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ON FUZZY *Q*-SPACES

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Abstract

By combining fuzzy extremally disconnectedness and fuzzy open hereditarily irresolvablity of fuzzy topological spaces, the notion of fuzzy Q-space is introduced in this article. Various properties of this notion are discussed and conditions under which fuzzy extremely disconnected spaces become fuzzy Q-spaces are obtained. Fuzzy Q-spaces are found to be fuzzy second category spaces but not fuzzy Baire spaces and fuzzy Q-spaces are not fuzzy F_σ-complemented spaces.

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1. Introduction

The application of mathematical concepts in various disciplines showcases the breadth of its impact and it depends firmly and closely how we introduce basic ideas that may lead to new theories in different directions. Through his significant theory on fuzzy sets, in 1965, Zadeh [35] made the first effective attempt in mathematical modelling to contain non-probabilistic uncertainty, i.e., uncertainty that is not caused by randomness of an event. A fuzzy set is one in which each element of the universe belongs to it, but with a value or degree of belongingness that falls between 0 and 1, and these values are referred to as the membership value of each element in that set. The potential of fuzzy notion was realized and has successfully been applied in variety of scientific domains including Statistics, Applied Mathematics, Dynamics and Mathematical Biology. In 1968, Chang [5] introduced the concept of fuzzy topological space and this introduction leads to subsequent tremendous growth of the numerous fuzzy topological concepts. In recent years, a considerable amount of research has been done on identifying new types of fuzzy topological spaces.

At the beginning of twentieth century, the problem of resolvability of topological spaces became a matter of intense research. Research in this area stems from the papers of Hewitt [9] and Katetov [11]. El'Kin [6] introduced the notion of open hereditarily irresolvablity of topological spaces, in 1969. Inspired by their works on resolvability, the concepts of resolvability, irresolvability and open hereditarily irresolvability in fuzzy setting were introduced and studied by Thangaraj and Balasubramanian [18].

The notion of extremally disconnectedness for topological spaces has widely been studied by many mathematicians and some of the nice applications of extremally disconnected spaces are found in Herrmann's [8] work. Jankovic [10] has applied the notion of extremally disconnectedness in his investigations of certain types of mappings. The concept of fuzzy extremally disconnected spaces was introduced and studied by Ghosh [7]. Sarma [13] studied the inter-relationships between fuzzy weak continuous and fuzzy semi-continuous functions between fuzzy topological spaces in the context of extremal disconnectedness.

While carrying out a study on various types of fuzzy topological spaces, we can find certain fuzzy topological spaces which are fuzzy open hereditarily irresolvable spaces but not fuzzy extremally disconnected spaces. For example, fuzzy Brown spaces are fuzzy open hereditarily irresolvable spaces but not fuzzy extremally disconnected spaces [26]. Also, we can find fuzzy topological spaces which are fuzzy extremally disconnected spaces but not open hereditarily irresolvable spaces. For example, fuzzy hyperconnected spaces in which fuzzy dense sets are having zero interior are fuzzy extremally disconnected spaces but not open hereditarily irresolvable spaces.

In classical topology, Nikolić [4] defined Q-spaces as those topological spaces which are both hereditarily irresolvable and fuzzy extremally disconnected spaces. Motivated on these lines, the concept of fuzzy Q-spaces is defined by combining fuzzy extremally disconnectedness and fuzzy open hereditarily irresolvablity of fuzzy topological spaces in this paper. It is aimed to find out those fuzzy topological spaces which are fuzzy second category but not fuzzy Baire. In this article, a work on identification of those fuzzy spaces which are fuzzy Q-spaces and those fuzzy spaces which are not fuzzy Q-spaces, has been carried out. Conditions under which fuzzy extremely disconnected spaces become fuzzy Q-spaces are explored.

2. Preliminaries

This part explains the concepts and findings that we need to know in order to comprehend the article. In this work, by (X, T) or simply by X, we denote a fuzzy topological space due to Chang [5].

Definition 2.1 [5]**.** Let X be a non-empty set and I, the unit interval $[0, 1]$. Then a fuzzy set λ in X is a mapping from X into I.

The *fuzzy set* 0_X is defined as $0_X(x) = 0,$ for all $x \in X$ and the *fuzzy set* 1_X is defined as $1_X(x) = 1,$ for all $x \in X$.

Definition 2.2 [5]**.** A fuzzy topology on a set *X* is a family *T* of fuzzy sets in *X* which satisfies the following conditions:

(a) $0_X \in T$ and $1_X \in T$.

(b) If $A, B \in T$, then $\text{cl}(\lambda) \neq 1$.

(c) If $A_i \in T$ for each $i \in J$, then $\vee_i A_i \in T$.

T is called a *fuzzy topology* for *X* and the ordered pair (X, T) is a fuzzy topological space, or fts for short. Every member of *T* is called a *T-open fuzzy set*. A fuzzy set is *T*-closed if and only if its complement is *T*-open. In the sequel, when no confusion is likely to arise, we call a *T*-open (*T*-closed) fuzzy set simply a *fuzzy open* (*closed*) *set*.

Definition 2.3 [5]**.** The *interior*, the *closure* and the *complement* of a fuzzy set λ in a fuzzy topological space (X, T) are defined, respectively, as follows:

(i) $\text{int}(\lambda) = \vee\{\mu/\mu \leq \lambda, \mu \in T\}$;

(ii) $\text{cl}(\lambda) = \wedge\{\mu/\lambda \leq \mu, 1 - \mu \in T\}$.

(iii) $\lambda'(x) = 1 - \lambda(x)$, for all $x \in X$.

For a family $(\lambda_i)_{i \in I}$ of fuzzy sets in (X, T), the *union* $\vee_{i \in I} \lambda_i$ and *intersection* $\wedge_{i \in I} \lambda_i$ are defined, respectively, as follows: For each $x \in X$,

(iv) $(\vee_{i \in I} \lambda_i)(x) = \sup_{i \in I} \lambda_i(x)$,

(v) $(\wedge_{i \in I} \lambda_i)(x) = \inf_{i \in I} \lambda_i(x)$.

Definition 2.4. Let λ be a fuzzy subset of *X*. Then, λ is called a

(i) *fuzzy regular-open* in (X, T) if $\lambda = \text{int cl}(\lambda)$ and

fuzzy regular-closed in (X, T) if $\lambda = \text{cl int}(\lambda)$ [1],

(ii) *fuzzy semi-open* in (X, T) if $\lambda \leq \text{cl int}(\lambda)$ and

fuzzy semi-closed in (X, T) if $\text{int cl}(\lambda) \leq \lambda$ [1],

(iii) *fuzzy pre-open* in (X, T) if $\lambda \le \operatorname{int}\operatorname{cl}(\lambda)$ and

fuzzy pre-closed in (X, T) if $\operatorname{cl}\operatorname{int}(\lambda) \le \lambda$ [3],

(iv) *fuzzy* α*-open* in (X, T) if $\lambda \le \operatorname{intclint}(\lambda)$ and

fuzzy α*-closed* in (X, T) if $\operatorname{clintcl}(\lambda) \le \lambda$ [14],

(v) *fuzzy* G_δ*-set* in (X, T) if $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$, where $\lambda_i \in T$ for $i \in I$; fuzzy F_σ-set in (X, T) if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $1 - \lambda_i \in T$ for $i \in I$ [2],

(vi) *fuzzy regular* G_δ*-set* in (X, T) if $\lambda = \wedge_{i=1}^{\infty} \operatorname{int}(\lambda_i)$, where $1 - \lambda_i \in T$; *fuzzy regular* F_σ*-set* in (X, T) if $\lambda = \vee_{i=1}^{\infty}\operatorname{cl}(\mu_i)$, where $\mu_i \in T$ [24].

Definition 2.5. Let λ be a fuzzy subset of *X*. Then, λ is called an

(i) *fuzzy dense set* in (X, T) if $\operatorname{cl}(\lambda) = 1$, in (X, T). That is, there exists no fuzzy closed set δ in (X, T) such that $\lambda < \delta < 1$ [16],

(ii) *fuzzy nowhere dense set* in (X, T) if $\operatorname{intcl}(\lambda) = 0$, in (X, T). That is, there exists no non-zero fuzzy open set γ in (X, T) such that $\gamma < \operatorname{cl}(\lambda)$ [16],

(iii) *fuzzy first category set* in (X, T) if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where the fuzzy sets (λ_i)'s are fuzzy nowhere dense in (X, T). Any other fuzzy set is said to be of *fuzzy second category* in (X, T) [16],

(iv) *fuzzy residual set* in (X, T) if the complement of a fuzzy set λ, that is, $1 - \lambda$ is a fuzzy first category set in (X, T) [19],

(v) *fuzzy somewhere dense set* in (X, T) if there exists a non-zero fuzzy open set γ in (X, T) such that $\gamma < \operatorname{cl}(\lambda)$. That is, $\operatorname{intcl}(\lambda) \ne 0$, in (X, T) [17] and the fuzzy complement of a fuzzy somewhere dense set in (X, T), that is, $1 - \lambda$ is called a *fuzzy cs dense set* in (X, T) [22],

(vi) *fuzzy Baire set* in (X, T) if $\lambda = \gamma \wedge \delta$, where the fuzzy set γ is a fuzzy open set and the fuzzy set δ is a fuzzy residual set in (X, T) [31].

