\text{int}(\delta_1) \leq 1 - \text{int}(\delta_2)$.*

Proof. Suppose that $\mu_1 \leq 1 - \mu_2$, where μ_1 and μ_2 are fuzzy regular open sets in (X, T) . Since (X, T) is a fuzzy Oz-space, by Theorem 4.2, μ_1 and μ_2 are fuzzy F_σ -sets in (X, T) . Now (X, T) being a fuzzy feebly F_σ -complemented space, for the fuzzy F_σ -sets μ_1 and μ_2 with $\mu_1 \leq 1 - \mu_2$ in (X, T) , by Proposition 3.1, there exist fuzzy G_δ -sets δ_1 and δ_2 in (X, T) such that $\mu_1 \leq \delta_1$, $\mu_2 \leq \delta_2$ and $\text{int}(\delta_1) \leq 1 - \text{int}(\delta_2)$.

The following proposition shows that fuzzy feebly F_σ -complemented spaces are not fuzzy hyperconnected spaces.

Proposition 4.12. *If a fuzzy topological space (X, T) is a fuzzy feebly F_σ -complemented space, then (X, T) is not a fuzzy hyperconnected space.*

Proof. Suppose that μ_1 and μ_2 are fuzzy F_σ -sets with $\mu_1 \leq 1 - \mu_2$ in (X, T) . Since (X, T) is a fuzzy feebly F_σ -complemented space, by Proposition 3.8, $\text{cl int}(\mu_2) \neq 1$ and $\text{cl int}(\mu_1) \neq 1$ in (X, T) . Hence, for the

fuzzy open sets $\text{int}(\mu_1)$ and $\text{int}(\mu_2)$ in (X, T) , $\text{cl}[\text{int}(\mu_2)] \neq 1$ and $\text{cl}[\text{int}(\mu_1)] \neq 1$, which show that (X, T) is not a fuzzy hyperconnected space.

5. Conclusion

In this paper, the notion of fuzzy feebly F_σ -complemented space is introduced and several characterizations of fuzzy feebly F_σ -complemented spaces are established. It is obtained that fuzzy weakly F_σ -complemented spaces are fuzzy feebly F_σ -complemented spaces. It is found that fuzzy feebly F_σ -complemented spaces are not fuzzy hyperconnected spaces. It is established that fuzzy F_σ -complemented and fuzzy extremally disconnected spaces in which fuzzy F_σ -sets are fuzzy open, are fuzzy feebly F_σ -complemented spaces. It is obtained that fuzzy F_σ -complemented and fuzzy F' -spaces are fuzzy feebly F_σ -complemented spaces and fuzzy feebly F_σ -complemented and fuzzy P spaces are fuzzy F -spaces. It is established that those fuzzy F' -spaces which are also fuzzy P -spaces, having fuzzy F_σ -complementedness property, are fuzzy normal spaces.

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