Definition 2.6. Let (X, T) be a fuzzy topological space. Then, the space X is called a

(i) *fuzzy submaximal space* if for each fuzzy set λ in (X, T) such that $\mathrm{cl}(\lambda) = 1$, $\lambda \in T$ [2],

(ii) *fuzzy hyperconnected space* if every non-null fuzzy open subset of X is fuzzy dense in X [12],

(iii) *fuzzy extremally disconnected space* if the closure of every fuzzy open set of X is fuzzy open in X [7],

(iv) *fuzzy open hereditarily irresolvable space* if $\mathrm{intcl}(\gamma) \neq 0$, for any non-zero fuzzy set γ in X, then $\mathrm{int}(\gamma) \neq 0$, in X [18],

(v) *fuzzy globally disconnected space* if each fuzzy semi-open set in X is a fuzzy open set in X [23],

(vi) *fuzzy P-space* if each fuzzy G_δ-set in X is fuzzy open in X [15],

(vii) *fuzzy almost P-space* if for each non-zero fuzzy G_δ-set δ in X, $\mathrm{int}(\delta) \neq 0$ in X [20],

(viii) *weak fuzzy P-space* if $\wedge_{i=1}^{\infty}(\lambda_i)$ is a fuzzy regular open set in (X, T), where (λ_i)'s are fuzzy regular open sets in (X, T) [20],

(ix) *fuzzy EZ-space* if for each non-zero fuzzy open set λ in X $\mathrm{cl}(\lambda) = \mathrm{cl}(\vee_{i=1}^{\infty}(\lambda_i))$, where (λ_i)'s are fuzzy clopen sets in (X, T). That is, for each fuzzy regular closed set μ in (X, T), $\mu = \mathrm{cl}(\eta)$, where η is a fuzzy open and fuzzy F_σ-set in (X, T) [30],

(x) *fuzzy Baire space* if $\mathrm{int}(\vee_{i=1}^{\infty}(\lambda_i)) = 0$, where the fuzzy sets (λ_i)'s are fuzzy nowhere dense sets in X [19],

(xi) *fuzzy second category space* if $\text{int}(\vee_{i=1}^{\infty}(\lambda_i)) \neq 0,$ where the fuzzy sets (λ_i)'s are fuzzy nowhere dense sets in X [19],

(xii) *fuzzy extraresolvable space* if λ_i and λ_j $(i \neq j)$ are fuzzy dense sets in X, then $\lambda_i \wedge \lambda_j$ is a fuzzy nowhere dense set in X [34],

(xiii) *fuzzy Brown space* if λ and μ are any two non-zero fuzzy open sets in X, then $\text{cl}(\lambda) \not\leq 1 - \text{cl}(\mu),$ in X [26],

(xiv) *fuzzy Baire-dominated space* if for each collection $\{\lambda_i\}$ $(i = 1 \text{ to } \infty)$ of fuzzy closed sets with $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ and $\wedge_{i=1}^{\infty}(\lambda_i) = 0,$ there exists a collection $\{\mu_i\}$ $(i = 1 \text{ to } \infty)$ of fuzzy Baire sets in (X, T) with $\lambda_i \leq \mu_i$ for each i and $\wedge_{i=1}^{\infty}(\mu_i) = 0,$ in (X, T) [31],

(xv) *fuzzy F_σ-complemented space* if for each fuzzy F_σ-set λ in X, there exists a fuzzy F_σ-set μ in X such that $\lambda \leq 1 - \mu$ and $\text{cl}(\lambda \vee \mu) = 1$ [29],

(xvi) *fuzzy Oz-space* if each fuzzy regular closed set is a fuzzy G_δ-set in X [27],

(xvii) *fuzzy regular Oz-space* if each fuzzy regular closed set in X is a fuzzy regular G_δ-set in X [28],

(xviii) *fuzzy regular space* iff each fuzzy open set λ of X is a union of fuzzy open sets (λ_α)'s of X such that $\text{cl}\lambda_\alpha \leq \lambda$ for each α [1],

(xix) *fuzzy maximal space* if X is a fuzzy extremally disconnected and fuzzy submaximal space [25].

Lemma 2.1 [1]. *For a fuzzy set δ of a fuzzy space X,*

(i) $1 - \text{int}(\delta) = \text{cl}(1 - \delta)$ *and* (ii) $1 - \text{cl}(\delta) = \text{int}(1 - \delta).$

Theorem 2.1 [7]. *A fuzzy topological space (X, T) is fuzzy extremally disconnected if and only if $FSO(X) \subset FPO(X).$*

Theorem 2.2 [7]**.** *For any fuzzy topological space* (X, T), *the following are equivalent*:

(a) *X is a fuzzy extremally disconnected space*.

(b) *For each fuzzy closed set* μ *in X*, $\text{int}(\mu)$ *is fuzzy closed in X*.

(c) *For each fuzzy open set* λ *in X*, $\text{cl}(\lambda) + \text{cl}[1 - \text{cl}(\lambda)] = 1$.

Theorem 2.3 [32]**.** *If* δ *is a fuzzy* G_δ*-set in a fuzzy almost P-space X, then there exists a fuzzy regular closed set* γ *in X such that* $\gamma \leq \text{cl}(\delta)$.

Theorem 2.4 [21]**.** *If a fuzzy topological space* (X, T) *is a fuzzy almost P-space*, *then* (X, T) *is a fuzzy second category space*.

Theorem 2.5 [26]**.** *If* (X, T) *is a fuzzy Brown space*, *then* (X, T) *is not a fuzzy extremally disconnected space*.

Theorem 2.6 [18]**.** *If* (X, T) *is a fuzzy open hereditarily irresolvable space*, *then* $\text{cl}(\lambda) = 1$, *for any non-zero fuzzy set* λ *in X implying that* $\text{clint}(\lambda) = 1$, *in X*.

Theorem 2.7 [34]**.** *If* (X, T) *is a fuzzy submaximal space*, *then* (X, T) *is a fuzzy open hereditarily irresolvable space*.

Theorem 2.8 [27]**.** *If* (X, T) *is a fuzzy Oz and fuzzy P-space*, *then* (X, T) *is a fuzzy extremally disconnected space*.

Theorem 2.9 [19]**.** *Let* (X, T) *be a fuzzy topological space*. *Then*, *the following are equivalent*:

(1) (X, T) *is a fuzzy Baire space*.

(2) $\text{int}(\lambda) = 0$, *for each fuzzy first category set* λ *in* (X, T).

(3) $\text{cl}(\mu) = 1$, *for each fuzzy residual set* μ *in* (X, T).

Theorem 2.10 [33]**.** *If* μ *is a fuzzy* G_δ*-set in a fuzzy topological space* (X, T) *such that* $\text{cl}(\mu) = 1$, *then* μ *is a fuzzy residual set in* (X, T).

Theorem 2.11 [31]**.** *If* (X, T) *is a fuzzy Baire-dominated and fuzzy extraresolvable space*, *then* (X, T) *is a fuzzy extremally disconnected space*.

Theorem 2.12 [30]**.** *If* (X, T) *is a fuzzy regular space*, *then each fuzzy open set is a fuzzy* F_σ*-set in* (X, T).

Theorem 2.13 [30]**.** *If* (X, T) *is a fuzzy extremally disconnected space*, *then* (X, T) *is a fuzzy EZ-space*.

Theorem 2.14 [30]**.** *If* δ *is a fuzzy set defined on X in a fuzzy EZ-space* (X, T) *with* $\text{int}(\delta) \neq 0$, *then* (X, T) *is a fuzzy open hereditarily irresolvable space*.

Theorem 2.15 [28]**.** *If* (X, T) *is a fuzzy regular Oz and weak fuzzy P-space*, *then* (X, T) *is a fuzzy extremally disconnected space*.

Theorem 2.16 [19]**.** *If* λ *is a fuzzy first category set in a fuzzy topological space* (X, T), *then there is a fuzzy* F_σ*-set* η *in* (X, T) *such that* $\lambda \leq \eta$.

3. Fuzzy Q-spaces

Nikolić [4] introduced the notion of Q-spaces in classical topology. Analogously, we introduce and study the notion of Q-spaces in fuzzy setting.

Definition 3.1. A fuzzy topological space X is called a *fuzzy* Q*-space* if the space X is a fuzzy open hereditarily irresolvable space which is also a fuzzy extremally disconnected space.

Example 3.1. Let $X = \{A, B, C\}$. Then the fuzzy sets α, β, γ, δ and η are defined on X as follows:

$\alpha : X \to I$ is defined by $\alpha(A) = 0.5$; $\alpha(B) = 0.6$; $\alpha(C) = 0.4$,

$\beta : X \to I$ is defined by $\beta(A) = 0.4$; $\beta(B) = 0.5$; $\beta(C) = 0.6$,

$\gamma : X \to I$ is defined by $\gamma(A) = 0.6$; $\gamma(B) = 0.4$; $\gamma(C) = 0.5$,

$\delta : X \to I$ is defined by $\delta(a) = 0.5;\ \delta(b) = 0.5;\ \delta(c) = 0.5,$

$\eta : X \to I$ is defined by $\eta(a) = 0.6;\ \eta(b) = 0.5;\ \eta(c) = 0.4.$

Then,

$$T = \{0, \alpha, \beta, \gamma, \alpha \vee \beta, \alpha \vee \gamma, \beta \vee \gamma, \alpha \wedge \beta, \alpha \wedge \gamma, \beta \wedge \gamma, \alpha \vee [\beta \wedge \gamma],$$
$$\beta \vee [\alpha \wedge \gamma], \gamma \vee [\alpha \wedge \beta], \alpha \wedge [\beta \vee \gamma], \beta \wedge [\alpha \vee \gamma], \gamma \wedge [\alpha \vee \beta],$$
$$\alpha \vee \beta \vee \gamma, \alpha \wedge \beta \wedge \gamma, 1\}$$

is a fuzzy topology on X. On computing, we find that closure of each fuzzy open set is fuzzy open in (X, T) and thus (X, T) is a fuzzy extremally disconnected space. On computing, we find that

$$\operatorname{intcl}(\delta) = \operatorname{int}(1 - [\beta \wedge [\alpha \vee \gamma]]) = \gamma \vee [\alpha \wedge \beta] \neq 0$$

and $\operatorname{int}(\delta) = \alpha \wedge [\beta \vee \gamma] \neq 0.$ Also, $\operatorname{intcl}(\eta) = \operatorname{int}(1 - \beta) = \alpha \wedge [\beta \vee \gamma] \neq 0$ and $\operatorname{int}(\eta) = \alpha \wedge [\beta \vee \gamma] \neq 0$ and thus (X, T) is a fuzzy open hereditarily irresolvable space. Hence (X, T) is a fuzzy Q-space.

Example 3.2. Let $X = \{a, b, c\}$. Define fuzzy sets α, β, γ and δ as follows:

$\alpha : X \to I$ is defined by $\alpha(a) = 0.5;\ \alpha(b) = 0.5;\ \alpha(c) = 0.6,$

$\beta : X \to I$ is defined by $\beta(a) = 0.5;\ \beta(b) = 0.6;\ \beta(c) = 0.5,$

$\gamma : X \to I$ is defined by $\gamma(a) = 0.6;\ \gamma(b) = 0.4;\ \gamma(c) = 0.5,$

$\delta : X \to I$ is defined by $\delta(a) = 0.6;\ \delta(b) = 0.5;\ \delta(c) = 0.4.$

Then,

$$T = \{0, \alpha, \beta, \gamma, \alpha \vee \beta, \alpha \vee \gamma, \beta \vee \gamma, \alpha \wedge \beta, \alpha \wedge \gamma,$$
$$\gamma \vee [\alpha \wedge \beta], \alpha \vee \beta \vee \gamma, 1\}$$

is a fuzzy topology on X. On computing, we can find that

$$\mathrm{cl}(\alpha) = 1; \quad \mathrm{int}(1 - \alpha) = 0;$$

$$\mathrm{cl}(\beta) = 1 - (\alpha \wedge \gamma) = \beta; \quad \mathrm{int}(1 - \beta) = \alpha \wedge \gamma;$$

$$\mathrm{cl}(\gamma) = 1; \quad \mathrm{int}(1 - \gamma) = 0;$$

$$\mathrm{cl}(\alpha \vee \beta) = 1; \quad \mathrm{int}(1 - [\alpha \vee \beta]) = 0;$$

$$\mathrm{cl}(\alpha \vee \gamma) = 1; \quad \mathrm{int}(1 - [\alpha \vee \gamma]) = 0;$$

$$\mathrm{cl}(\beta \vee \gamma) = 1; \quad \mathrm{int}(1 - [\beta \vee \gamma]) = 0;$$

$$\mathrm{cl}(\alpha \wedge \beta) = 1 - (\alpha \wedge \beta) = \alpha \wedge \beta; \quad \mathrm{int}(1 - [\alpha \wedge \beta]) = \alpha \wedge \beta;$$

$$\mathrm{cl}(\alpha \wedge \gamma) = 1 - \beta = \alpha \wedge \gamma; \quad \mathrm{int}(1 - [\alpha \wedge \gamma]) = \beta;$$

$$\mathrm{cl}(\alpha \vee \beta \vee \gamma) = 1; \quad \mathrm{int}(1 - [\alpha \vee \beta \vee \gamma]) = 0;$$

$$\mathrm{cl}(\delta) = 1; \quad \mathrm{int}(\delta) = 0.$$

On computation, we find that closure of each fuzzy open set in X is fuzzy open in X and thus (X, T) is a fuzzy extremally disconnected space. For a fuzzy set δ, $\mathrm{intcl}(\delta) = \mathrm{int}(1) = 1 \neq 0$ and $\mathrm{int}(\delta) = 0$, showing that (X, T) is not a fuzzy open hereditarily irresolvable space. Hence (X, T) is not a fuzzy Q-space.

Example 3.3. Let $X = \{A, B, C\}$. Now α, β and γ are the fuzzy sets defined on X as below:

$\alpha : X \to I$ is defined by $\alpha(A) = 0.6$; $\alpha(B) = 0.4$; $\alpha(C) = 0.5$,

$\beta : X \to I$ is defined by $\beta(A) = 0.5$; $\beta(B) = 0.7$; $\beta(C) = 0.6$,

$\gamma : X \to I$ is defined by $\gamma(A) = 0.4$; $\gamma(B) = 0.5$; $\gamma(C) = 0.7$.

Clearly,

$$T = \{0, \alpha, \beta, \gamma, \alpha \vee \beta, \alpha \vee \gamma, \beta \vee \gamma, \alpha \wedge \beta, \alpha \wedge \gamma, \beta \wedge \gamma, \alpha \vee [\beta \wedge \gamma], \gamma \vee [\alpha \wedge \beta], \beta \wedge [\alpha \vee \gamma], \alpha \vee \beta \vee \gamma, 1\}$$

is a fuzzy topology on *X*. By computation, we can find that $\beta, \gamma, \alpha \vee \beta, \alpha \vee \gamma, \beta \vee \gamma, \beta \wedge \gamma, \alpha \vee [\beta \wedge \gamma], \gamma \vee [\alpha \wedge \beta], \beta \wedge [\alpha \vee \gamma]$, and $\alpha \vee \beta \vee \gamma$ are fuzzy dense sets in (X, T) and $\mathrm{cl}(\alpha) = 1 - (\alpha \wedge \gamma)$, $\mathrm{cl}(\alpha \wedge \beta) = 1 - (\alpha \wedge \beta)$, $\mathrm{cl}(\alpha \wedge \gamma) = 1 - \alpha$.

Now

$$\mathrm{intcl}(\alpha) = \alpha \neq 0 \text{ and } \mathrm{int}(\alpha) = \alpha \neq 0;$$

$$\mathrm{intcl}(\alpha \wedge \beta) = \alpha \wedge \beta \neq 0 \quad \text{and} \quad \mathrm{int}(\alpha \wedge \beta) = \alpha \wedge \beta \neq 0;$$

$$\mathrm{intcl}(\alpha \wedge \gamma) = \alpha \wedge \gamma \neq 0 \quad \text{and} \quad \mathrm{int}(\alpha \wedge \gamma) = \alpha \wedge \gamma \neq 0.$$

For the fuzzy dense sets $\eta (= \beta, \gamma, \alpha \vee \beta, \alpha \vee \gamma, \beta \vee \gamma, \beta \wedge \gamma, \alpha \vee [\beta \wedge \gamma], \gamma \vee [\alpha \wedge \beta], \beta \wedge [\alpha \vee \gamma], \alpha \vee \beta \vee \gamma)$, $\mathrm{intcl}(\eta) = 1 \neq 0$ and $\mathrm{int}(\eta) = \eta \neq 0$. Thus (X, T) is a fuzzy open hereditarily irresolvable space. Now, for a fuzzy open set α, $\mathrm{cl}(\alpha)$ is not a fuzzy open set in (X, T), showing that (X, T) is not a fuzzy extremally disconnected space. Hence (X, T) is not a fuzzy *Q*-space.

Proposition 3.1. *If* λ *is a fuzzy somewhere dense set in a fuzzy Q-space X, then* $\mathrm{clint}(\lambda) \leq \mathrm{intcl}(\lambda)$, *in X.*

Proof. Suppose that λ is a fuzzy somewhere dense set in (X, T). Then, $\mathrm{intcl}(\lambda) \neq 0$, in *X*. Since (X, T) is a fuzzy *Q*-space, (X, T) is a fuzzy open hereditarily irresolvable space and $\mathrm{intcl}(\lambda) \neq 0$, implying that $\mathrm{int}(\lambda) \neq 0$, in *X*. Now (X, T) being a fuzzy *Q*-space, (X, T) is a fuzzy extremally disconnected space and thus, for a non-zero fuzzy open set $\mathrm{int}(\lambda)$, $\mathrm{clint}(\lambda)$ is a fuzzy open set in *X*. Then, $\mathrm{clint}(\lambda) = \mathrm{int}[\mathrm{clint}(\lambda)] \leq \mathrm{intcl}(\lambda)$ and thus it follows that $\mathrm{clint}(\lambda) \leq \mathrm{intcl}(\lambda)$, in *X*.

Corollary 3.1. *If* $\lambda \leq \mathrm{intcl}(\lambda)$, *for a fuzzy set* λ *in a fuzzy Q-space X, then* $\mathrm{clint}(\lambda) \leq \mathrm{intcl}(\lambda)$, *in X.*

Proof. Suppose that $\lambda \leq \text{intcl}(\lambda)$, for a fuzzy set λ in *X*. This implies that $\text{intcl}(\lambda) \neq 0$, in *X* and then λ is a fuzzy somewhere dense set in the fuzzy *Q*-space *X*. By Proposition 3.1, $\text{clint}(\lambda) \leq \text{intcl}(\lambda)$, in *X*.

Corollary 3.2. *If* λ *is a fuzzy open set in a fuzzy Q-space X, then* $\text{clint}(\lambda) \leq \text{intcl}(\lambda)$, *in X.*

Proof. For a fuzzy open set λ in *X*, $\lambda \leq \text{intcl}(\lambda)$ and by Corollary 3.1, $\text{clint}(\lambda) \leq \text{intcl}(\lambda)$, in *X*.

Corollary 3.3. *If* λ *is a fuzzy pre-open set in a fuzzy Q-space X, then* $\text{clint}(\lambda) \leq \text{intcl}(\lambda)$, *in X.*

Proof. Suppose that λ is a fuzzy pre-open set in *X*. Then, $\lambda \leq \text{intcl}(\lambda)$, in *X*. This implies that $\text{intcl}(\lambda) \neq 0$ and thus λ is a fuzzy somewhere dense set in the fuzzy *Q*-space *X*. Then, by Proposition 3.1, $\text{clint}(\lambda) \leq \text{intcl}(\lambda)$, in *X*.

Corollary 3.4. *If* λ *is a fuzzy semi-open set in a fuzzy Q-space X, then* $\text{clint}(\lambda) \leq \text{intcl}(\lambda)$, *in X.*

Proof. Let λ be a fuzzy semi-open set in *X*. Now *X* being a fuzzy *Q*-space, *X* is a fuzzy extremally disconnected space and thus, by Theorem 2.1, the fuzzy semi-open open set λ is a fuzzy pre-open open set in *X*. By Corollary 3.1, it follows that $\text{clint}(\lambda) \leq \text{intcl}(\lambda)$, in *X*.

Proposition 3.2. *If* λ *is a fuzzy semi-open set in a fuzzy Q-space X, then* $\text{intclint}(\lambda) = \text{intcl}(\lambda) = \text{clint}(\lambda) = \text{clintcl}(\lambda)$, *in X.*

Proof. Let λ be a fuzzy semi-open set in *X*. Then, $\lambda \leq \text{clint}(\lambda)$, in *X*. This implies that $\text{cl}(\lambda) \leq \text{cl}(\text{clint}(\lambda))$ and then $\text{cl}(\lambda) \leq \text{clint}(\lambda)$, in (X, T). Now $\text{intcl}(\lambda) \leq \text{cl}(\lambda) \leq \text{clint}(\lambda)$ gives that $\text{intcl}(\lambda) \leq \text{clint}(\lambda)$. Since (X, T) is a fuzzy *Q*-space, for the fuzzy semi-open set λ, by Corollary 3.4, $\text{clint}(\lambda) \leq \text{intcl}(\lambda)$ and thus $\text{intcl}(\lambda) \leq \text{clint}(\lambda) \leq \text{intcl}(\lambda)$, in *X* and hence, $\text{clint}(\lambda) = \text{intcl}(\lambda)$, in *X*.

Now $\text{clint}(\lambda) = \text{intcl}(\lambda)$ implies that

$$\text{int}[\text{clint}(\lambda)] = \text{int}[\text{intcl}(\lambda)] = \text{intcl}(\lambda),$$

and $\text{intcl}(\lambda) = \text{clint}(\lambda)$ implies that $\text{cl}[\text{intcl}(\lambda)] = \text{cl}[\text{clint}(\lambda)] = \text{clint}(\lambda)$. Hence, it follows that $\text{intclint}(\lambda) = \text{intcl}(\lambda) = \text{clint}(\lambda) = \text{clintcl}(\lambda)$, in *X*.

Singal and Rajvansi [14] established by an example that fuzzy semi-open sets need not be fuzzy α-open sets in fuzzy topological spaces. The following proposition establishes that in fuzzy *Q*-spaces, fuzzy semi-open sets become fuzzy α-open sets.

Proposition 3.3. *If* λ *is a fuzzy semi-open set in a fuzzy Q-space X, then* λ *is a fuzzy* α*-open set in X.*

Proof. Let λ be a fuzzy semi-open set in *X*. Then, $\lambda \leq \text{clint}(\lambda)$, in *X*. Since (X, T) is a fuzzy *Q*-space, by Proposition 3.2, $\text{clint}(\lambda) = \text{intclint}(\lambda)$, in *X*. This implies that $\lambda \leq \text{clint}(\lambda) = \text{intclint}(\lambda)$ and thus $\lambda \leq \text{intclint}(\lambda)$. Hence λ is a fuzzy α-open set in *X*.

The following proposition establishes that fuzzy regular open sets become fuzzy pre-closed sets in fuzzy *Q*-spaces.

Proposition 3.4. *If* λ *is a fuzzy regular open set in a fuzzy Q-space X, then* λ *is a fuzzy pre-closed set in X.*

Proof. Let λ be a fuzzy regular open set in *X*. Then, $\lambda = \text{intcl}(\lambda)$, in *X* and this implies that $\text{intcl}(\lambda) \neq 0$. Thus λ is a fuzzy somewhere dense set in *X*. Then, by Proposition 3.1, $\text{clint}(\lambda) \leq \text{intcl}(\lambda)$ and $\text{clint}(\lambda) \leq \lambda$, in *X*. Hence λ is a fuzzy pre-closed set in *X*.

Corollary 3.5. *If* μ *is a fuzzy regular closed set in a fuzzy Q-space X, then* μ *is a fuzzy pre-open set in X.*

Proof. Let μ be a fuzzy regular closed set in *X*. Then, $1 - \mu$ is a fuzzy regular open set in *X* and by Proposition 3.4, $1 - \mu$ is a fuzzy pre-closed set in *X*. Hence μ is a fuzzy pre-open set in *X*.

The following proposition provides a condition for a fuzzy extremally disconnected space to become a fuzzy Q-space.

Proposition 3.5. *If* $\mathrm{int}(\lambda) \neq 0$, *for a fuzzy set* λ *in a fuzzy extremally disconnected space X*, *then X is a fuzzy Q-space*.

Proof. Let λ be a non-zero fuzzy set in X. Now $\mathrm{cl}(\lambda)$ is a fuzzy closed set in X. Since X is a fuzzy extremally disconnected space, by Theorem 2.2, $\mathrm{int}[\mathrm{cl}(\lambda)]$ is a fuzzy closed set in X and then $\mathrm{intcl}(\lambda) \neq 0$, in X. By hypothesis, for a non-zero fuzzy set λ, $\mathrm{int}(\lambda) \neq 0$, in X. Thus, for a non-zero fuzzy set λ with $\mathrm{intcl}(\lambda) \neq 0$ in X, $\mathrm{int}(\lambda) \neq 0$ shows that X is a fuzzy open hereditarily irresolvable space and hence X is a fuzzy Q-space.

Remark 3.1. Proposition 3.5 can also be proved by using Theorems 2.13 and 2.14.

Proof. Let (X, T) be a fuzzy extremally disconnected space in which for each fuzzy set λ, $\mathrm{int}(\lambda) \neq 0$, in X. By Theorem 2.13, (X, T) is a fuzzy EZ-space. Thus (X, T) is a fuzzy EZ-space in which $\mathrm{int}(\lambda) \neq 0$, for each fuzzy set λ in X and by Theorem 2.14, (X, T) is a fuzzy open hereditarily irresolvable space. Thus, (X, T) is a fuzzy Q-space.

Proposition 3.6. *If* λ *is a non-zero fuzzy* G_δ*-set in a fuzzy Q-space X*, *then* $\mathrm{int}(\lambda) \neq 0$, *in X*.

Proof. Let λ be a non-zero fuzzy G_δ-set in (X, T). Now $cl(\lambda)$ is a fuzzy closed set in (X, T). Since (X, T) is a fuzzy Q-space, (X, T) is a fuzzy extremally disconnected space and by Theorem 2.2, $\mathrm{int}[\mathrm{cl}(\lambda)]$ is a fuzzy closed set in (X, T) and then $\mathrm{intcl}(\lambda) \neq 0$. Since (X, T) is a fuzzy Q-space, (X, T) is a fuzzy open hereditarily irresolvable space and $\mathrm{intcl}(\lambda) \neq 0$, implying that $\mathrm{int}(\lambda) \neq 0$, in (X, T).

The following proposition establishes that fuzzy F_σ-sets are not fuzzy dense sets in fuzzy Q-spaces.

Proposition 3.7. *If* λ *is a non-zero fuzzy* F_σ*-set in a fuzzy Q-space X, then* $\mathrm{cl}(\lambda) \neq 1,$ *in X.*

Proof. Let λ be a fuzzy F_σ-set in *X*. Then, $1 - \lambda$ is a fuzzy G_δ-set in *X*. Since *X* is a fuzzy *Q*-space, by Proposition 3.6, $\mathrm{int}(1 - \lambda) \neq 0.$ Then, by Lemma 2.1, $1 - \mathrm{cl}(\lambda) \neq 0$ and hence $\mathrm{cl}(\lambda) \neq 1,$ in *X*.

Proposition 3.8. *If* λ *is a fuzzy* G_δ*-set in a fuzzy Q-space X, then there exists a fuzzy* G_δ*-set* μ *in X such that* $\mathrm{cl}(\lambda) \leq \mu.$

Proof. Let λ be a fuzzy G_δ-set in $(X, T).$ Then, $\lambda = \wedge_{i=1}^{\infty}(\lambda_i),$ where $\lambda_i \in T.$ Now $\mathrm{cl}(\lambda) = \mathrm{cl}(\wedge_{i=1}^{\infty}(\lambda_i)) \leq \wedge_{i=1}^{\infty}[\mathrm{cl}(\lambda_i)].$ Since *X* is a fuzzy *Q*-space, *X* is a fuzzy extremally disconnected space and the fuzzy sets (λ_i)'*s* are fuzzy open in *X*, $[\mathrm{cl}(\lambda_i)]$'*s* are fuzzy open sets in *X*. Thus, $\wedge_{i=1}^{\infty}\mathrm{cl}(\lambda_i)$ is a fuzzy G_δ-set in *X*. Let $\mu = \wedge_{i=1}^{\infty}[\mathrm{cl}(\lambda_i)].$ Hence, for a fuzzy G_δ-set λ in a fuzzy *Q*-space *X*, there exists a fuzzy G_δ-set μ in *X* such that $\mathrm{cl}(\lambda) \leq \mu.$

Corollary 3.6. *If* μ *is a fuzzy* F_σ*-set in a fuzzy Q-space X, then there exists a fuzzy* F_σ*-set* η *in X such that* $\eta \leq \mathrm{int}(\mu).$

Proof. Let μ be a fuzzy F_σ-set in *X*. Then $1 - \mu$ is a fuzzy G_δ-set in *X*. Since *X* is a fuzzy *Q*-space, by Proposition 3.8, there exists a fuzzy G_δ-set δ in *X* such that $\mathrm{cl}(1 - \mu) \leq \delta.$ By Lemma 2.1, $\mathrm{cl}(1 - \mu) = 1 - \mathrm{int}(\mu) \leq \delta.$ This gives that $1 - \delta \leq \mathrm{int}(\mu).$ Let $\eta = 1 - \delta.$ Hence it follows that, for a fuzzy F_σ-set $\mu,$ there exists a fuzzy F_σ-set η in *X* such that $\eta \leq \mathrm{int}(\mu).$

Proposition 3.9. *If* λ *is a fuzzy* G_δ*-set in a fuzzy Q-space X, then there exist fuzzy* G_δ*-sets* $\mu, \eta, \alpha, \beta, \ldots$ *in X such that* $\lambda \leq \mu \leq \eta \leq \alpha \leq \beta \leq \cdots.$

Proof. Let λ be a fuzzy G_δ-set in a fuzzy *Q*-space *X*. By Proposition 3.8, there exists a fuzzy G_δ-set μ in *X* such that $\mathrm{cl}(\lambda) \leq \mu.$ Then, $\lambda \leq \mathrm{cl}(\lambda) \leq \mu.$ Again, for a fuzzy G_δ-set μ in *X*, there exists a fuzzy G_δ-

set η in *X* such that $\text{cl}(\mu) \leq \eta$. Then, $\lambda \leq \mu \leq \text{cl}(\mu) \leq \eta$ and $\lambda \leq \mu \leq \eta$. By continuing in this way, we can find fuzzy G_δ-sets $\alpha, \beta, \ldots$ in *X* such that $\lambda \leq \mu \leq \eta \leq \alpha \leq \beta \leq \cdots$.

Corollary 3.7. *If* μ *is a fuzzy* F_σ*-set in a fuzzy Q-space X, then there exist fuzzy* F_σ*-sets* $\eta, \alpha, \beta, \ldots$ *in X such that* $\mu \geq \eta \geq \alpha \geq \beta \geq \cdots$.

Proposition 3.10. *If* λ *is a fuzzy* G_δ*-set in a fuzzy Q-space X, then* $\text{clint}(\lambda) \leq \text{intcl}(\lambda)$, *in X.*

Proof. Let λ be a fuzzy G_δ-set in *X*. Since *X* is a fuzzy *Q*-space, by Proposition 3.6, $\text{int}(\lambda) \neq 0$, in *X*. Now $\text{int}(\lambda) \leq \text{intcl}(\lambda)$ implies that $\text{intcl}(\lambda) \neq 0$ and thus λ is a fuzzy somewhere dense set in *X* and then by Proposition 3.1, $\text{clint}(\lambda) \leq \text{intcl}(\lambda)$, in *X*.

Proposition 3.11. *If* λ *is a fuzzy* G_δ*-set in a fuzzy Q-space X, then either* λ *is a fuzzy residual set with* $\text{int}(\lambda) \neq 0$ *or* λ *is a fuzzy cs dense set in X.*

Proof. Let λ be a fuzzy G_δ-set in *X*. Then, either $\text{cl}(\lambda) = 1$ or $\text{cl}(\lambda) \neq 1$, in *X*.

If $\text{cl}(\lambda) = 1$, in *X*, then λ is a fuzzy dense and fuzzy G_δ-set in *X*. By Theorem 2.10, λ is a fuzzy residual set in *X*. Since *X* is a fuzzy *Q*-space, $\text{int}(\lambda) \neq 0$, in *X*. Thus, λ is a fuzzy residual set with $\text{int}(\lambda) \neq 0$, in *X*.

If $\text{cl}(\lambda) \neq 1$, in *X*, then $\text{clint}(\lambda) \leq \text{cl}(\lambda)$ gives that $\text{clint}(\lambda) < 1$ and then $1 - \text{clint}(\lambda) > 0$. By Lemma 2.1, $\text{intcl}(1 - \lambda) > 0$. That is, $\text{intcl}(1 - \lambda) \neq 0$, in *X*. Thus $1 - \lambda$ is a fuzzy somewhere dense set and then λ is a fuzzy cs dense set in *X*.

Proposition 3.12. *If* λ *is a fuzzy first category set in a fuzzy Q-space X, then* λ *is not a fuzzy dense set in X.*

Proof. Let λ be a fuzzy first category set in *X*. Then by Theorem 2.16, there exists a fuzzy F_σ-set η in *X* such that $\lambda \leq \eta$. Then, $\text{cl}(\lambda) \leq \text{cl}(\eta)$, in

X. Since X is a fuzzy Q-space, by Proposition 3.6, for a fuzzy F_σ-set η, $\text{cl}(\eta) \neq 1$, in X. This shows that $\text{cl}(\lambda) \neq 1$ and thus λ is not a fuzzy dense set in X.

4. Fuzzy *Q*-spaces and Other Fuzzy Topological Spaces

In this section, we relate fuzzy Q-spaces to some other well known fuzzy topological spaces.

The following proposition shows that fuzzy Q-spaces are fuzzy almost P-spaces.

Proposition 4.1. *If a fuzzy topological space X is a fuzzy Q-space, then X is a fuzzy almost P-space.*

Proof. Let λ be a non-zero fuzzy G_δ-set in X. Since X is a fuzzy Q-space, by Proposition 3.6, $\text{int}(\lambda) \neq 0$, in X. Hence, for a fuzzy G_δ-set λ, $\text{int}(\lambda) \neq 0$ in X, showing that X is a fuzzy almost P-space.

Example 4.1. Let $X = \{A, B, C\}$. Define fuzzy sets α, β, γ and η as follows:

$\alpha : X \to I$ is defined by $\alpha(A) = 0.6;\ \alpha(B) = 0.4;\ \alpha(C) = 0.5$,

$\beta : X \to I$ is defined by $\beta(A) = 0.5;\ \beta(B) = 0.7;\ \beta(C) = 0.6$,

$\gamma : X \to I$ is defined by $\gamma(A) = 0.4;\ \gamma(B) = 0.5;\ \gamma(C) = 0.7$,

$\eta : X \to I$ is defined by $\eta(A) = 0.5;\ \eta(B) = 0.5;\ \eta(C) = 0.6$.

Clearly,

$$T = \{0, \alpha, \beta, \gamma, \alpha \vee \beta, \alpha \vee \gamma, \beta \vee \gamma, \alpha \wedge \beta, \alpha \wedge \gamma, \beta \wedge \gamma,$$
$$\alpha \vee [\beta \wedge \gamma], \gamma \vee [\alpha \wedge \beta], \beta \wedge [\alpha \vee \gamma], \alpha \vee \beta \vee \gamma, 1\}$$

is a fuzzy topology on *X*. On computation, we find that $\beta, \gamma, \alpha \vee \beta, \alpha \vee \gamma, \beta \vee \gamma, \beta \wedge \gamma, \alpha \vee [\beta \wedge \gamma], \gamma \vee [\alpha \wedge \beta], \beta \wedge [\alpha \vee \gamma]$ and $\alpha \vee \beta \vee \gamma$ are fuzzy dense sets in (X, T) and $\text{cl}(\alpha) = 1 - (\alpha \wedge \gamma)$, $\text{cl}(\alpha \wedge \beta) = 1 - (\alpha \wedge \beta)$, $\text{cl}(\alpha \wedge \gamma) = 1 - \alpha$.

On computation, we find that $\eta = (\alpha \vee \beta) \wedge (\alpha \vee \gamma) \wedge (\beta \vee \gamma)$ and η is a G_δ-set in (X, T) and $\text{int}(\eta) = \beta \wedge [\alpha \vee \gamma] \neq 0$ and hence (X, T) is a fuzzy almost *P*-space.

Now,

$$\text{intcl}(\alpha) = \alpha \neq 0 \text{ and } \text{int}(\alpha) = \alpha \neq 0;$$

$$\text{intcl}(\alpha \wedge \beta) = \alpha \wedge \beta \neq 0 \quad \text{and} \quad \text{int}(\alpha \wedge \beta) = \alpha \wedge \beta \neq 0;$$

$$\text{intcl}(\alpha \wedge \gamma) = \alpha \wedge \gamma \neq 0 \quad \text{and} \quad \text{int}(\alpha \wedge \gamma) = \alpha \wedge \gamma \neq 0.$$

For the fuzzy dense sets $\eta (= \beta, \gamma, \alpha \vee \beta, \alpha \vee \gamma, \beta \vee \gamma, \beta \wedge \gamma, \alpha \vee [\beta \wedge \gamma], \gamma \vee [\alpha \wedge \beta], \beta \wedge [\alpha \vee \gamma], \alpha \vee \beta \vee \gamma)$, $\text{intcl}(\eta) = 1 \neq 0$ and $\text{int}(\eta) = \eta \neq 0$. Thus (X, T) is a fuzzy open hereditarily irresolvable space. For a fuzzy open set α, $\text{cl}(\alpha)$ is not a fuzzy open set in (X, T), implying that (X, T) is not a fuzzy extremally disconnected space. Hence (X, T) is not a fuzzy *Q*-space.

Example 4.2. Let $X = \{A, B, C\}$. Define fuzzy sets α, β, γ and δ as follows:

$\alpha : X \to I$ is defined by $\alpha(A) = 0.5; \alpha(B) = 0.6; \alpha(C) = 0.4$,

$\beta : X \to I$ is defined by $\beta(A) = 0.4; \beta(B) = 0.7; \beta(C) = 0.5$,

$\gamma : X \to I$ is defined by $\gamma(A) = 0.5; \gamma(B) = 0.8; \gamma(C) = 0.6$,

$\delta : X \to I$ is defined by $\delta(A) = 0.4; \delta(B) = 0.5; \delta(C) = 0.4$.

Thus $T = \{0, \alpha, \beta, \gamma, \alpha \vee \beta, \alpha \wedge \beta, 1\}$ is clearly a fuzzy topology on *X*. By computation, we find that for the fuzzy G_δ-sets β and $\alpha \wedge \beta$, $\text{int}(\beta) \neq 0$ and $\text{int}(\alpha \wedge \beta) \neq 0$, in (X, T) and thus (X, T) is a fuzzy almost *P*-space.

By computation, we find that $\alpha, \beta, \gamma, \alpha \vee \beta, \alpha \wedge \beta$ are fuzzy dense sets in (X, T) and hence for each fuzzy open set $\eta (= \alpha, \beta, \gamma, \alpha \vee \beta, \alpha \wedge \beta)$, $\text{cl}(\eta) = 1 \in T$ and thus (X, T) is a fuzzy extremally disconnected space.

Now, for a fuzzy set δ in *X*, $\text{intcl}(\delta) = \text{int}(1) \neq 0$ and $\text{int}(\delta) = 0$ in *X* and thus (X, T) is not a fuzzy open hereditarily irresolvable space. Hence (X, T) is not a fuzzy *Q*-space.

Proposition 4.2. *If* λ *is a fuzzy* G_δ*-set in a fuzzy Q-space X, then there exists a fuzzy regular closed set* γ *in X such that* $\gamma \leq \text{cl}(\lambda)$.

Proof. Let λ be a fuzzy G_δ-set in a fuzzy *Q*-space *X*. Then by Proposition 4.1, the fuzzy *Q*-space *X* is a fuzzy almost *P*-space. Thus, λ is a fuzzy G_δ-set in the fuzzy almost *P*-space *X* and by Theorem 2.3, there exists a fuzzy regular closed set γ in *X* such that $\gamma \leq \text{cl}(\lambda)$.

Corollary 4.1. *If* λ *is a fuzzy* G_δ*-set in a fuzzy Q-space X, then there exists a fuzzy pre-open set* γ *in X such that* $\gamma \leq \text{cl}(\lambda)$.

Proof. Let λ be a fuzzy G_δ-set in a fuzzy *Q*-space *X*. By Proposition 4.2, there exists a fuzzy regular closed set γ in *X* such that $\gamma \leq \text{cl}(\lambda)$. By Corollary 3.5, the fuzzy regular closed set γ is a fuzzy pre-open set in *X*. Hence, for a fuzzy G_δ-set λ, there exists a fuzzy pre-open set γ in *X* such that $\gamma \leq \text{cl}(\lambda)$.

In [19], it has been established that fuzzy Baire spaces are fuzzy second category spaces, and fuzzy second category spaces need not be fuzzy Baire spaces. In this regard, it is aimed to identify those fuzzy topological spaces which are fuzzy second category but not fuzzy Baire.

Proposition 4.3. *If a fuzzy topological space X is a fuzzy Q-space, then X is a fuzzy second category space.*

Proof. Let the fuzzy space X be a fuzzy Q-space. Then, by Proposition 4.1, X is a fuzzy almost P-space and by Theorem 2.4, X is a fuzzy second category space.

Proposition 4.4. *If a fuzzy topological space X is a fuzzy Q-space, then X is not a fuzzy Baire space.*

Proof. Let λ be a fuzzy first category set in X. Now, for a fuzzy set λ in X, $cl(\lambda)$ is a fuzzy closed set in X. Since X is a fuzzy Q-space, X is a fuzzy extremally disconnected space and by Theorem 2.2, for a fuzzy closed set $\mathrm{cl}(\lambda)$, $\mathrm{int}[\mathrm{cl}(\lambda)]$ is a fuzzy closed set in X and then $\mathrm{intcl}(\lambda) \neq 0$, in X. Now X being a fuzzy Q-space, X is a fuzzy open hereditarily irresolvable space and $\mathrm{intcl}(\lambda) \neq 0$ gives that $\mathrm{int}(\lambda) \neq 0$ in X. Thus, for a fuzzy first category set λ, $\mathrm{int}(\lambda) \neq 0$ in X shows, by Theorem 2.9, that X is not a fuzzy Baire space.

Remark 4.1. From Propositions 4.3 and 4.4, we can infer that fuzzy Q-spaces are fuzzy second category spaces but not fuzzy Baire spaces.

Proposition 4.5. *If a fuzzy topological space X is a fuzzy Brown space, then X is not a fuzzy Q-space.*

Proof. Let X be a fuzzy Brown space. Then by Theorem 2.5, X is not a fuzzy extremally disconnected space and hence X is not a fuzzy Q-space.

The following proposition provides a condition for a fuzzy Q-space to become a fuzzy Baire space.

Proposition 4.6. *If* $\mathrm{cl}[\wedge_{i=1}^{\infty}(\lambda_i)] = 1$, *where* (λ_i)'s *are fuzzy dense sets in a fuzzy Q-space X, then X is a fuzzy Baire space.*

Proof. Suppose that (λ_i)'s $(i = 1 \text{ to } \infty)$ are fuzzy dense sets in a fuzzy Q-space X such that $\mathrm{cl}(\wedge_{i=1}^{\infty}(\lambda_i)) = 1$. Now X being a fuzzy Q-space, X is a fuzzy open hereditarily irresolvable space and by Theorem 2.6, for each $\mathrm{cl}(\lambda_i) = 1$, in *X*, $\mathrm{clint}(\lambda_i) = 1$, in *X*. This implies that $1 - \mathrm{clint}(\lambda_i) = 0$ and by Lemma 2.1, $\mathrm{intcl}(1 - \lambda_i) = 1 - \mathrm{clint}(\lambda_i) = 0$. Then $(1 - \lambda_i)$'s are fuzzy nowhere dense sets in *X*. Now

$$\mathrm{int}[\vee_{i=1}^{\infty}(1 - \lambda_i)] = \mathrm{int}[1 - \wedge_{i=1}^{\infty}(\lambda_i)] = 1 - cl[\wedge_{i=1}^{\infty}(\lambda_i)] = 1 - 1 = 0$$

and thus $\mathrm{int}[\vee_{i=1}^{\infty}(1 - \lambda_i)] = 0$, where $(1 - \lambda_i)$'s are fuzzy nowhere dense sets in *X* establishing that *X* is a fuzzy Baire space.

Remark 4.2. If $\mathrm{cl}[\wedge_{i=1}^{\infty}(\lambda_i)] = 1$, where (λ_i)'s are fuzzy dense sets in a fuzzy *Q*-space *X*, then *X* is a fuzzy Baire and fuzzy second category space.

Proposition 4.7. *If a fuzzy topological space X is a fuzzy Q-space*, *then X is a fuzzy Oz-space*.

Proof. Suppose that λ is a fuzzy regular closed set in *X*. Then,

$$\mathrm{clint}(\lambda) = \lambda. \tag{A}$$

Now $\mathrm{int}(\lambda)$ is a fuzzy open set in *X*. Since *X* is a fuzzy *Q*-space, *X* is a fuzzy extremally disconnected space and then, $\mathrm{cl}(\mathrm{int}(\lambda))$ is a fuzzy open set in *X*. Now, from (A), it follows that λ is a fuzzy open set in *X*. Also *X* being a fuzzy extremally disconnected space, $\mathrm{cl}(\lambda)$ is a fuzzy open set in *X* and $\lambda \leq \mathrm{cl}(\lambda)$ gives that $\lambda = \lambda \wedge \mathrm{cl}(\lambda)$ and $\lambda, \mathrm{cl}(\lambda) \in T$ implies that λ is a fuzzy G_δ-set in *X*. Hence, fuzzy regular closed set λ is a fuzzy G_δ-set in *X* showing that *X* is a fuzzy *Oz*-space.

The following propositions give conditions under which fuzzy *Oz*-spaces become fuzzy *Q*-spaces.

Proposition 4.8. *If a fuzzy topological space X is a fuzzy open hereditarily irresolvable*, *fuzzy Oz and fuzzy P-space*, *then X is a fuzzy Q-space*.

Proof. Suppose that the fuzzy topological space *X* is a fuzzy open hereditarily irresolvable, fuzzy *Oz* and fuzzy *P*-space. By Theorem 2.8, the fuzzy *Oz* and fuzzy *P*-space *X* is a fuzzy extremally disconnected space. Thus, *X* is a fuzzy open hereditarily irresolvable and fuzzy extremally disconnected space. Hence *X* is a fuzzy *Q*-space.

Proposition 4.9. *If a fuzzy topological space X is a fuzzy open hereditarily irresolvable, fuzzy Baire-dominated and fuzzy extraresolvable space, then X is a fuzzy Q-space.*

Proof. Suppose that the fuzzy topological space *X* is a fuzzy open hereditarily irresolvable, fuzzy Baire-dominated and fuzzy extraresolvable space. By Theorem 2.11, the fuzzy Baire-dominated and fuzzy extraresolvable space *X* is a fuzzy extremally disconnected space. Thus, *X* is a fuzzy open hereditarily irresolvable space which is also a fuzzy extremally disconnected space and hence the space *X* is a fuzzy *Q*-space.

Proposition 4.10. *If a fuzzy topological space X is a fuzzy hyperconnected and fuzzy irresolvable space, then* (X, T) *is a fuzzy Q-space.*

Proof. Suppose that λ is a fuzzy open set in *X*. Since *X* is a fuzzy hyperconnected space, λ is a fuzzy dense set in (X, T) and $\text{cl}(\lambda) = 1_X \in T$. This implies that $\text{cl}(\lambda)$ is a fuzzy open set in *X*. Hence *X* is a fuzzy extremally disconnected space.

Let λ be a non-zero fuzzy set in the space *X* such that $\text{intcl}(\lambda) \neq 0$. Now $\text{intcl}(\lambda)$ is a non-zero fuzzy open set in *X*. Since *X* is a fuzzy hyperconnected space, $cl[\text{int}\, cl(\lambda)] = 1$, in *X*. Then, $\text{cl}[\text{intcl}(\lambda)] \leq \text{cl}[\text{cl}(\lambda)] = \text{cl}(\lambda)$ and thus $1 \leq \text{cl}(\lambda)$. That is, $\text{cl}(\lambda) = 1$, in *X*. Since *X* is a fuzzy irresolvable space, for the fuzzy dense set λ, $\text{cl}(1 - \lambda) \neq 1$, in *X*. By Lemma 2.1, $1 - \text{int}(\lambda) \neq 1$ and thus $\text{int}(\lambda) \neq 0$ in *X*. That is, for a non-zero fuzzy set λ in *X* with $\text{intcl}(\lambda) \neq 0$, we have $\text{int}(\lambda) \neq 0$, in *X* and this shows that *X* is a fuzzy open hereditarily irresolvable space. Thus, *X* is a fuzzy open hereditarily irresolvable space which is also a fuzzy extremally disconnected space. Hence the space *X* is a fuzzy *Q*-space.

Remark 4.3. If $\text{int}(\lambda) = 0$ and $\text{cl}(\lambda) = 1$, for a fuzzy set λ in a fuzzy hyperconnected space *X*, then *X* is not a fuzzy *Q*-space. $\text{intcl}(\lambda) = \text{int}(1) = 1 \neq 0$ and $\text{int}(\lambda) = 0$ prove that *X* is not a fuzzy open hereditarily irresolvable space and hence the space *X* is not a fuzzy *Q*-space.

Proposition 4.11. *If a fuzzy topological space X is a fuzzy regular and fuzzy Q-space, then X is not a fuzzy hyperconnected space.*

Proof. Suppose that λ is a fuzzy open set in *X*. Since *X* is a fuzzy regular space, by Theorem 2.12, the fuzzy open set λ is a fuzzy F_σ-set in *X*. Also, since *X* is a fuzzy *Q*-space, by Proposition 3.7, for a fuzzy F_σ-set λ, $\text{cl}(\lambda) \neq 1$, in *X*. Thus, a fuzzy open set λ is not a fuzzy dense set in *X*. Hence the space *X* is not a fuzzy hyperconnected space.

Proposition 4.12. *If a fuzzy topological space X is a fuzzy open hereditarily irresolvable, fuzzy regular Oz and weak fuzzy P-space, then X is a fuzzy Q-space.*

Proof. Suppose that *X* is a fuzzy open hereditarily irresolvable, fuzzy regular *Oz* and weak fuzzy *P*-space. Then the fuzzy regular *Oz* and weak fuzzy *P*-space *X* is, by Theorem 2.15, a fuzzy extremally disconnected space. Thus *X* is a fuzzy open hereditarily irresolvable space which is also a fuzzy extremally disconnected space and hence the space *X* is a fuzzy *Q*-space.

Proposition 4.13. *If a fuzzy topological space X is a fuzzy maximal space, then the space X is a fuzzy Q-space.*

Proof. Suppose that *X* is a fuzzy maximal space. Then, *X* is a fuzzy extremally disconnected and fuzzy submaximal space. The fuzzy submaximal space *X* is, by Theorem 2.7, a fuzzy open hereditarily irresolvable space. Thus, *X* is a fuzzy open hereditarily irresolvable and fuzzy extremally disconnected space. Hence the space *X* is a fuzzy *Q*-space.

Remark 4.4. It should be noted that the converse of Proposition 4.13 need not be true. That is, a fuzzy *Q*-space need not be a fuzzy maximal space. The following example shows that the space *X* is a fuzzy *Q*-space but not a fuzzy maximal space.

Example 4.3. Let $X = \{A, B, C\}$. Then the fuzzy sets $\alpha, \beta, \gamma, \delta, \eta$ and θ are defined on X as follows:

$\alpha : X \to I$ is defined by $\alpha(A) = 0.5;\ \alpha(B) = 0.6;\ \alpha(C) = 0.4,$

$\beta : X \to I$ is defined by $\beta(A) = 0.4;\ \beta(B) = 0.5;\ \beta(C) = 0.6,$

$\gamma : X \to I$ is defined by $\gamma(A) = 0.6;\ \gamma(B) = 0.4;\ \gamma(C) = 0.5,$

$\delta : X \to I$ is defined by $\delta(A) = 0.5;\ \delta(B) = 0.5;\ \delta(C) = 0.5,$

$\eta : X \to I$ is defined by $\eta(A) = 0.6;\ \eta(B) = 0.5;\ \eta(C) = 0.4,$

$\theta : X \to I$ is defined by $\theta(A) = 0.7;\ \theta(B) = 0.6;\ \theta(C) = 0.5.$

Thus,

$$T = \{0, \alpha, \beta, \gamma, \alpha \vee \beta, \alpha \vee \gamma, \beta \vee \gamma, \alpha \wedge \beta, \alpha \wedge \gamma, \beta \wedge \gamma, \alpha \vee [\beta \wedge \gamma], \beta \vee [\alpha \wedge \gamma], \gamma \vee [\alpha \wedge \beta], \alpha \wedge [\beta \vee \gamma], \beta \wedge [\alpha \vee \gamma], \gamma \wedge [\alpha \vee \beta], \alpha \vee \beta \vee \gamma, \alpha \wedge \beta \wedge \gamma, 1\}$$

is a fuzzy topology on X. On computation, we find that closure of each fuzzy open set is fuzzy open in X and thus X is a fuzzy extremally disconnected space. Also, on computation, we find that

$\operatorname{intcl}(\delta) = \operatorname{int}(1 - [\beta \wedge [\alpha \vee \gamma]]) = \gamma \vee [\alpha \wedge \beta] \neq 0$ and

$\operatorname{int}(\delta) = \alpha \wedge [\beta \vee \gamma] \neq 0;$

$\operatorname{intcl}(\eta) = \operatorname{int}(1 - \beta) = \alpha \wedge [\beta \vee \gamma] \neq 0$ and $\operatorname{int}(\eta) = \alpha \wedge [\beta \vee \gamma] \neq 0;$

$\operatorname{intcl}(\theta) = \operatorname{int}(1) = 1 \neq 0$ and $\operatorname{int}(\theta) = \alpha \vee \gamma \neq 0.$

Thus, X is a fuzzy open hereditarily irresolvable space. Hence X is a fuzzy Q-space. Now, for the fuzzy dense θ in X, $\operatorname{int}(\theta) = \alpha \vee \gamma \neq \theta$ and thus θ is not a fuzzy open set in X. This means that X is not a fuzzy submaximal space and hence X is not a fuzzy maximal space.

The following proposition establishes that fuzzy *Q*-spaces are not fuzzy F_σ-complemented spaces.

Proposition 4.14. *If a fuzzy topological space X is a fuzzy Q-space, then the space X is not a fuzzy* F_σ*-complemented space.*

Proof. Suppose that *X* is a fuzzy F_σ-complemented space. Then, for each fuzzy F_σ-set λ in *X*, there exists a fuzzy F_σ-set μ in *X* such that $\lambda \leq 1 - \mu$ and $\mathrm{cl}(\lambda \vee \mu) = 1$. Now λ and μ are fuzzy F_σ-sets in *X* implying that $\lambda \vee \mu$ is a fuzzy F_σ-set in *X* and $\mathrm{cl}(\lambda \vee \mu) = 1$. Thus, the fuzzy F_σ-set $\lambda \vee \mu$ is a fuzzy dense set in the fuzzy *Q*-space *X*. But this is a contradiction, by Proposition 3.7, which states that fuzzy F_σ-sets are not fuzzy dense sets in fuzzy *Q*-spaces. Hence the space *X* is not a fuzzy F_σ-complemented space.

Proposition 4.15. *If a fuzzy topological space X is a fuzzy globally disconnected and fuzzy open hereditarily irresolvable space, then X is a fuzzy Q-space.*

Proof. Suppose that λ is a fuzzy open set in *X*. Then, $\lambda \leq \mathrm{intcl}(\lambda)$, in *X*. This implies that $\mathrm{cl}(\lambda) \leq \mathrm{cl}(\mathrm{intcl}(\lambda))$ and then, $\mathrm{cl}(\lambda)$ is a fuzzy semi-open set in *X*. Since *X* is a fuzzy globally disconnected space, $\mathrm{cl}(\lambda)$ is a fuzzy ope1n set in *X*. Hence *X* is a fuzzy extremally disconnected space. Thus, *X* is a fuzzy open hereditarily irresolvable space which is also a fuzzy extremally disconnected space and hence the space *X* is a fuzzy *Q*-space.

5. Conclusion

In this paper, the notion of fuzzy *Q*-spaces is introduced in terms of fuzzy extremally disconnectedness and fuzzy open hereditarily irresolvablity of fuzzy topological spaces. In fuzzy *Q*-spaces, fuzzy semi-open sets are found to be fuzzy α-open sets and fuzzy regular open sets as fuzzy pre-closed sets. It is also obtained that fuzzy G_δ-sets are having non-zero interior and fuzzy F_σ-sets are not fuzzy dense sets in fuzzy *Q*-spaces. It is established that fuzzy first category sets are not fuzzy dense sets in fuzzy *Q*-spaces.

A condition for a fuzzy extremely disconnected space to become a fuzzy *Q*-space is established in this paper. It is obtained that fuzzy *Q*-spaces are fuzzy second category but not fuzzy Baire spaces and fuzzy *Q*-spaces are not fuzzy F_σ-complemented spaces. It is found that fuzzy *Q*-spaces are fuzzy almost *P*-spaces, and fuzzy maximal spaces are fuzzy *Q*-spaces. It is established that fuzzy globally disconnected and fuzzy open hereditarily irresolvable spaces are fuzzy *Q*-spaces, and fuzzy Brown spaces are not fuzzy *Q*-spaces.

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References

[1] K. K. Azad, On fuzzy semi continuity, fuzzy almost continuity and fuzzy weakly continuity, J. Math. Anal. Appl. 82 (1981), 14-32.

[2] G. Balasubramanian, Maximal fuzzy topologies, Kybernetika 31(5) (1995), 459-464.

[3] A. S. Bin Shahna, On fuzzy strong semi-continuity and fuzzy pre-continuity, Fuzzy Sets and Systems 44 (1991), 303-308.

[4] Bojan Nikolić, Degenerated topological spaces, Novi Sad. J. Math. 49(2) (2019), 95-107.

[5] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968), 182-190.

[6] A. G. El'Kin, Ultrafilters and undecomposable spaces, Vestnik Mosk. Univ. Math. 24(5) (1969), 51-56.

[7] B. Ghosh, Fuzzy extremally disconnected spaces, Fuzzy Sets and Systems 46(2) (1992), 245-250.

[8] R. A. Herrmann, RC-convergence, Proc. Amer. Math. Soc. 75(2) (1979), 311-317.

[9] E. Hewitt, A problem of set-theoretic topology, Duke Math. J. 10(2) (1943), 309-333.

[10] D. S. Jankovic, A note on mappings of extremally disconnected spaces, Acta. Math. Hung. 46(1-2) (1985), 83-92.

[11] M. Katetov, On topological spaces containing no disjoint dense sets, Rec. Math. [Mat. Sbornik] N. S. 63(1) (1947), 3-12.

[12] Miguel Caldas, Govindappa Navalagi and Ratnesh Saraf, On fuzzy weakly semi-open functions, Proyecciones 21(1) (2002), 51-63.

[13] Ratna Dev Sharma, Extremal disconnectedness in fuzzy topological spaces, Mat. Vesnik 46 (1994), 17-23.

[14] M. K. Singal and Niti Rajvanshi, Fuzzy alpha-sets and alpha-continuous maps, Fuzzy Sets and Systems 48 (1992), 383-390.

[15] G. Thangaraj and G. Balasubramanian, On fuzzy basically disconnected spaces, J. Fuzzy Math. 9(1) (2001), 103-110.

[16] G. Thangaraj and G. Balasubramanian, On somewhat fuzzy continuous functions, J. Fuzzy Math. 11(2) (2003), 725-736.

[17] G. Thangaraj, Resolvability and irresolvability in fuzzy topological spaces, News Bull. Cal. Math. Soc. 31(4-6) (2008), 11-14.

[18] G. Thangaraj and G. Balasubramanian, On fuzzy resolvable and fuzzy irresolvable spaces, Fuzzy Sets, Rough Sets and Multi-valued Operations and Applications 1(2) (2009), 173-180.

[19] G. Thangaraj and S. Anjalmose, On fuzzy Baire spaces, J. Fuzzy Math. 21(3) (2013), 667-676.

[20] G. Thangaraj, C. Anbazhagan and P. Vivakanandan, On fuzzy *P*-spaces, weak fuzzy *P*-spaces and fuzzy almost *P*-spaces, Gen. Math. Notes 18(2) (2013), 128-139.

[21] G. Thangaraj and C. Anbazhagan, On fuzzy almost *P*-spaces, Inter. J. Innov. Sci. Engin. and Tech. 2(4) (2015), 389-407.

[22] G. Thangaraj and S. Senthil, On somewhere fuzzy continuous functions, Anal. Fuzzy Math. Inform. 15(2) (2018), 181-198.

[23] G. Thangaraj and S. Muruganantham, On fuzzy globally disconnected spaces, J. Tri. Math. Soc. 21 (2019), 37-46.

[24] G. Thangaraj and S. Soundararajan, On fuzzy regular Volterra Spaces, J. Comput. Math. 4(1) (2020), 17-25.

[25] G. Thangaraj and S. Lokeswari, On fuzzy maximal spaces, Anal. Fuzzy Math. Inform. 19(3) (2020), 281-293.

[26] G. Thangaraj and M. Ponnusamy, Some remarks on fuzzy brown spaces, High Technology Letters 30(5) (2024), 653-672.

[27] G. Thangaraj and M. Ponnusamy, On fuzzy *Oz*-spaces, Adv. and Appl. Math. Sci. 22(5) (2023), 939-954.

[28] G. Thangaraj and M. Ponnusamy, Fuzzy regular *Oz*-spaces, Ann. Fuzzy Math. Inform. 25(1) (2023), 1-14.

[29] G. Thangaraj and L. Vikraman, On fuzzy F_{σ}-complemented spaces, Adv. and Appl. Math. Sci. 22(9) (2023), 1991-2008.

[30] G. Thangaraj and L. Vikraman, Fuzzy EZ-spaces and related concepts, Stoch. Model. and Comput. Sci. 3(1) (2023), 168-180.

[31] G. Thangaraj and M. Raji, On fuzzy Baire-dominated spaces, IOSR J. Math. 2(3) (2024), 1-7.

[32] G. Thangaraj and A. Vinothkumar, Some remarks on fuzzy almost *P*-spaces, Mukt Shabd J. 11(7) (2022), 386-415.

[33] G. Thangaraj and R. Palani, A short note on fuzzy residual sets and fuzzy functions, Int. J. Adv. Math. 2 (2017), 1-9.

[34] D. Vijayan, A study on various forms of resolvability in fuzzy topological spaces, Ph. D Thesis, Thiruvalluvar University, Tamilnadu, India 2016.

[35] L. A. Zadeh, Fuzzy sets, Inform. and Control. 8 (1965), 338-353.